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Zariski topology on the spectrum of fuzzy classical primary submodules

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Abstract

Let R be a commutative ring with identity and M a unitary R-module. The fuzzy classical primary spectrum $Fcp.spec(M)$ is the collection of all fuzzy classical primary submodules $\mathcal{A}$ of M, the recent generalization of fuzzy primary ideals and fuzzy classical prime submodules. In this paper, we topologize $\mathcal{FM}(M)$ with a topology having the fuzzy primary Zariski topology on the fuzzy classical primary spectrum $Fcp.spec(M)$ as a subspace topology, and investigate the properties of this topological space.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. The fuzzy classical primary submodule in module theory plays crucial role in algebra. The concept of fuzzy classical primary submodule which is a generalization of fuzzy primary ideals and fuzzy classical prime submodules. The Zariski topology on the prime spectrum of an R-module have

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been introduced by Lu [13] and these have been studied by several authors [1, 2, 6, 7, 8, 9, 15, 16]. In 2008, Ameri and Mahjoob [3] investigated some properties of the Zarisky topology of prime L-submodules. As it is well known that Ameri and Mahjoob in 2009 [4] introduced the notion of the Zarisky topology of prime fuzzy hyperideals. In 2013, Darani and Motmaen [10] introduced and studied the concept of the Zariski topology on the spectrum of graded classical prime submodules and these have been studied by several authors [18]. The concept of Zariski topology on prime fuzzy submodules was introduced Ameri and Mahjoob [5] in 2017. In 2021, Goswami and Saikia [11] gave the concept of the spectrum of weakly prime submodules and investigated related properties.

In this paper, we rely on the fuzzy classical primary submodules, and then introduce and study a new topology on the fuzzy classical primary spectrum $Fcp.spec(M)$ is the collection of all fuzzy classical primary submodules $\mathcal{A}$ of M, which generalizes the Zariski topology of fuzzy prime submodule, called fuzzy primary Zariski topology and investigate several properties of the topology.

2. Basic definitions and preliminary results

In this section, a brief overview of the concepts of fuzzy sets and fuzzy modules are required in this study.

A function $\mu : M \to [0, 1]$ is called a **fuzzy set** [17] of a non empty set M.

The concept of fuzzy ideals of a ring was introduced in [12] as a generalization of the notion of fuzzy subrings.

Definition 2.1 ([12])**.** A fuzzy set μ of a ring R is called a **fuzzy ideal** of R if

(1) $\mu(ab) \geq \mu(a) \vee \mu(b)$ for all $a, b \in R$;
(2) $\mu(a - b) \geq \mu(a) \wedge \mu(b)$ for all $a, b \in R$.

The set of all fuzzy sets (fuzzy ideals) of R is denoted by $\mathcal{FS}(R)$ $(\mathcal{FI}(R))$. Let μ be a fuzzy set of a ring R. The radical of μ is denoted by $\Re(\mu)$ and is defined by $(\Re(\mu))(r) = \bigvee_{n\in\mathbf{N}} \mu(r^n)$ for every element $r \in R$.

Definition 2.2. Let μ be a fuzzy ideal of a ring R. A fuzzy set μ is called a **fuzzy primary ideal** of R if for every fuzzy ideals ν and η of R with $\nu \odot \eta \leq \mu$, then either $\nu \leq \mu$ or $\eta \leq \Re(\mu)$

The concept of fuzzy modules of an R-module M was introduced in [14] as a generalization of the notion of fuzzy ideals.

Definition 2.3 ([14])**.** A fuzzy set $\mathcal{A}$ of an R-module M is called a **fuzzy module** of M if

(1) $\mathcal{A}(0) = 1$;
(2) $\mathcal{A}(am) \geq \mathcal{A}(m)$ for all $a \in R$ and $m \in M$;
(3) $\mathcal{A}(m - n) \geq \mathcal{A}(m) \wedge \mathcal{A}(n)$ for all $m, n \in M$.

Condition (3) of the above definition is equivalent to $\mathcal{A}(m+n) \geq \mathcal{A}(m) \wedge \mathcal{A}(n)$, and $\mathcal{A}(m) = \mathcal{A}(-m)$ for all $m, n \in M$. The set of all fuzzy sets (fuzzy modules) of an R-module M is denoted by $\mathcal{FS}(M)$ ($\mathcal{FM}(M)$).

