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# Quasihomeomorphisms and lattice equivalent topological spaces

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Abstract. This paper deals with lattice-equivalence of topological spaces. We are concerned with two questions: the first one is when two topological spaces are lattice equivalent; the second one is what additional conditions have to be imposed on lattice equivalent spaces in order that they be homeomorphic. We give a contribution to the study of these questions. Many results of Thron [Lattice-equivalence of topological spaces, Duke Math. J. 29 (1962), 671-679] are recovered, clarified and commented.

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## 1. Introduction

Let $X$ be a topological space; we denote by $\Gamma(X)$ the lattice of closed sets of $X$. Two topological spaces $X$ and $Y$ are said to be *lattice equivalent* if there is a bijective map from $\Gamma(X)$ to $\Gamma(Y)$ which together with its inverse is order-preserving [8].

The question of characterizing when two topological spaces are lattice equivalent is still open.

A lattice equivalence $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ is said to be *induced by a homeomorphism* if there is a homeomorphism $f : X \longrightarrow Y$ such that $\varphi(C) = f(C)$, for each $C \in \Gamma(X)$.

In [8], Thron was concerned in the problem of determining what additional conditions have to be imposed on lattice equivalent spaces in order that they be homeomorphic. A complete answer to this problem is still very far off.

It is worth noting that, over the years, several researchers dealt with the concept of lattice equivalent topological spaces and representations of an abstract lattice as the family of closed sets on a topological space (see for instance [4], [8], [9]).

In 1966, Finch [4] proved that a lattice equivalence $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ is induced by a homeomorphism if and only if the following conditions are satisfied:

($i$) For each $x \in X$, there exists $y_x \in Y$ such that $\varphi(\overline{\{x\}}) = \overline{\{y_x\}}$.

($ii$) For each $y \in Y$, there exists $x_y \in X$ such that $\varphi^{-1}(\overline{\{y\}}) = \overline{\{x_y\}}$.

($iii$) Let $x \in X$ and $y \in Y$. Set $X_x = \{t \in X : \overline{\{x\}} = \overline{\{t\}}\}$ and $Y_y = \{t \in Y : \overline{\{y\}} = \overline{\{t\}}\}$. Then $\varphi^{-1}(\overline{\{y\}}) = \overline{\{x\}}$ implies that $|X_x| = |Y_y|$ (where $|X_x|$ denotes the cardinality number of the set $X_x$).

In 1972, Yip Kai Wing [9] was interested in quasi-homeomorphisms and lattice-equivalences of topological spaces. This work seems to be very close to our paper; but it is important to announce that none of our results is in Yip's paper [9]. In this connection, the first section will be entirely devoted to quasi-homeomorphisms and comments on Yip's results in his note [9].

The present paper is devoted to shed some light on lattice equivalent topological spaces.

Let us first recall some notions which were introduced by the Grothendieck school (see for example [5] and [6]), such as quasihomeomorphisms and strongly dense subsets.

If $X$ is a topological space, we denote by $\mathfrak{O}(X)$ the set of all open subsets of $X$. Recall that a continuous map $q : X \longrightarrow Y$ is said to be a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ defines a bijection $\mathfrak{O}(Y) \longrightarrow \mathfrak{O}(X)$. We say that a subset $S$ of a topological space $X$ is *locally closed* if it is an intersection of an open set and a closed set of $X$. A subset $S$ of a topological space $X$ is said to be *strongly dense* in $X$, if $S$ meets every nonempty locally closed subset of $X$. Thus a subset $S$ of $X$ is strongly dense if and only if the canonical injection $S \hookrightarrow X$ is a quasihomeomorphism. It is well known that a continuous map $q : X \longrightarrow Y$ is a quasihomeomorphism if and only if the topology on $X$ is the inverse image by $q$ of that on $Y$ and the subset $q(X)$ is strongly dense in $Y$ [5].

Let $X$ be a topological space. If $f : X \longrightarrow Y$ is continuous, then we define

$$\Gamma(f) : \Gamma(Y) \longrightarrow \Gamma(X), \text{ by } \Gamma(f)(C) = f^{-1}(C).$$

In particular, if $q : X \longrightarrow Y$ is a quasihomeomorphism. Then the map $\Gamma(q) : \Gamma(Y) \longrightarrow \Gamma(X)$ is a lattice equivalence (see for example [1, Proposition 1.9]). The following definition is natural.

