Applied General Topology
© Universidad Politécnica de Valencia
Volume 11, No. 1, 2010
pp. 57-65

On almost cl-supercontinuous functions

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Abstract. Recently the class of almost cl-supercontinuous functions between topological spaces has been studied in some detail. We consider this class of functions from the point of view of change(s) of topology. In particular, we conclude that this class of functions coincides with the usual class of continuous functions when the domain and codomain have been retopologized appropriately. Some of the consequences of this fact are considered in this paper.

2000 AMS Classification: Primary 54C60, 54C05, 54C10; Secondary 54D10, 54D20.

Keywords: change of topology, cl-supercontinuity, almost cl-supercontinuity, continuity, almost continuity, perfect continuity, almost perfect continuity, slightly continuous.

1. Introduction

There can be no argument that the notion of continuity is one of the most important concepts in the whole of mathematics. Over the years many generalizations and variants of continuous functions between topological spaces have been introduced. Very recently, Kohli and Singh [7], have considered the class of "almost cl-supercontinuous " functions. This is a new name for the class of "almost clopen " functions introduced by Ekici [4] as a generalization of the class of "clopen continuous " mappings defined by Reilly and Vamanamurthy [11], and studied in some detail by Singh [13], under the name of cl-supercontinuous functions.

In particular, Kohli and Singh [7] noted that almost cl-supercontinuity is a variant of continuity which is " independent of continuity ". A primary purpose of this paper is to argue exactly the opposite of this view. We advocate that the distinction made by Kohli and Singh [7] between the concepts of

*The second author gratefully acknowledges the award of a Fellowship for Visiting Scientists by the Scientific and Technological Research Council of Turkey (TUBITAK)

almost cl-supercontinuity and continuity must be interpreted very strictly. It is our contention that almost cl-supercontinuity is just continuity in disguise. Indeed, if the domain and codomain spaces of an almost cl-supercontinuous function f are retopologized in a suitable fashion (see Theorem 4.1), then f is simply a continuous function. This observation puts the notion of almost cl-supercontinuity in a more natural context, and it allows alternative proofs of some of the results of Kohli and Singh [7]. What we are claiming, in the language of category theory, is that an almost cl-supercontinuous function f arises because the wrong source and target have been chosen for the morphism f in the category of topological spaces and continuous functions.

In section 2 we consider the basic properties of semi-regular topologies and of quasi-topologies, which we shall need in this paper. The relevant definitions of the classes of functions we consider in this paper are provided in section 3. Section 4 examines the class of almost cl-supercontinuous functions studied by Kohli and Singh [7], especially from the perspective of change(s) of topology.

Kohli and Singh [7, 2.1] and Singh [13, Introduction] make a case for a change in nomenclature, and rename the clopen continuous functions defined by Reilly and Vamanamurthy [11] as cl-supercontinuous functions. Names are largely a matter of personal taste, although the original term at least suggests what the definition might be. However, the use of the cl- prefix notation leads to the possibility of serious confusion. From [13, Definition 2.1] we see that the subsets called cl-open by Singh [13] are precisely the quasi-open sets of the quasi-topology, see our section 2, and the cl-closed sets are the quasi-closed sets. The difference between clopen sets and cl-open sets is significant, but the notations are too similar. Similarly [13, Definitions 2.4 and 2.6] show that cl-adherence and cl-convergence are precisely adherence and convergence in the quasi-topology, so why not use the terms quasi-adherence and quasi-convergence. This usage of the cl-prefix leads to [13, Definition 2.9] where a function is defined to be cl-open if it takes clopen sets to open sets. But as pointed out by Singh [13, Introduction p293] "in the topological folklore the phrase " clopen map" is used for the functions which map clopen sets to open sets". This desire for purity of nomenclature has created the absurd situation where "clopen maps" should now be termed "cl-open maps". Despite our preference for the original term "clopen continuous", we shall use the term "cl-supercontinuous" throughout this paper.

Our notation and terminology are standard, see for example Dugundji [3]. No separation properties are assumed for topological spaces unless explicitly stated. We denote the interior of the subset A of the topological space (X, τ) by $\tau intA$, and the closure of A by τclA.