Theorem 2.4. *Let $\mathcal{A}$ be a fuzzy set over an R-module R such that $\mathcal{A}(0) = 1$. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is a fuzzy module of R.*
(2) *$\mathcal{A}$ is a fuzzy ideal of R.*

Proof. By Definition 2.3 it suffices to prove that (2) implies (1). Assume that (2) holds. Let a and b be any elements of R. Since $\mathcal{A}$ is a fuzzy ideal of R with $\mathcal{A}(0) = 1$, we have $\mu(ab) \geq \mu(a) \vee \mu(b) \geq \mu(b)$ and $\mu(a-b) \geq \mu(a) \wedge \mu(b)$. Therefore, we obtain that $\mathcal{A}$ is a fuzzy module of R. □

Let N be a non empty subset of an R-module M. For each $\alpha \in [0, 1)$, a **characteristic function** of N is denoted by $\alpha\mathcal{C}_N$ and is defined as

$$(\alpha\mathcal{C}_N)(m) = \begin{cases} 1 & ; m \in N \\ \alpha & ; \text{otherwise.} \end{cases}$$

We note that the R-module M can be considered a bipolar fuzzy set of itself and we write $\mathcal{M} = \alpha\mathcal{C}_M$ ($\mathcal{R} = \alpha\mathcal{C}_R$), i.e., $\mathcal{M}(m) = 1$ for all $m \in M$.

In the following theorem, we establish a relationship between bipolar fuzzy modules and submodules of an R-module.

Theorem 2.5. *Let α be any element of $[0, 1)$ and let M be an R-module. Then the following conditions are equivalent.*

(1) *N is a submodule of M.*
(2) *The characteristic function $\alpha\mathcal{C}_N$ of N is a fuzzy module over M.*

Proof. First assume that N is a submodule of M. Since 0 is an element of N, we have $(\alpha\mathcal{C}_N)(0) = 1$. Let m and n be any elements of M and $r \in R$. If $m, n \in N$, then $(\alpha\mathcal{C}_N)(m) = 1 = (\alpha\mathcal{C}_N)(n)$ and since $m - n, rm \in N$, we have $(\alpha\mathcal{C}_N)(rm) = 1 = (\alpha\mathcal{C}_N)(m)$ and $(\alpha\mathcal{C}_N)(m-n) = 1 = (\alpha\mathcal{C}_N)(m) \wedge (\alpha\mathcal{C}_N)(n)$. Otherwise, if $m \notin N$ or $n \notin N$, then $(\alpha\mathcal{C}_N)(m) = \alpha$ or $(\alpha\mathcal{C}_N)(m) = \alpha$ and so we have $(\alpha\mathcal{C}_N)(m-n) \geq \alpha = (\alpha\mathcal{C}_N)(m) \wedge (\alpha\mathcal{C}_N)(n)$. It is obvious that $(\alpha\mathcal{C}_N)(rm) \geq \alpha = (\alpha\mathcal{C}_N)(m)$. Therefore $\alpha\mathcal{C}_N$ is a fuzzy module over M and hence (1) implies (2).

Conversely, assume that (2) holds. Let m and n be any elements of M and $r \in R$ such that $m, n \in N$. Set $x = rm$ and $y = m - n$. Then $(\alpha\mathcal{C}_N)(x) = (\alpha\mathcal{C}_N)(rm) \geq (\alpha\mathcal{C}_N)(m) = 1$ and $(\alpha\mathcal{C}_N)(y) = (\alpha\mathcal{C}_N)(m-n) \geq (\alpha\mathcal{C}_N)(m) \wedge (\alpha\mathcal{C}_N)(n) = 1 \wedge 1 = 1$. Hence we have $(\alpha\mathcal{C}_N)(x) = 1$ and $(\alpha\mathcal{C}_N)(y) = 1$, and so $m - n, rm \in N$. Therefore N is a submodule of M and hence (1) implies (2). □

From the above result, we have the following corollary:

Corollary 2.6. *Let M be an R-module. Then the following conditions are equivalent.*

(1) *N is a submodule of M.*
(2) *The characteristic function $0\mathcal{C}_N$ of N is a fuzzy module over M.*