**Definition 1.1.** *A lattice equivalence $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ between two topological spaces is said to be* induced by a quasihomeomorphism *if there is either a quasihomeomorphism $q : Y \longrightarrow X$ such that $\varphi = \Gamma(q)$ or a quasihomeomorphism $p : X \longrightarrow Y$ such that $\varphi^{-1} = \Gamma(p)$.*

Our main result is a characterization of lattice equivalences induced by a quasihomeomorphism (see Theorem 3.4). This result is very close to the one done by Finch [4] characterizing lattice equivalences induced by a homeomorphism. As a consequence, many results of Thron are recovered, clarified and commented.

## 2. Grothendieck's Quasihomeomorphisms and Yip's Quasihomeomorphisms

As we have said in the introduction the concept of quasihomeomorphisms was introduced in Algebraic geometry by Grothendieck ([6], [5]). Also, it was shown that this concept arises naturally in the theory of some foliations associated to closed connected manifolds( see the papers [2] and [3]).

The following definition is given in [9].

**Definition 2.1.** *A continuous map $q : X \longrightarrow Y$ between topological spaces is said to be a quasihomeomorphism if the following conditions are satisfied:*

($i$) *For any closed set $C$ in $X$, $q^{-1}[\overline{q(C)}] = C$.*
($ii$) *For any closed set $F$ in $Y$, $\overline{q[q^{-1}(F)]} = F$.*

Fortunately, the two notions "Grothendieck's quasihomeomorphism" and "Yip's quasihomeomorphism" coincides, as it is shown in the following.

**Proposition 2.2.** *Let $q : X \longrightarrow Y$ be a continuous map between topological spaces. Then $q$ is a Yip's quasihomeomorphism if and only if it is a Grothendieck's quasihomeomorphism.*

*Proof.* Suppose that $q$ is a Grothendieck's quasihomeomorphism. Then $\Gamma(q) : \Gamma(Y) \longrightarrow \Gamma(X)$ is a lattice isomorphism, by [1, Proposition 1.9]. Hence $q$ is a quasihomeomorphism in the sense of Yip, by [9, Theorem 1].

Conversely, if $q$ is a Yip's quasihomeomorphism, then $\Gamma(q) : \Gamma(Y) \longrightarrow \Gamma(X)$ is a lattice isomorphism, by [9, Theorem 1]; so that $\Gamma(q)$ is bijective; proving that $q$ is a Grothendieck's quasihomeomorphism. □

**Remark 2.3.** Let $q : X \longrightarrow Y$ be a quasihomeomorphism between topological spaces. In [9, Theorem 2], Yip proved that $q(X)$ is dense in $Y$. In fact, it follows from [5, Chapter 0, Proposition 2.7.1] that $q(X)$ is even strongly dense in $Y$.

In [9, Theorem 3], the author investigated closed quasihomeomorphisms between $T_0$-spaces. The following result is more precise.

**Proposition 2.4.** *Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then the following statements are equivalent:*

($i$) *$q$ is a surjective mapping;*
($ii$) *$q$ is a closed mapping;*
($iii$) *$q$ is an open mapping.*

*Proof.* ($ii$) $\Longrightarrow$ ($i$) and ($iii$) $\Longrightarrow$ ($i$). Let $q$ be closed (resp.open). Then, $q(X)$ is a closed (resp.open) subset of $Y$. Now, since $q^{-1}(q(X)) = q^{-1}(Y)$, we get $q(X) = Y$, by the definition of a quasihomeomorphism. Thus $q$ is onto.

$(i) \Longrightarrow (ii)$ and $(iii)$. Let $C$ be a closed (resp. an open) subset of $X$. By definition, there is a closed (resp. an open) subset $D$ of $Y$ such that $C = q^{-1}(D)$. But, since $q$ is onto, we get $q(C) = D$, proving that $q(C)$ is closed (resp. open). □

**Example 2.5.** Let us construct a quasihomeomorphism that is not a surjection. Consider an infinite set $X$ and a point $\alpha$ not belonging to $X$. Set $Y := X \cup \{\alpha\}$. Equip $Y$ with the topology whose closed sets are $Y$ and the finite subsets of $X$. Hence $X$ is a strongly dense subspace of $Y$; so that the canonical embedding $i : X \hookrightarrow Y$ is a quasihomeomorphism. Since $i$ is not onto, it is not closed and not open.