2. Two topologies

In a topological space (X, τ) a set A is called τ regular open if $A = \tau int(\tau clA)$ and τ regular closed if $A = \tau cl(\tau intA)$. We let $RO(X, \tau)$ denote the collection of all regular open sets in (X, τ). Since the intersection of two regular open sets is regular open, the collection of all τ regular open sets forms a base for a smaller topology τ_s on X, called the semi-regularization of τ. The space (X,τ) is said to be semi-regular if $\tau_s = \tau$. Any regular space is semi-regular, but the converse is false.

In 1968, Velicko [16] introduced the notion of δ-open and δ-closed sets in a space (X,τ). A point $x \in X$ is said to be a δ-cluster point of the subset A of (X,τ) if $A \cap U \neq \varnothing$ for every τ regular open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and denoted by $[A]_\delta$. If $A = [A]_\delta$ then A is called δ-closed, and the complement of a δ-closed set is called δ-open. For any space (X,τ) the collection of all δ-open sets forms a topology τ_δ on X, and the definitions of δ-adherent point, δ-closure, δ-convergence of filterbases and so on are the usual definitions applied to the topology τ_δ. It turns out that using δ-open sets is another way to describe the semi-regularization topology, that is, $\tau_\delta = \tau_s$ for any space (X,τ). Semi-regularization topologies are considered in some detail by Mrsevic, Reilly and Vamanamurthy [8], especially from the change of topology perspective.

Let (X,τ) be a topological space. The quasi-component of a point $x \in X$ is the intersection of all clopen subsets of X which contain the point x. The quasi-topology on X is the topology having as base the collection of all clopen subsets of (X,τ). Since any clopen subset of X is regular open (and regular closed), the quasi-topology is smaller than the semi-regularization topology in general. The closure of each point in the quasi-topology is precisely the quasi-component of that point. We shall call the open (resp. closed) subsets of the quasi-topology quasi-open (resp. quasi-closed) and denote the quasi-topology by τ_q. For a given topological space (X,τ), the space (X,τ_q) is called by Staum [15] the ultraregular kernel of X. Recall that a space (X,τ) is zero-dimensional if it has a base of clopen sets for the topology τ. We observe that (X,τ) is zero-dimensional if and only if $\tau_q = \tau$. We have noted that for any topological space (X,τ) we have $\tau_q \subset \tau_s \subset \tau$. Quasi-topologies are considered by Dontchev, Ganster and Reilly [2, Section 3], and by Singh [13, Section 5] who used the notation τ^* for τ_q.

An unusual feature of quasi-topologies is their behaviour with respect to the lower separation properties, T_0, T_1 and Hausdorff. In fact, any quasi-topology has either all or none of these three separation properties. To be precise, if (X,τ) is such that (X,τ_q) is T_0, then (X,τ_q) is Hausdorff. Let a and b be distinct points of X, so that there is a quasi-open subset M of X containing one of a and b but not the other. Assume that $a \in M$. Then there is a clopen subset D of X such that $a \in D \subset M$. Now $b \in X - D$ and $X - D$ is clopen and disjoint from D, so that $\{D, X - D\}$ is a Hausdorff separation of a and b in (X,τ_q). In particular, (X,τ_q) is Hausdorff if and only if (X,τ_q) is T_0.

3. Definitions

Here we provide a list of definitions of the variations of continuity that we consider in this paper.

A function $f:(X,\tau)\to(Y,\sigma)$ *between topological spaces is defined to be*