Let μ and $\mathcal{A}$ be a fuzzy set over a ring R and fuzzy set over an R-module M, respectively. Define the **composition** $\mu \odot \mathcal{A}$, and **product** $\mu\mathcal{A}$ respectively as follows:

$$(\mu \odot \mathcal{A})(x) = \begin{cases} \bigvee\limits_{x=rm} (\mu(r) \wedge \mathcal{A}(m)) & ; \text{if } x = rm \text{ for some } r \in R, m \in M \\ 0 & ; \text{otherwise,} \end{cases}$$

and

$$(\mu\mathcal{A})(x) = \begin{cases} \bigvee\limits_{x=\sum\limits_{i}^{n} r_i m_i} \bigwedge\limits_{i=1}^{n} \mu(r_i) \wedge \bigwedge\limits_{i=1}^{n} \mathcal{A}(m_i) & ; x = \sum\limits_{i}^{n} r_i m_i \exists r_i \in R, m_i \in M \\ 0 & ; \text{otherwise.} \end{cases}$$

Next, let x be an element of an R-module M and $\alpha \in (0, 1]$. Define the fuzzy set x_α over M as follows:

$$x_\alpha(a) = \begin{cases} \alpha & ; x = a \\ 0 & ; \text{otherwise.} \end{cases}$$

Then x_α is called a **fuzzy point or fuzzy singleton**. Let $\mathcal{A}$ be a fuzzy set over an R-module M. Next, let $\langle \mathcal{A} \rangle$ denote the intersection of all fuzzy modules over M which contain $\mathcal{A}$. Then $\langle \mathcal{A} \rangle$ is a fuzzy module over M, called the **fuzzy module generated by** $\mathcal{A}$.

Definition 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be any fuzzy sets of an R-module M. For every fuzzy set μ of R define $(\mathcal{A} : \mathcal{B})$ and $(\mathcal{A} : \mu)$, as follows:

$$(\mathcal{A} : \mathcal{B}) = \bigvee \{\mu \in \mathcal{FS}(R) : \mu \odot \mathcal{B} \leq \mathcal{A}\}$$

and

$$(\mathcal{A} : \mu) = \bigvee \{\mathcal{B} \in \mathcal{FS}(M) : \mu \odot \mathcal{B} \leq \mathcal{A}\}.$$

3. Topologies on fuzzy classical primary submodules

The given definition of fuzzy classical primary submodule is a generalization of the notion of classical prime and classical primary submodules in module theory.

Definition 3.1. Let $\mathcal{A}$ be a fuzzy submodule of an R-module M. A fuzzy set $\mathcal{A}$ is called a **fuzzy classical primary submodule** of M if for every elements a and b of R and every element x of M with $a_\zeta b_\xi x_\alpha \in \mathcal{A}$, then either $a_\zeta x_\alpha \in \mathcal{A}$ or $b_\zeta^n x_\alpha \in \mathcal{A}$ for some positive integer n.

We now present the following example satisfying above definition.

Example 3.2. Let $\mathbf{Z}$ be the set of all integers. Suppose $M = R = \mathbf{Z}$ is a commutative ring. Define the fuzzy set $\mathcal{A}$ of $\mathbf{Z}$ as follows:

$$\mathcal{A}(x) = \begin{cases} 1 & ; \text{if } x \in 4\mathbf{Z} \\ 0 & ; \text{if } x \notin 4\mathbf{Z}. \end{cases}$$

Then it is easily seen that $\mathcal{A}$ is a fuzzy classical primary submodule of an an R-module M.

Let M be an R-module. In the sequel $Fcp.spec(M)$ denotes the set of all fuzzy classical primary submodules of an R-module M. We call $Fcp.spec(M)$, the fuzzy classical primary spectrum of M. For every fuzzy submodule $\mathcal{A}$ of M, the fuzzy classical variety of $\mathcal{A}$ is denoted by $\mathcal{V}(\mathcal{A})$, and is defined as the set of all fuzzy classical primary submodule containing $\mathcal{A}$, i.e., $\mathcal{V}(\mathcal{A}) = \{\mathcal{B} \in Fcp.spec(M) : \mathcal{A} \leq \mathcal{B}\}$.

Theorem 3.3. *For any family of fuzzy submodules $\{\mathcal{A}_i\}_{i\in I}$ of an R-module M. Then the following properties hold.*

(1) $\mathcal{V}(0_1) = Fcp.spec(M)$ *and* $\mathcal{V}(\mathcal{M}) = \varnothing$.