Before stating the next result, we need to recall the notion of sober space (also introduced by the school of Grothendieck [5] or [6]). A subspace $Y$ of $X$ is called *irreducible*, if each nonempty open subset of $Y$ is dense in $Y$ (equivalently, if $C_1$ and $C_2$ are two closed subsets of $X$ such that $Y \subseteq C_1 \cup C_2$, then $Y \subseteq C_1$ or $Y \subseteq C_2$). Let $C$ be a closed subset of a space $X$; we say that $C$ *has a generic point* if there exists $x \in C$ such that $C = \overline{\{x\}}$. Recall that a topological space $X$ is said to be *sober* if any nonempty irreducible closed subset of $X$ has a unique generic point.

Let $X$ be a topological space and $\mathbf{S}(X)$ the set of all nonempty irreducible closed subsets of $X$ [5]. Let $U$ be an open subset of $X$; set $\widetilde{U} = \{C \in \mathbf{S}(X) \mid U \cap C \neq \varnothing\}$; then the collection $\{\widetilde{U} \mid U \text{ is an open subset of } X\}$ provides a topology on $\mathbf{S}(X)$ and the following properties hold [5]:

- $(i)$ The map $\eta_X : X \longrightarrow \mathbf{S}(X)$, which takes $x$ to $\overline{\{x\}}$, is a quasihomeomorphism.
- $(ii)$ $\mathbf{S}(X)$ is a sober space.
- $(iii)$ Let $f : X \longrightarrow Y$ be a continuous map. Let $\mathbf{S}(f) : \mathbf{S}(X) \longrightarrow \mathbf{S}(Y)$ be the map defined by $\mathbf{S}(f)(C) = \overline{f(C)}$, for each irreducible closed subset $C$ of $X$. Then $\mathbf{S}(f)$ is continuous.
- $(iv)$ The topological space $\mathbf{S}(X)$ is called the soberification of $X$, and the assignment $\mathbf{S}$, defines a functor from the category **TOP** of topological spaces to itself.

**Proposition 2.6.** *Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then the following properties hold.*

1. *If $X$ is a $T_0$-space, then $q$ is injective.*
2. *If $X$ is sober and $Y$ is a $T_0$-space, then $q$ is a homeomorphism.*

*Proof.* (1) Let $x_1$, $x_2$ be two distinct points of $X$ with $q(x_1) = q(x_2)$. Then there exists an open subset $U$ of $X$ such that, for example, $x_1 \in U$ and $x_2 \notin U$. Since there exists an open subset $V$ of $Y$ satisfying $q^{-1}(V) = U$, we get $q(x_1) \in V$ and $q(x_2) \notin V$, which is impossible. It follows that $q$ is injective.

(2) Firstly, it is obviously seen that if $S$ is a closed subset of $Y$, then $S$ is irreducible if and only if so is $q^{-1}(S)$.

Now, let us prove that $q$ is surjective. For this end, let $y \in Y$. According to the above observation, $q^{-1}(\overline{\{y\}})$ is a nonempty irreducible closed subset of $X$. Hence $q^{-1}(\overline{\{y\}})$ has a generic point $x$. Thus we have the containments

$$\overline{\{x\}} \subseteq q^{-1}(\overline{\{q(x)\}}) \subseteq q^{-1}(\overline{\{y\}}) = \overline{\{x\}}.$$

So that $q^{-1}(\overline{\{q(x)\}}) = q^{-1}(\overline{\{y\}})$. It follows, from the fact that $q$ is a quasihomeomorphism, that $\overline{\{q(x)\}} = \overline{\{y\}}$. Since $Y$ is a $T_0$-space, we get $q(x) = y$. This proves that $q$ is a surjective map, and thus $q$ is bijective. But it is easily seen that bijective quasihomeomorphisms are homeomorphisms. □

**Remark 2.7.** Let $q : X \longrightarrow Y$ be a quasihomeomorphism. If $Y$ is sober and $X$ is a $T_0$-space, then $q$ need not be a homeomorphism. To see this, it suffices to consider a $T_0$-space $X$ which is not sober. Then the canonical embedding $\eta_X : X \longrightarrow \mathbf{S}(X)$ is a quasihomeomorphism which is not a homeomorphism.

