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On set star-Lindelöf spaces

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Abstract

A space X is said to be set star-Lindelöf if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$. The class of set star-Lindelöf spaces lie between the class of Lindelöf spaces and the class of star-Lindelöf spaces. In this paper, we investigate the relationship between set star-Lindelöf spaces and other related spaces by providing some suitable examples and study the topological properties of set star-Lindelöf spaces.

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1. Introduction and Preliminaries

Arhangel'skii [1] defined a cardinal number $sL(X)$ of X: the minimal infinite cardinality τ such that for every subset $A \subset X$ and every open cover $\mathcal{U}$ of $\overline{A}$, there is a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \tau$ and $A \subseteq \overline{\bigcup \mathcal{V}}$. If $sL(X) = \omega$, then the space X is called *sLindelöf space*. Following this idea, Kočinac and Konca [7] introduced and studied the new types of selective covering properties called set-covering properties (for a similar studies, see [4, 14, 15, 16, 17]). A space X is said to have the set-Menger [7] property if for each nonempty subset A of X and each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of collections of open sets in X such that for each $n \in \mathbb{N}$, $\overline{A} \subseteq \bigcup \mathcal{U}_n$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $A \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$. The author [13] noticed

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that the set-Menger property is nothing but another view of Menger covering property. Recently, the author [12] defined and studied set starcompact and set strongly starcompact spaces (also see [8]).

In this paper, we consider the classes of set star-Lindelöf spaces and set strongly star-Lindelöf spaces already introduced in [9] and recently studied in [4]. Note that in fact in the class of T_1 spaces, set strongly star-Lindelöfness is equivalent to the property having countable extent [[4], Proposition 3.1]. If A is a subset of a space X and $\mathcal{U}$ is a collection of subsets of X, then $\mathrm{St}(A,\mathcal{U})=\bigcup\{U \in \mathcal{U} : U \cap A \neq \varnothing\}$. We usually write $\mathrm{St}(x,\mathcal{U}) = \mathrm{St}(\{x\},\mathcal{U})$.

Throughout the paper, by "a space" we mean "a topological space", $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ denotes the set of natural numbers, set of real numbers, and set of rational numbers, respectively, the cardinality of a set is denoted by $|A|$. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. An open cover $\mathcal{U}$ of a subset $A \subset X$ means elements of $\mathcal{U}$ open in X such that $A \subseteq \bigcup\mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$.

We first recall the classical notions of spaces that are used in this paper.

Definition 1.1 ([5])**.** A space X is said to be

(1) starcompact if for each open cover $\mathcal{U}$ of X, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \mathrm{St}(\bigcup\mathcal{V},\mathcal{U})$.
(2) strongly starcompact if for each open cover $\mathcal{U}$ of X, there is a finite subset F of X such that $X = \mathrm{St}(F,\mathcal{U})$.

Definition 1.2 ([12, 8])**.** A space X is said to be

(1) set starcompact if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup\mathcal{U}$, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \mathrm{St}(\bigcup\mathcal{V},\mathcal{U})$.
(2) set strongly starcompact if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup\mathcal{U}$, there is a finite subset F of $\overline{A}$ such that $A \subseteq \mathrm{St}(F,\mathcal{U})$.

Definition 1.3. A space X is said to be

(1) star-Lindelöf [5] if for each open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \mathrm{St}(\bigcup\mathcal{V},\mathcal{U})$.
(2) strongly star-Lindelöf [5] if for each open cover $\mathcal{U}$ of X, there is a countable subset F of X such that $X = \mathrm{St}(F,\mathcal{U})$.

Note that the star-Lindelöf spaces have a different name such as 1-star-Lindelöf and $1\frac{1}{2}$-star-Lindelöf in different papers (see [5, 10]) and the strongly star-Lindelöf space is also called star countable in [10, 21]. It is clear that, every strongly star-Lindelöf space is star-Lindelöf.

Recall that a collection $\mathcal{A} \subseteq P(\omega)$ is said to be almost disjoint if each set $A \in \mathcal{A}$ is infinite and the sets $A \bigcap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, put $\psi(\mathcal{A}) = \mathcal{A} \bigcup \omega$ and topologize $\psi(\mathcal{A})$ as follows: for each element $A \in \mathcal{A}$ and each finite set $F \subset \omega$, $\{A\} \bigcup (A \setminus F)$ is a basic open neighborhood of A and the natural numbers are isolated. The spaces of this type are called Isbell-Mrówka ψ-spaces [2, 11] or $\psi(\mathcal{A})$ space. For other terms and symbols, we follow [6].

The following result was proved in [8].

Theorem 1.4 ([8]). *Every countably compact space is set strongly starcompact.*

Note that in the class of Hausdorff spaces strongly starcompactness, set strongly starcompactness and countable compactness are equivalent [4, Proposition 2.2].

2. Set star-Lindelöf and related spaces

In this section, we give some examples showing the relationship among set star-Lindelöf spaces, set strongly star-Lindelöf spaces, and other related spaces. First we define our main definition.

Definition 2.1. A space X is said to be

1. set star-Lindelöf if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$.
2. set strongly star-Lindelöf if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there is a countable subset F of $\overline{A}$ such that $A \subseteq \mathrm{St}(F, \mathcal{U})$.