(2) $\bigcap_{i\in I} \mathcal{V}(\mathcal{A}_i) = \mathcal{V}\left(\sum_{i\in I} \mathcal{A}_i\right)$.

(3) $\mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2) = \mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2)$.

Proof. (1). Obvious.

(2). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \bigcap_{i\in I} \mathcal{V}(\mathcal{A}_i)$. Then we have $\mathcal{B} \in \mathcal{V}(\mathcal{A}_i)$ for all $i \in I$, i.e., $\mathcal{A}_i \leq \mathcal{B}$. Next let x be an element of M. We also consider

$$\begin{aligned} \left(\sum_{i\in I} \mathcal{A}_i\right)(x) &= \bigvee_{x=\sum_{i\in I} x_i} \left(\bigwedge_{x=\sum_{i\in I} x_i} \mathcal{A}_i(x_i) \right) \\ &\leq \bigvee_{x=\sum_{i\in I} x_i} \left(\bigwedge_{x=\sum_{i\in I} x_i} \mathcal{B}(x_i) \right) \\ &= \bigvee_{x=\sum_{i\in I} x_i} \mathcal{B}(x_i) \\ &= \mathcal{B}(x) \end{aligned}$$

ans so $\sum_{i\in I}\mathcal{A}_i \leq \mathcal{B}$. Thus we have $\mathcal{B} \in \mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right)$, which implies that $\bigcap_{i\in I}\mathcal{V}(\mathcal{A}_i) \subseteq \mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right)$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right)$. It is easy to see that $\mathcal{A}_i \leq \sum_{i\in I}\mathcal{A}_i \leq \mathcal{B}$, i.e., $\mathcal{B} \in \mathcal{V}(\mathcal{A}_i)$ for all $i \in I$. Therefore $\mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right) \subseteq \bigcap_{i\in I}\mathcal{V}(\mathcal{A}_i)$ and hence $\bigcap_{i\in I}\mathcal{V}(\mathcal{A}_i) = \mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right)$.

(3). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2)$. Then we have $\mathcal{A}_1 \leq \mathcal{B}$ or $\mathcal{A}_2 \leq \mathcal{B}$, it follows that $\mathcal{A}_1 \wedge \mathcal{A}_2 \leq \mathcal{B}$. Thus $\mathcal{B} \in \mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2)$ and so $\mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2) \subseteq \mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2)$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2)$. This implies that $\mathcal{A}_1 \wedge \mathcal{A}_2 \leq \mathcal{B}$, i.e., $\mathcal{A}_1 \leq \mathcal{B}$ or $\mathcal{A}_2 \leq \mathcal{B}$. Also, $\mathcal{B} \in \mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2)$. Therefore, we obtain that $\mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2) \subseteq \mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2)$ and hence $\mathcal{V}(\mathcal{A}_1) \cup \mathcal{V}(\mathcal{A}_2) = \mathcal{V}(\mathcal{A}_1 \wedge \mathcal{A}_2)$. □

Corollary 3.4. *Let μ and ν be any fuzzy ideal of a ring R. Then $\mathcal{V}(\mu \odot \mathcal{M}) \cup \mathcal{V}(\nu \odot \mathcal{M}) = \mathcal{V}(\mu \odot \nu \odot \mathcal{M})$.*

Set $X = Fcp.spec(M)$. For every fuzzy submodule $\mathcal{A}$ of an R-module M we define $\mathcal{E}(\mathcal{A})$ and τ as follows:

$$\mathcal{E}(\mathcal{A}) = X - \mathcal{V}(\mathcal{A}) \text{ and } \tau = \{\mathcal{E}(\mathcal{A}) : \mathcal{A} \in \mathcal{FM}(M)\}.$$

In the next theorem we will show that the pair (X, τ) is a topological space.

Theorem 3.5. *Let M be an R-module. Then the following statements hold:*

(1) *The pair (X, τ) is a topological space.*
(2) *X is a T_0 topological space.*

Proof. 1. Since $\mathcal{V}(0_1) = X$ and $\mathcal{V}(\mathcal{M}) = \varnothing$, we have $\mathcal{E}(0_1) = X - X = \varnothing$ and $\mathcal{E}(\mathcal{M}) = X - \varnothing = X$, .i.e., $\varnothing, X \in \tau$.