## 3. Lattice equivalence

A *Brouwerian lattice* is a complete lattice $L$ for which $x \bigvee(\bigwedge C) = \bigwedge\{x \bigvee y \mid y \in C\}$ for all $x \in L$ and all $C \subseteq L$. A morphism of Brouwerian lattices is a mapping $f : L \longrightarrow M$ that preserves all infima and all finite suprema.

Let **CBL** denotes the category of Brouwerian lattices and Brouwerian lattice maps. Then $\Gamma : \mathbf{TOP} \longrightarrow \mathbf{CBL}$ is a contravariant functor.

**Remark 3.1.** Let $f : X \longrightarrow Y$ be a continuous map. Then $f$ is rendered invertible by the functor $\Gamma$ (i.e., $\Gamma(f)$ is a lattice equivalence) if and only if $f$ is a quasihomeomorphism.

The following example shows that a lattice equivalence that is induced by a quasihomeomorphism is not necessarily induced by a homeomorphism.

**Example 3.2.** Let $X$ be a topological space which is not sober. Then the canonical quasihomeomorphism $\eta_X : X \longrightarrow \mathbf{S}(X)$ induces a lattice equivalence between $\mathbf{S}(X)$ and $X$ which is not induced by a homeomorphism.

In order to give a complete characterization of lattice equivalence of topological spaces induced by a quasihomeomorphism, we give the following definition.

**Definition 3.3.** *Let $X$, $Y$ be two topological spaces and $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ a lattice equivalence. We say that $\varphi$ is a* point-closure lattice equivalence *if one of the following properties is satisfied:*

(*i*) *For each $x \in X$, there exists $y_x \in Y$ such that $\varphi(\overline{\{x\}}) = \overline{\{y_x\}}$.*
(*ii*) *For each $y \in Y$, there exists $x_y \in X$ such that $\varphi^{-1}(\overline{\{y\}}) = \overline{\{x_y\}}$.*

Our main result is the following.

**Theorem 3.4.** *Let $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ be a lattice equivalence of topological spaces. Then the following statements are equivalent:*

(*i*) *$\varphi$ is induced by a quasihomeomorphism;*
(*ii*) *$\varphi$ is a point-closure lattice equivalence.*

*Proof.*
$(i) \Longrightarrow (ii)$ Suppose that there is a quasihomeomorphism $q : Y \longrightarrow X$ such that $\varphi = \Gamma(q)$. Then for each $D \in \Gamma(Y)$, we have $\varphi^{-1}(D) = C$ with $C \in \Gamma(X)$ and $q^{-1}(C) = D$. Hence

$$D \subseteq q^{-1}(\overline{q(D)}) \subseteq q^{-1}(C) = D.$$

Thus $\varphi^{-1}(D) = \overline{q(D)}$. Therefore, for each $y \in Y$, we have $\varphi^{-1}(\overline{\{y\}}) = \overline{q(\overline{\{y\}})}$. But $\overline{q(\overline{\{y\}})} = \overline{\{q(y)\}}$, since $q$ is continuous; so that $\varphi^{-1}(\overline{\{y\}}) = \overline{\{q(y)\}}$.

Now, if we suppose that there is a quasihomeomorphism $p : X \longrightarrow Y$ such that $\varphi^{-1} = \Gamma(p)$, then we get $\varphi(\overline{\{x\}}) = \overline{\{p(x)\}}$, for each $x \in X$.

It follows that $\varphi$ is a point-closure lattice equivalence.

$(ii) \Longrightarrow (i)$ Suppose, for instance, that for each $x \in X$, there exists $y_x \in Y$ such that $\varphi(\overline{\{x\}}) = \overline{\{y_x\}}$. For each $x \in X$ choose $q(x) \in Y$ such that $\varphi(\overline{\{x\}}) = \overline{\{q(x)\}}$. This allows us to define a mapping $q : X \longrightarrow Y$. We are aiming to prove that $q$ is a quasihomeomorphism and $\varphi^{-1} = \Gamma(q)$.