(1) *almost continuous [12] if for each* $x\in X$ *and for each regular open set* V *containing* $f(x)$ *there is an open set* U *containing* x *such that* $f(U)\subset V$.
(2) δ*-continuous [10] if for each* $x\in X$ *and for each regular open set* V *containing* $f(x)$ *there is a regular open set* U *containing* x *such that* $f(U)\subset V$.
(3) *supercontinuous [9] if for each* $x\in X$ *and for each open set* V *containing* $f(x)$ *there is a regular open set* U *containing* x *such that* $f(U)\subset V$.
(4) *cl-supercontinuous [13] (or clopen continuous [11]) if for each* $x\in X$ *and for each open set* V *containing* $f(x)$ *there is a clopen set* U *containing* x *such that* $f(U)\subset V$.
(5) *almost cl-supercontinuous [7] (or almost clopen [4]) if for each* $x\in X$ *and for each regular open set* V *containing* $f(x)$ *there is a clopen set* U *containing* x *such that* $f(U)\subset V$.
(6) *slightly continuous [6] if* $f^{-1}(M)$ *is open in* X *for each clopen set* M *in* Y.

4. Change of topology

The fundamental defining characteristic of the class of almost cl-supercontinuous functions is given by the following result, especially the equivalence of (1) and (4), which is proved immediately from the definitions. We observe that Ekici [4 Theorem 6 (1), (10), (11), (12)] obtained this theorem, but did not subsequently use it in his paper. On the other hand, it is the cornerstone of our approach to this topic.

Theorem 4.1. *Let* f *be a function from a topological space* (X,τ) *to a topological space* (Y,σ). *Then the following are equivalent:*

(1) $f:(X,\tau)\to(Y,\sigma)$ *is almost cl-supercontinuous,*
(2) $f:(X,\tau_q)\to(Y,\sigma)$ *is almost continuous,*
(3) $f:(X,\tau)\to(Y,\sigma_s)$ *is cl-supercontinuous,*
(4) $f:(X,\tau_q)\to(Y,\sigma_s)$ *is continuous.*

The equivalence of Theorem 4.1 (1) and (4) shows that almost cl-supercontinuity is a μ-continuity property in the sense of Gauld, Mrsevic, Reilly and Vamanamurthy [5]. The containment $\sigma_s\subset\sigma$ and the equivalence of (1) and (3) in Theorem 4.1 show that, in general, cl-supercontinuity is stronger than almost cl-supercontinuity, but that for semi-regular codomains they are equivalent.

It seems that the diagram of implications given by Kohli and Singh [7, page 3] can create confusion. We reproduce their diagram here.

$$
\begin{array}{ccc}
\text{strongly continuous} & & \\
\Downarrow & & \\
\text{perfectly continuous} & \Rightarrow & \text{almost perfectly continuous} \\
\Downarrow & & \Downarrow \\
\text{cl-supercontinuous} & \Rightarrow & \text{almost cl-supercontinuous} \\
\Downarrow & & \Downarrow \\
z\text{-supercontinuous} & \Rightarrow & \text{almost } z\text{-supercontinuous} \\
\Downarrow & & \Downarrow \\
\text{supercontinuous} & \Rightarrow & \delta\text{-continuous} \\
\Downarrow & & \Downarrow \\
\text{continuous} & \Rightarrow & \text{almost continuous}
\end{array}
$$

We take this diagram to mean that one can find a function between topological spaces (X,τ) and (Y,σ) having one of these properties but not one of the stronger properties. One must regard the topologies on X and Y as fixed for this interpretation. On the other hand, Theorem 4.1 above shows that four of these continuity type concepts coincide if one is willing to change the topology on X or on Y or on both X and Y. In particular, it shows that the concept of almost cl-supercontinuity coincides with the classical notion of continuity under appropriate changes of topology on domain and co-domain. This diagram of implications [7, page 3] strongly suggests that such a situation cannot occur.

Theorem 5.1 of Singh [13] is a corollary of the method of proof of Theorem 4.1.

Theorem 4.2. *The function* $f:(X,\tau)\to(Y,\sigma)$ *is cl-supercontinuous if and only if* $f:(X,\tau_q)\to(Y,\sigma)$ *is continuous.*

This result reveals that whenever $\tau_q=\tau$ then any continuous function $f:(X,\tau)\to(Y,\sigma)$ is cl-supercontinuous. Thus if (X,τ) is zero-dimensional we have that any continuous function with domain (X,τ) is cl-supercontinuous. The topological properties of the codomain space are not germaine to this discussion.