2. Let $\mathcal{A}$ and $\mathcal{B}$ be any fuzzy submodules of M. Thus by Theorem 3.3(3), we have

$$\begin{aligned}
\mathcal{E}(\mathcal{A}) \cap \mathcal{E}(\mathcal{B}) &= (X - \mathcal{V}(\mathcal{A})) \cap (X - \mathcal{V}(\mathcal{B})) \\
&= X \cap \left(\mathcal{V}(\mathcal{A})' \cap \mathcal{V}(\mathcal{B})'\right) \\
&= X \cap (\mathcal{V}(\mathcal{A}) \cup \mathcal{V}(\mathcal{B}))' \\
&= X - \mathcal{V}(\mathcal{A} \wedge \mathcal{B}) \\
&= \mathcal{E}(\mathcal{A} \wedge \mathcal{B}).
\end{aligned}$$

3. For any family of fuzzy submodules $\{\mathcal{A}_i\}_{i\in I}$ of M. Then by Theorem 3.7(2), we have

$$\begin{aligned}\bigcup_{i\in I}\mathcal{E}(\mathcal{A}_i) &= \bigcup_{i\in I}(X-\mathcal{V}(\mathcal{A}_i))\\ &= X-\bigcap_{i\in I}\mathcal{V}(\mathcal{A}_i)\\ &= X-\mathcal{V}\left(\sum_{i\in I}\mathcal{A}_i\right)\\ &= \mathcal{E}\left(\sum_{i\in I}\mathcal{A}_i\right).\end{aligned}$$

2 and 3 show that τ is closed under arbitrary union and finite intersection. Thus the pair (X,τ) satisfies in axioms of a topological space. Therefore we have (X,τ) is a topological space.

(2) Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct points of X. If $\mathcal{A}\not\leq\mathcal{B}$, then obviously $\mathcal{B}\in\mathcal{E}(\mathcal{A})$ and $\mathcal{A}\notin\mathcal{E}(\mathcal{A})$ showing that X is a T_0 topological space. □

In this case, the topology τ on X is called the **fuzzy primary Zariski topology**. For every fuzzy submodule $\mathcal{A}$ of M, the set

$$\mathcal{V}^*(\mathcal{A})=\left\{\mathcal{B}\in Fcp.spec(M):\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}\right\}.$$

Then we have the following lemma.

Lemma 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be any fuzzy submodules of an R-module M. If $\mathcal{A}\leq\mathcal{B}$, then $\mathcal{V}^*(\mathcal{B})\leq\mathcal{V}^*(\mathcal{A})$.*

Proof. Let $\mathcal{C}$ be a fuzzy submodule of M such that $\mathcal{C}\in\mathcal{V}^*(\mathcal{B})$. Then we have $\sqrt{(\mathcal{B}:\mathcal{M})}\leq\sqrt{(\mathcal{C}:\mathcal{M})}$. Since $\mathcal{A}\leq\mathcal{B}$, we have $\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$, i.e., $\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{C}:\mathcal{M})}$. Therefore $\mathcal{C}\in\mathcal{V}^*(\mathcal{A})$ and hence $\mathcal{V}^*(\mathcal{B})\leq\mathcal{V}^*(\mathcal{A})$. □

Then we have the next results.

Theorem 3.7. *For any family of fuzzy submodules $\{\mathcal{A}_i\}_{i\in I}$ of an R-module M. Then the following properties hold.*

(1) $\mathcal{V}^*(0_1)=Fcp.spec(M)$ *and* $\mathcal{V}^*(\mathcal{M})=\varnothing$.

(2) $\bigcap_{i\in I}\mathcal{V}^*(\mathcal{A}_i)=\mathcal{V}^*\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right)$.

(3) $\mathcal{V}^*(\mathcal{A}_1)\cup\mathcal{V}^*(\mathcal{A}_2)=\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)$.

Proof. (1). Obvious.