It is enough to prove that $\varphi^{-1}(G) = q^{-1}(G)$, for each $G \in \Gamma(Y)$.

Let $x \in \varphi^{-1}(G)$. Then $\overline{\{x\}} \subseteq \varphi^{-1}(G)$. Thus $\overline{\{q(x)\}} \subseteq G$; in particular $q(x) \in G$; therefore $x \in q^{-1}(G)$. Conversely, let $x \in q^{-1}(G)$; then $q(x) \in G$. Hence $\overline{\{q(x)\}} \subseteq G$; consequently, $\overline{\{x\}} \subseteq \varphi^{-1}(G)$. Therefore, $x \in \varphi^{-1}(G)$. It follows that $\varphi^{-1} = \Gamma(q)$.

If we suppose that for each $y \in Y$, there exists a $x_y \in X$ such that $\varphi^{-1}(\overline{\{y\}}) = \overline{\{x_y\}}$, then by the above argument, there is a quasihomeomorphism $p : Y \longrightarrow X$ such that $\varphi = \Gamma(p)$. □

**Corollary 3.5.** *If $X$ is a topological space and $Y$ is a $T_1$-space, then each lattice equivalence between them is induced by a quasihomeomorphism.*

*Proof.* It is easy to check that condition (*ii*) of Definition 3.3 is fulfilled. Then we can apply Theorem 3.4. □

The following result establishes some links between lattice equivalences induced by a homeomorphism and those induced by a quasihomeomorphism.

For the proof of the next theorem, we need a lemma.

**Lemma 3.6.** *Let $E$, $F$ be two topological spaces such that $F$ is a $T_0$-space. If $f,\ g : E \longrightarrow F$ are two continuous maps such that $\Gamma(f) = \Gamma(g)$, then $f = g$.*

*Proof.* Let $x \in E$. Then

$$f^{-1}(\overline{\{f(x)\}}) = g^{-1}(\overline{\{f(x)\}}) \text{ and } g^{-1}(\overline{\{g(x)\}}) = f^{-1}(\overline{\{g(x)\}}).$$

This yields $g(x) \in \overline{\{f(x)\}}$ and $f(x) \in \overline{\{g(x)\}}$. Thus $\overline{\{f(x)\}} = \overline{\{g(x)\}}$; so that $f(x) = g(x)$, since $Y$ is a $T_0$-space. □

**Theorem 3.7.** *Let $X$, $Y$ be two $T_0$-spaces and $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ a lattice equivalence of topological spaces. Then the following statements are equivalent:*

(*i*) *$\varphi$ is induced by a homeomorphism;*
(*ii*) *There are two quasihomeomorphisms $q : Y \longrightarrow X$ and $p : X \longrightarrow Y$ such that $\varphi = \Gamma(q)$ and $\varphi^{-1} = \Gamma(p)$.*

*Proof.*
[$(i) \Longrightarrow (ii)$]. Straightforward.
[$(ii) \Longrightarrow (i)$].
Let $q : Y \longrightarrow X$ and $p : X \longrightarrow Y$ be two quasihomeomorphisms such that $\varphi = \Gamma(q)$ and $\varphi^{-1} = \Gamma(p)$.
We have

$$\varphi\varphi^{-1} = 1_{\Gamma(Y)} = \Gamma(1_Y) \text{ and } \varphi^{-1}\varphi = 1_{\Gamma(X)} = \Gamma(1_X).$$

Hence $\Gamma(pq) = \Gamma(1_Y)$ and $\Gamma(qp) = \Gamma(1_X)$. Thus, according to Lemma 3.6, we get $qp = 1_X$ and $pq = 1_Y$. Therefore, $\varphi$ is induced by the homeomorphism $p$. □

The following example shows that the $T_0$ axiom cannot be deleted in Theorem 3.7.