Our Theorem 4.1 can be used to provide elegant alternative proofs of some of the results of Kohli and Singh [7]. For example, consider the matter of composition of functions [7, Theorem 3.2].

First we need to observe the following equivalences.

(1) $f:(X,\tau)\to(Y,\sigma)$ is δ-continuous if and only if $f:(X,\tau_s)\to(Y,\sigma_s)$ is continuous [10, Theorem 2.5].

(2) $f:(X,\tau)\to(Y,\sigma)$ is cl-supercontinuous if and only if $f:(X,\tau_q)\to(Y,\sigma)$ is continuous [13, Theorem 5.1].

(3) $f:(X,\tau)\to(Y,\sigma)$ is almost continuous if and only if $f:(X,\tau)\to(Y,\sigma_s)$ is continuous [8, Proposition 12].

(4) $f : (X, \tau) \to (Y, \sigma)$ is supercontinuous if and only if $f : (X, \tau_s) \to (Y, \sigma)$ is continuous [9, Theorem 2.1].

(5) $f : (X, \tau) \to (Y, \sigma)$ is slightly continuous if and only if $f : (X, \tau) \to (Y, \sigma_q)$ is continuous [Theorem 5.3 (a) of Singh [13]].

Change of topology allows us to prove the next theorem simply by observing that the composition of two continuous functions is continuous. There is no need to give proofs going back to first principles like those presented by Kohli and Singh [7] and Ekici [4, Theorem 13]. Note that part of Theorem 2.10 of Singh [13] is a special case of (1) of Theorem 4.3, while Theorem 2.17 of Singh [13] is (5) of our Theorem 4.3.

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \psi)$ be functions.*

(1) If f is cl-supercontinuous and g is continuous then $g \circ f$ is cl-supercontinuous.

(2) If f is cl-supercontinuous and g is almost continuous then, $g \circ f$ is almost cl-supercontinuous.

(3) If f is almost cl-supercontinuous and g is δ-continuous, then $g \circ f$ is almost cl-supercontinuous.

(4) If f is almost cl-supercontinuous and g is supercontinuous, then $g \circ f$ is cl-supercontinuous.

(5) If f is slightly continuous and g is cl-supercontinuous, then $g \circ f$ is continuous.

(6) If f is slightly continuous and g is almost cl-supercontinuous, then $g \circ f$ is almost continuous.

Proof.

(1) $f : (X, \tau_q) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \psi)$ are continuous, so that $g \circ f : (X, \tau_q) \to (Z, \psi)$ is continuous.

(2) $f : (X, \tau_q) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \psi_s)$ are continuous, so that $g \circ f : (X, \tau_q) \to (Z, \psi_s)$ is continuous.

(3) $f : (X, \tau_q) \to (Y, \sigma_s)$ and $g : (Y, \sigma_s) \to (Z, \psi_s)$ are continuous, so that $g \circ f : (X, \tau_q) \to (Z, \psi_s)$ is continuous.

(4) $f : (X, \tau_q) \to (Y, \sigma_s)$ and $g : (Y, \sigma_s) \to (Z, \psi)$ are continuous, so that $g \circ f : (X, \tau_q) \to (Z, \psi)$ is continuous.

(5) $f : (X, \tau) \to (Y, \sigma_q)$ and $g : (Y, \sigma_q) \to (Z, \psi)$ are continuous, so that $g \circ f : (X, \tau) \to (Z, \psi)$ is continuous.

(6) $f : (X, \tau) \to (Y, \sigma_q)$ and $g : (Y, \sigma_q) \to (Z, \psi_s)$ are continuous, so that $g \circ f : (X, \tau) \to (Z, \psi_s)$ is continuous.

□

In section 4 of their paper, Kohli and Singh [7] consider separation properties. Their Definition 4.1 defines three classes of spaces, namely ultra-Hausdorff, ultra-T_1 and ultra-T_0 spaces. We observe that

1) (X, τ) is ultra-Hausdorff if and only if (X, τ_q) is Hausdorff,

2) (X, τ) is ultra-T_1 if and only if (X, τ_q) is T_1, and

3) (X, τ) is ultra-T_0 if and only if (X, τ_q) is T_0.