(2). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B}\in\bigcap_{i\in I}\mathcal{V}^*(\mathcal{A}_i)$. Then we have $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_i)$ for all $i\in I$, i.e., $\sqrt{(\mathcal{A}_i:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$. Since $(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\leq\sqrt{(\mathcal{A}_i:\mathcal{M})}\odot\mathcal{M}\leq\sqrt{(\mathcal{B}:\mathcal{M})}\odot\mathcal{M}$, we have $\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\leq$

$\sqrt{(\mathcal{B}:\mathcal{M})}\odot\mathcal{M}$, it follows that,

$$\begin{aligned}\sqrt{\left(\left(\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right):\mathcal{M}\right)} &\leq \sqrt{\left(\sqrt{(\mathcal{B}:\mathcal{M})}\odot\mathcal{M}:\mathcal{M}\right)}\\ &\leq \sqrt{\sqrt{(\mathcal{B}:\mathcal{M})}}\\ &= \sqrt{(\mathcal{B}:\mathcal{M})}.\end{aligned}$$

It is easy to see that $\mathcal{B}\in\mathcal{V}^*\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right)$ and so $\bigcap_{i\in I}\mathcal{V}^*(\mathcal{A}_i)\subseteq\mathcal{V}^*\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right)$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B}\in\mathcal{V}^*\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right)$. Thus we have

$$\sqrt{\left(\left(\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right):\mathcal{M}\right)}\leq\sqrt{(\mathcal{B}:\mathcal{M})}.$$

Clearly, we have $(((\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}):\mathcal{M})=(\mathcal{A}_i:\mathcal{M})$ for all $i\in I$. Also for each $i\in I$, we obtain that

$$\begin{aligned}\sqrt{(\mathcal{A}_i:\mathcal{M})} &= \sqrt{(((\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}):\mathcal{M})}\\ &\leq \sqrt{\left(\left(\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right):\mathcal{M}\right)}\\ &\leq \sqrt{(\mathcal{B}:\mathcal{M})}\\ &= \sqrt{(\mathcal{B}:\mathcal{M})}.\end{aligned}$$

Therefore we obtain that $\mathcal{B}\in\bigcap_{i\in I}\mathcal{V}^*(\mathcal{A}_i)$ and hence $\mathcal{V}^*\left(\sum_{i\in I}(\mathcal{A}_i:\mathcal{M})\odot\mathcal{M}\right)\subseteq\bigcap_{i\in I}\mathcal{V}^*(\mathcal{A}_i)$.

(3). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1)\cup\mathcal{V}^*(\mathcal{A}_2)$. Then we have $\sqrt{(\mathcal{A}_1:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$ or $\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$. If $\sqrt{(\mathcal{A}_1:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$, then $\sqrt{(\mathcal{A}_1\wedge\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{A}_1:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$, it follows that, $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)$. Similarly, if $\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$, then $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)$. Then $\sqrt{(\mathcal{A}_1\wedge\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$. Since $\mathcal{A}_1\wedge\mathcal{A}_2\leq\mathcal{A}_1$ and $\mathcal{A}_1\wedge\mathcal{A}_2\leq\mathcal{A}_2$, we have $\sqrt{(\mathcal{A}_1:\mathcal{M})}\leq\sqrt{(\mathcal{A}_1\wedge\mathcal{A}_2:\mathcal{M})}$ and $\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{A}_1\wedge\mathcal{A}_2:\mathcal{M})}$, which implies that,

$$\sqrt{(\mathcal{A}_1:\mathcal{M})}\odot\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{A}_1\wedge\mathcal{A}_2:\mathcal{M})}.$$

Now since $\sqrt{(\mathcal{B}:\mathcal{M})}$ is prime and $\sqrt{(\mathcal{A}_1:\mathcal{M})}\odot\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$, it follows that $\sqrt{(\mathcal{A}_1:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$ or $\sqrt{(\mathcal{A}_2:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$. Clearly, we have $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1)$ or $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_2)$, i.e., $\mathcal{B}\in\mathcal{V}^*(\mathcal{A}_1)\cup\mathcal{V}^*(\mathcal{A}_2)$. Therefore $\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)\leq\mathcal{V}^*(\mathcal{A}_1)\cup\mathcal{V}^*(\mathcal{A}_2)$ and hence $\mathcal{V}^*(\mathcal{A}_1)\cup\mathcal{V}^*(\mathcal{A}_2)=\mathcal{V}^*(\mathcal{A}_1\wedge\mathcal{A}_2)$. □

For every fuzzy submodule $\mathcal{A}$ of an R-module M we define $\mathcal{E}^*(\mathcal{A})$ and τ^* as follows:

$$\mathcal{E}^*(\mathcal{A})=X-\mathcal{V}^*(\mathcal{A})\text{ and }\tau^*=\{\mathcal{E}^*(\mathcal{A}):\mathcal{A}\in\mathcal{FM}(M)\}.$$

In the next theorem we will show that the pair (X,τ^*) is a topological space.