**Example 3.8.** Let $X$, $Y$ be two sets with distinct cardinalities. We equip $X$ and $Y$ with the indiscrete topology. The unique lattice equivalence between $X$ and $Y$ is $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$, defined by $\varphi(\varnothing) = \varnothing$ and $\varphi(X) = Y$. It is easily seen that for any two continuous maps $q : Y \longrightarrow X$ and $p : X \longrightarrow Y$, we have $\varphi = \Gamma(q)$ and $\varphi^{-1} = \Gamma(p)$. Moreover, $p$, $q$ are quasihomeomorphisms. But since the two sets have distinct cardinalities, they cannot be homeomorphic.

Recall that a topological space $X$ is said to be a *$T_D$-space* if for each $x \in X$, $\{x\}$ is locally closed. It is easily seen that $T_D$ is between $T_0$ and $T_1$.
The following result will be used in the next corollary.

**Proposition 3.9.** *Every quasihomeomorphism between two $T_D$-spaces is a homeomorphism.*

*Proof.* Let $q : X \longrightarrow Y$ be a quasihomeomorphism between two $T_D$-spaces. Hence $q$ is injective, by Proposition 2.6. On the other hand, $q(X)$ is strongly dense in $Y$ and every point-set is locally closed; so that $q(X) = Y$. Thus $q$ is a bijective quasihomeomorphism. Therefore, $q$ is a homeomorphism. □

For the proof of the next corollary, we need a lemma.

**Lemma 3.10.** *Let $X$, $Y$ be two topological spaces and $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ a lattice equivalence. Let $G$ be a closed subset of $Y$. Then the following statements are equivalent:*

(*i*) *$G$ is irreducible in $Y$;*
(*ii*) *$\varphi^{-1}(G)$ is irreducible in $X$.*

*Proof.* Since $\varphi^{-1}$ is also a lattice equivalence, it is enough to show the implication $(i) \Longrightarrow (ii)$.

Let $F_1,\ F_2$ be two closed subsets of $X$ such that $\varphi^{-1}(G) \subseteq F_1 \cup F_2$. Hence $G \subseteq \varphi(F_1 \cup F_2)$. But $\varphi(F_1 \cup F_2) = \varphi(F_1) \cup \varphi(F_2)$. Thus $G \subseteq \varphi(F_1) \cup \varphi(F_2)$. Now, since $G$ is irreducible in $Y$, $G \subseteq \varphi(F_1)$ or $G \subseteq \varphi(F_2)$. This yields $\varphi^{-1}(G) \subseteq F_1$ or $\varphi^{-1}(G) \subseteq F_2$, proving that $\varphi^{-1}(G)$ is irreducible in $X$. □

**Corollary 3.11.** *Let $X$, $Y$ be two $T_0$-spaces and $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ a lattice equivalence.*

(1) *If $X$ or $Y$ is a $T_D$-space (resp. sober space), then $\varphi$ is induced by a quasihomeomorphism $q : X \longrightarrow Y$ or $p : Y \longrightarrow X$.*
(2) *If $X$ and $Y$ are $T_D$-spaces (resp. sober spaces), then the mapping $q$ of (1) is a homeomorphism (thus $\varphi$ is induced by a homeomorphism).*

*Proof.*
(1) Suppose for example, that $X$ is a $T_D$-space and $Y$ is a $T_0$-space.

Thron showed in [8] that for each $x \in X$, there exists a unique $y_x \in Y$ such that $\varphi(\overline{\{x\}}) = \overline{\{y_x\}}$. Then $\varphi$ is a point-closure lattice equivalence and according to the proof of Theorem 3.4, $\varphi$ is induced by a quasihomeomorphism $q : X \longrightarrow Y$ ( $q$ takes $x$ to $y_x$).

Now, suppose that $X$ is sober and $Y$ is $T_0$. Let $y \in Y$; then, since $\overline{\{y\}}$ is an irreducible closed subset of $Y$, $\varphi^{-1}(\overline{\{y\}})$ is an irreducible closed subset of $X$, by Lemma 3.10. Hence $\varphi^{-1}(\overline{\{y\}})$ has a unique generic point. Thus $\varphi$ is a point-closure lattice equivalence; so that there is a quasihomeomorphism $p : Y \longrightarrow X$ which induces $\varphi$ ($p$ takes $y \in Y$ to the unique generic point of $\varphi^{-1}(\overline{\{y\}})$).