In section 2 above we noted that (X, τ_q) is Hausdorff if and only if (X, τ_q) is T_0. This fact provides a change of topology proof of Proposition 4.2 of [7].

Definition 4.3 of [7] defines two classes of spaces. We note that

1) (X, τ) is δT_1 if and only if (X, τ_s) is T_1, and

2) (X, τ) is δT_0 if and only if (X, τ_s) is T_0.

Mrsevic, Reilly and Vamanamurthy [8, Proposition 1] have observed that (X, τ) is Hausdorff if and only if (X, τ_s) is Hausdorff. This fact explains why there is no analagous definition of δT_2 spaces.

We now provide an alternative proof of Theorem 4.5 of Kohli and Singh [7].

Proposition 4.4. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an almost cl-supercontinuous injection. If* (Y, σ) *is* δT_0, *then* (X, τ) *is ultra-Hausdorff.*

Proof. Note that $f : (X, \tau_q) \to (Y, \sigma_s)$ is a continuous injection, and that (Y, σ_s) is T_0. Thus (X, τ_q) is T_0, so that (X, τ_q) is Hausdorff. That is, (X, τ) is ultra-Hausdorff. □

Theorem 4.7 of Kohli and Singh [7] is a standard result restated in this new setting.

Proposition 4.5. *Let* $f, g : (X, \tau) \to (Y, \sigma)$ *be almost cl-supercontinuous functions and* (Y, σ) *be Hausdorff. Then* $E = \{x \in X : f(x) = g(x)\}$ *is quasi-closed in* (X, τ).

Proof. We have that $f, g : (X, \tau_q) \to (Y, \sigma_s)$ are continuous, by Theorem 4.1 (1) and (4), and (Y, σ_s) is Hausdorff. So by Dugundji [3, page 140 1.5 (1)] we have that E is closed in (X, τ_q), and therefore E is quasi-closed in (X, τ). □

Another standard theorem applied in this context yields the following result.

Proposition 4.6. *If* $f : (X, \tau) \to (Y, \sigma)$ *is almost cl-supercontinuous and* (Y, σ) *is Hausdorff, then* $G(f)$, *the graph of* f, *is closed in* $(X \times Y, \tau_q \times \sigma_s)$.

Proof. We have that $f : (X, \tau_q) \to (Y, \sigma_s)$ is continuous and (Y, σ_s) is Hausdorff. Thus, by Dugundji [3, page 140 1.5 (3)] the graph $G(f)$ of f is closed in $(X \times Y, \tau_q \times \sigma_s)$. □

This result is a simpler version of much of the work in section 6 of Kohli and Singh [7].

Recall that a topological space (X, τ) is called mildly compact [15], or a clustered space [14], if every clopen cover of X has a finite subcover. Furthermore, (X, τ) is called nearly compact [1] if every regular open cover of X has a finite subcover. In fact, (X, τ) is mildly compact if and only if (X, τ_q) is compact, while (X, τ) is nearly compact if and only if (X, τ_s) is compact, Carnahan [1,Theorem 4.1]. We can now use the preservation of compactness by continuous functions together with our Theorem 4.1 to provide an alternative proof of Theorem 41 (1) of Ekici [4].

Theorem 4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost cl-supercontinuous surjection. If (X, τ) is mildly compact, then (Y, σ) is nearly compact.*

Proof. Since (X, τ) is mildly compact, (X, τ_q) is compact. As $f : (X, \tau) \to (Y, \sigma)$ is almost cl-supercontinuous, we have by Theorem 4.1 that $f : (X, \tau_q) \to (Y, \sigma_s)$ is continuous. Furthermore, f is surjective, so that (Y, σ_s) is compact. Hence (Y, σ) is nearly compact. □

By now we have provided sufficient evidence to be able to claim unequivocally that change of topology approaches are significant. They provide new insights into some of the developments taking place in general topology. They permit elegant alternative proofs of existing results, and allow the creation of new results.

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Received October 2009

Accepted March 2010

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