Theorem 3.8. *Let M be an R-module. Then the following statements hold:*

(1) *The pair (X,τ^*) is a topological space.*
(2) *X is a T_0 topological space.*

Proof. The proof follows from Theorem 3.5. □

For any R-module M and $\mathcal{A},\mathcal{B}\in\mathcal{FM}(M)$ we have the next result.

Proposition 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be any fuzzy submodules of an R-module M. If $\sqrt{(\mathcal{A}:\mathcal{M})}=\sqrt{(\mathcal{B}:\mathcal{M})}$, then $\mathcal{V}^*(\mathcal{A})=\mathcal{V}^*(\mathcal{B})$. Moreover, the converse is true if both $\mathcal{A}$ and $\mathcal{B}$ are classical primary.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be any fuzzy submodules of M such that $\sqrt{(\mathcal{A}:\mathcal{M})}=\sqrt{(\mathcal{B}:\mathcal{M})}$. Next let $\mathcal{C}$ be a fuzzy submodule of M such that $\mathcal{C}\in\mathcal{V}^*(\mathcal{A})$. Then we have $\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{C}:\mathcal{M})}$, i.e., $\sqrt{(\mathcal{B}:\mathcal{M})}\leq\sqrt{(\mathcal{C}:\mathcal{M})}$. Thus $\mathcal{C}\in\mathcal{V}^*(\mathcal{B})$ and so $\mathcal{V}^*(\mathcal{A})\subseteq\mathcal{V}^*(\mathcal{B})$. Similarly, we obtain that $\mathcal{V}^*(\mathcal{B})\subseteq\mathcal{V}^*(\mathcal{A})$. For the converse, suppose that $\mathcal{A},\mathcal{B}\in\mathcal{FM}(M)$ is classical primary and $\mathcal{V}^*(\mathcal{A})=\mathcal{V}^*(\mathcal{B})$. Since $\mathcal{A}\in\mathcal{V}^*(\mathcal{A}),\mathcal{B}\in\mathcal{V}^*(\mathcal{B})$ and $\mathcal{V}^*(\mathcal{A})=\mathcal{V}^*(\mathcal{B})$, we have $\sqrt{(\mathcal{B}:\mathcal{M})}\leq\sqrt{(\mathcal{A}:\mathcal{M})}$ and $\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}$. Therefore, we obtain that $\sqrt{(\mathcal{A}:\mathcal{M})}=\sqrt{(\mathcal{B}:\mathcal{M})}$. □

For a fuzzy prime ideal p of R, by $Fcp.spec_p(M)$ we mean the set of all $\mathcal{A}\in\mathcal{FM}(M)$ such that $\sqrt{(\mathcal{A}:\mathcal{M})}=p$. In other words

$$Fcp.spec_p(M)=\Big\{\mathcal{A}\in Fcp.spec(M):\sqrt{(\mathcal{A}:\mathcal{M})}=p\Big\}.$$

Theorem 3.10. *Let μ and $\mathcal{A}$ be any fuzzy ideal and any fuzzy submodule of R and M, respectively. Then the following properties hold.*

(1) $\mathcal{V}^*(\mathcal{A})=\bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})}\leq p}Fcp.spec_p(M)$.
(2) $\mathcal{V}^*(\mu^m\odot\mathcal{M})=\mathcal{V}(\mu^n\odot\mathcal{M})$ *for some positive integers m,n.*
(3) $\mathcal{V}\left(\sqrt{(\mathcal{A}:\mathcal{M})}\odot\mathcal{M}\right)\subseteq\mathcal{V}^*(\mathcal{A})\subseteq\mathcal{V}^*((\mathcal{A}:\mathcal{M})\odot\mathcal{M})$.

Proof. (1). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B}\in\mathcal{V}^*(\mathcal{A})$. Then we have $\sqrt{(\mathcal{A}:\mathcal{M})}\leq\sqrt{(\mathcal{B}:\mathcal{M})}=p$ and so

$$\mathcal{B} \in Fcp.spec_p(M) \subseteq \bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})} \leq p} Fcp.spec_p(M).$$