(2) Every quasihomeomorphism between two $T_D$-spaces is a homeomorphism, by Proposition 3.9. Also, every quasihomeomorphism between two sober spaces is a homeomorphism, by Proposition 2.6. □

**Corollary 3.12** ([8, Theorem 2.1 ]). *Every lattice equivalence between two $T_D$-spaces is induced by a homeomorphism.*

**Corollary 3.13** ([8, Corollary 2.1 ]). *Every lattice equivalence between a $T_0$-space and a $T_2$-space is induced by a homeomorphism.*

*Proof.* Let $X$ be a $T_0$-space, $Y$ a $T_2$-space (thus a $T_D$-space) and $\varphi : \Gamma(Y) \longrightarrow \Gamma(X)$ a lattice equivalence. Then by Corollary 3.11, there is a quasihomeomorphism $q : Y \longrightarrow X$ such that $\varphi^{-1} = \Gamma(q)$.

Now, $Y$ is a sober space (since it is $T_2$) and $X$ is $T_0$. This forces $q$ to be a homeomorphism, by Proposition 2.6. Therefore, $\varphi$ is induced by a homeomorphism. □

**Example 3.14. A lattice equivalence between a $T_1$-space (thus a $T_D$-space) and a sober space which is not induced by a homeomorphism.** For, let $Y$ be an infinite set equipped with the cofinite topology. Let $\alpha \notin Y$, and $X = Y \cup \{\alpha\}$. We equip $X$ with the topology whose closed sets are $X$ and the finite subsets of $Y$. Hence $Y$ is a strongly dense subspace of $X$; so that the canonical embedding $Y \hookrightarrow X$ is a quasihomeomorphism; thus it induces a lattice equivalence $\varphi$. Clearly, $\varphi$ is not induced by a homeomorphism, since $Y$ is a $T_1$-space and $X$ is not.

Note that Example 3.18 provides nontrivial examples of lattice equivalences between a $T_1$-space and a sober space which are not induced by a homeomorphism.

It is worth noting that the $T_D$-axiom is the weakest requirement under which [8, Theorem 2.1] is true, as shown by Thron in the following

**Theorem 3.15** ([8, Theorem 2.2])**.** *If $X$ is not a $T_D$-space, then there exists a lattice equivalence between $X$ and some other space $Y$, which is not induced by a homeomorphism.*

Looking carefully at the proof of the above theorem, we remark that the lattice equivalence given by the author is not induced by a homeomorphism in both cases, when $X$ is $T_0$ or not; nevertheless, it is induced by a quasihomeomorphism. This rises the natural question whether a lattice equivalence is always induced by a quasihomeomorphism. Unfortunately, the answer is negative, as shown by the following nice example.

**Example 3.16. A lattice equivalence of topological spaces that is not induced by a quasihomeomorphism.**

Let $X$ and $Y$ be two disjoint infinite sets equipped with the cofinite topology. Let $\alpha$, $\beta \notin X \cup Y$ and $\alpha \neq \beta$. Set $X^{'} = X \cup \{\alpha\}$ and $Y^{'} = Y \cup \{\beta\}$. We equip $X^{'}$ (resp. $Y^{'}$) with the topology whose closed sets are $X^{'}$ (resp. $Y^{'}$) and the finite subset of $X$ (resp. of $Y$).

Recall that the free union $E + F$ of disjoint spaces $E$ and $F$ is the set $E \cup F$ with the topology: $U \subseteq E + F$ is open if and only if $U \cap E$ is open in $E$ and $U \cap F$ is open in $F$.

Now, consider $\Lambda = X^{'} + Y$ and $\Delta = Y^{'} + X$. It is clear that there exists a unique morphism of lattices $\varphi : \Gamma(\Lambda) \longrightarrow \Gamma(\Delta)$ which satisfies the following properties:

– $(i)$ $\varphi(X^{'}) = X, \varphi(C) = C,$ for all finite subset $C$ of $X$.

– $(ii)$ $\varphi(Y) = Y^{'},\ \varphi(D) = D,$ for all finite subset $D$ of $Y$.

Clearly, $\varphi$ is a lattice equivalence of topological spaces.