It is easy to see that $\mathcal{V}^*(\mathcal{A}) \subseteq \bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})} \leq p} Fcp.spec_p(M)$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})} \leq p} Fcp.spec_p(M)$. Thus there exists a fuzzy prime ideal p of R such that $\sqrt{(\mathcal{A} : \mathcal{M})} \leq p$ and $\mathcal{B} \in Fcp.spec_p(M)$. Clearly, we have $\sqrt{(\mathcal{B} : \mathcal{M})} = p$, i.e., $\sqrt{(\mathcal{A} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$, it follows that, $\mathcal{B} \in \mathcal{V}^*(\mathcal{A})$. Therefore we obtain that $\bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})} \leq p} Fcp.spec_p(M) \subseteq \mathcal{V}^*(\mathcal{A})$ and hence $\mathcal{V}^*(\mathcal{A}) = \bigcup_{\sqrt{(\mathcal{A}:\mathcal{M})} \leq p} Fcp.spec_p(M)$.

(2). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}(\mu^n \odot \mathcal{M})$. Then we have $\mu^n \odot \mathcal{M} \leq \mathcal{B}$, i.e., $\sqrt{(\mu^m \odot \mathcal{M} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$. This implies that $\mathcal{B} \in \mathcal{V}^*(\mu^m \odot \mathcal{M})$ and so $\mathcal{V}(\mu^n \odot \mathcal{M}) \subseteq \mathcal{V}^*(\mu^m \odot \mathcal{M})$. On the other hand, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}^*(\mu^m \odot \mathcal{M})$. Thus $\sqrt{(\mu^m \odot \mathcal{M} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$. Obviously, $\mu^m \leq (\mu^m \odot \mathcal{M} : \mathcal{M})$. Since $\sqrt{(\mu^m \odot \mathcal{M} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$ and $\mu^m \leq (\mu^n \odot \mathcal{M} : \mathcal{M})$, we have $\mu^m \leq \sqrt{(\mathcal{B} : \mathcal{M})}$, which implies that, $\mu^{mt} \odot \mathcal{M} \leq \mathcal{B}$. It is easy to see that $\mathcal{B} \in \mathcal{V}(\mu^n \odot \mathcal{M})$. Therefore $\mathcal{V}^*(\mu^m \odot \mathcal{M}) \subseteq \mathcal{V}(\mu^n \odot \mathcal{M})$ and hence $\mathcal{V}^*(\mu^m \odot \mathcal{M}) = \mathcal{V}(\mu^n \odot \mathcal{M})$.

(3). Let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}^*(\mathcal{A})$. Then we have $\sqrt{(\mathcal{A} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$. Since $(\mathcal{A} : \mathcal{M}) \odot \mathcal{M} \leq \mathcal{A}$, we have

$$\sqrt{((\mathcal{A} : \mathcal{M}) \odot \mathcal{M} : \mathcal{M})} \leq \sqrt{(\mathcal{A} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}.$$

This implies that $\mathcal{B} \in \mathcal{V}^*((\mathcal{A} : \mathcal{M}) \odot \mathcal{M})$ and so $\mathcal{V}^*(\mathcal{A}) \subseteq \mathcal{V}^*((\mathcal{A} : \mathcal{M}) \odot \mathcal{M})$. Next, let $\mathcal{B}$ be a fuzzy submodule of M such that $\mathcal{B} \in \mathcal{V}\left(\sqrt{(\mathcal{A} : \mathcal{M})} \odot \mathcal{M}\right)$. Thus $\sqrt{(\mathcal{A} : \mathcal{M})} \odot \mathcal{M} \leq \mathcal{B}$. Obviously, $\sqrt{(\mathcal{A} : \mathcal{M})} \leq (\mathcal{B} : \mathcal{M})$. Since $(\mathcal{B} : \mathcal{M}) \leq \sqrt{(\mathcal{B} : \mathcal{M})}$, we have $\sqrt{(\mathcal{A} : \mathcal{M})} \leq \sqrt{(\mathcal{B} : \mathcal{M})}$, which implies that, $\mathcal{B} \in \mathcal{V}^*(\mathcal{A})$. Therefore $\mathcal{V}\left(\sqrt{(\mathcal{A} : \mathcal{M})} \odot \mathcal{M}\right) \subseteq \mathcal{V}^*(\mathcal{A})$ and hence $\mathcal{V}\left(\sqrt{(\mathcal{A} : \mathcal{M})} \odot \mathcal{M}\right) \subseteq \mathcal{V}^*(\mathcal{A}) \subseteq \mathcal{V}^*((\mathcal{A} : \mathcal{M}) \odot \mathcal{M})$. □

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