Suppose that $\varphi$ is induced by a quasihomeomorphism. Without loss of generality, we may suppose that there is a quasihomeomorphism $q : \Delta \longrightarrow \Lambda$ such that $\varphi = \Gamma(q)$. Hence $q^{-1}(Y) = \varphi(Y) = Y^{'}$; so that $q(\beta) \in Y$. On the other hand, $\overline{\{\beta\}} = Y^{'}$. The continuity of $q$ implies that

$$q(Y^{'}) = q(\overline{\{\beta\}}) \subseteq \overline{q(\{\beta\})} = \{q(\beta)\}.$$

Thus

$$Y^{'} \subseteq q^{-1}(\{q(\beta)\}) = \varphi(\{q(\beta)\}) = \{q(\beta)\},$$

a contradiction, since $Y$ is infinite.

Therefore, $\varphi$ is not induced by a quasihomeomorphism.

In [8, Corollary 2.1], Thron has written that "If $X$ is a $T_0$-space and $Y$ is a $T_2$-space, then they are homeomorphic if and only if they are lattice-equivalent".

The following result shows that in [8, Corollary 2.1], "$T_2$-space" cannot be replaced by "$T_1$-space".

We need to recall the notion of Jacobson space [5]. A topological space $X$ is said to be *a Jacobson space* if the subset of all closed points of $X$ is strongly dense in $X$.

**Theorem 3.17.**

(1) *If a $T_0$-space $X$ is lattice equivalent to a $T_1$-space $Y$, then $X$ is a Jacobson space.*
(2) *There exist a $T_0$-space $X$ and a $T_1$-space $Y$ which are lattice equivalent but not homeomorphic (hence any lattice equivalence between them is not induced by a homeomorphism).*

*Proof.*

(1) By Corollary 3.5, the lattice equivalence between $X$ and $Y$ is induced by a quasihomeomorphism $q : Y \longrightarrow X$. According to Proposition 2.6, the induced quasihomeomorphism $q_1 : Y \longrightarrow q(Y)$ is bijective. Hence $Y$ is homeomorphic to the subspace $q(Y)$ of $X$ and $q(Y)$ is strongly dense in $X$. It suffices to prove that $q(Y)$ is the set $X_0$ of all closed points of $X$. Indeed, $X_0 \subseteq q(Y)$, since $q(Y)$ is strongly dense in $X$.

On the other hand, let $y \in Y$; then $\{q(y)\}$ is closed in $q(Y)$ since $Y$ is homeomorphic to $q(Y)$. Hence $\overline{\{q(y)\}} \cap q(Y) = \{q(y)\}$. Let $z \in \overline{\{q(y)\}}$; then $\overline{\{z\}} \cap q(Y) \neq \varnothing$. Thus $\overline{\{z\}} \cap q(Y) = \{q(y)\}$. It follows that $\overline{\{q(y)\}} = \overline{\{z\}}$. Therefore, $z = q(y)$, since $X$ is a $T_0$-space.

(2) It suffices to take a Jacobson $T_0$-space $X$ which is not $T_1$. Let $Y$ be the subspace of $X$ whose elements are the closed points of $X$. Hence the canonical embedding of $Y$ into $X$ is a quasihomeomorphism. Thus $X$ and $Y$ are lattice equivalent; however, they are not homeomorphic. □

**Example 3.18.** It is easy to give explicit examples of Jacobson $T_0$-spaces which are not $T_1$. Let $R$ be a Hilbert ring which is not a field; i.e., a ring such that the intersection with $R$ of a maximal ideal of the polynomial ring $R[t]$ is maximal (take for example $R = K[t_1, ..., t_n]$ the polynomial ring on $n$ indeterminates over a field $K$). Let $X = Spec(R)$ equipped with the hull-kernel topology. Then $X$ is a Jacobson space which is not $T_1$.

Here, if we let $Y := Max(R)$ be the set of all maximal ideals of $R$, then $Y$ is a $T_1$ strongly dense subspace of $X$. Thus the canonical quasihomeomorphism $i : Y \longrightarrow X$ induces a lattice equivalence between the topological spaces $X$ and $Y$. On the other hand, the space $X$ is sober by [7, Proposition 4]. This yields a lattice equivalence between a $T_1$-space and a sober space which is not induced by a homeomorphism.

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