Note that in the class of T_1 spaces the set strongly star-Lindelöfness is equivalent to the property to have a countable extent [4, Proposition 3.1]. Note that there is a misprint in the statement of the definition of relatively* set star strongly-compact in [4]: the authors write that set F is a finite subset of A but the original definition asks that F is contained in $\overline{A}$ and Bonanzinga and Maesano use exactly this last fact during all the paper.

We have the following diagram from the definitions and [4, Proposition 3.1]. However, the following examples show that the converse of these implications are not true.

$$\begin{array}{ccccccc}
 & & & & \textit{set strongly starcompact} & \to & \textit{set starcompact} \\
 & & & & \downarrow & & \downarrow \\
\textit{Lindelof} & \to & \textit{countable extent} & \leftrightarrow^{T_1} & \textit{set strongly star}-\textit{Lindelof} & \to & \textit{set star}-\textit{Lindelof} \\
 & & & & \downarrow & & \downarrow \\
 & & & & \textit{strongly star}-\textit{Lindelof} & \to & \textit{star}-\textit{Lindelof}
\end{array}$$

Example 2.2. (i) The discrete space ω has countable extent but it is not set starcompact space.

(ii) The space $[0, \omega_1)$ has countable extent but it is not Lindelöf.

(iii) Let Y be a discrete space with cardinality $\mathfrak{c}$. Let $X = Y \cup \{y^*\}$, where $y^* \notin Y$ topologized as follows: each $y \in Y$ is an isolated point and a set U

containing y^* is open if and only if $X \setminus U$ is countable. Then X has countable extent but it is not countably compact.

Bonanzinga [3] proved that every Isbell-Mrówka space is a Tychonoff strongly star-Lindelöf space with uncountable extent (hence, it is not set strongly star-Lindelöf). Note that in [3] strongly star-Lindelöf is called star-Lindelöf.

The following lemma was proved by Song [18].

Lemma 2.3 ([18, Lemma 2.2]). *A space X having a dense Lindelöf subspace is star-Lindelöf.*

The following example shows that the Lemma 2.3 does not hold if we replace star-Lindelöf space by a set star-Lindelöf space.

Example 2.4. There exists a Tychonoff space X having a dense Lindelöf subspace such that X is not set star-Lindelöf.

Proof. Let $D(\mathfrak{c}) = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D(\mathfrak{c}) \cup \{d^*\}$ be one-point compactification of $D(\mathfrak{c})$. Let

$$X = (Y \times [0, \omega)) \cup (D(\mathfrak{c}) \times \{\omega\})$$

be the subspace of the product space $Y \times [0, \omega]$. Then $Y \times [0, \omega)$ is a dense Lindelöf subspace of X and by Lemma 2.3, X is star-Lindelöf.

In [4, Proposition 3.4] shows that if X is a space such that there exists a closed and discrete subspace D of X having uncountable cardinality and a disjoint family $\mathcal{U} = \{O_a : a \in D\}$ of open neighborhoods of points $a \in D$, then X is not set star-Lindelöf. So, we conclude that X is not set star-Lindelöf. □

Bonanzinga and Maesano [4, Example 3.5] constructed an example of a Tychonoff separable (hence set star-Lindelöf) non set strongly star-Lindelöf space.

Remark 2.5. (1) In [12], Singh gave an example of a Tychonoff set starcompact space X that is not set strongly starcompact.

(2) It is known that there are star-Lindelöf spaces that are not strongly star-Lindelöf (see [5, Example 3.2.3.2] and [5, Example 3.3.1]).

Now we give some conditions under which star-Lindelöfness coincides with set star-Lindelöfness and strongly star-Lindelöfness coincide with set strongly star-Lindelöfness.

Recall that a space X is paraLindelöf if every open cover $\mathcal{U}$ of X has a locally countable open refinement.

Song and Xuan [19] proved the following result.

Theorem 2.6 ([19, Theorem 2.24]). *Every regular paraLindelöf star-Lindelöf spaces are Lindelöf.*

We have the following theorem from Theorem 2.6 and the diagram.

Theorem 2.7. *If X is a regular paraLindelöf space, then the following statements are equivalent:*

(1) X is Lindelöf;
(2) X is set strongly star-Lindelöf;
(3) $e(X) = \omega$;
(4) X is set star-Lindelöf;
(5) X is strongly star-Lindelöf;
(6) X is star-Lindelöf.

A space is said to be metaLindelöf if every open cover of it has a point-countable open refinement.

Bonanzinga [3] proved the following result.

Theorem 2.8 ([3])**.** *Every strongly star-Lindelöf metaLindelöf spaces are Lindelöf.*

We have the following theorem from Theorem 2.8 and the diagram.

Theorem 2.9. *If X is a metaLindelöf space, then the following statements are equivalent:*

(1) X is Lindelöf;
(2) X is set strongly star-Lindelöf;
(3) $e(X) = \omega$;
(4) X is strongly star-Lindelöf.

3. Properties of set star-Lindelöf spaces

In this section, we study the topological properties of set star-Lindelöf spaces.

Theorem 3.1. *If X is a set star-Lindelöf space, then every open and closed subset of X is set star-Lindelöf.*

Proof. Let X be a set star-Lindelöf space and $A \subseteq X$ be an open and closed set. Let B be any subset of A and $\mathcal{U}$ be a collection of open sets in (A, τ_A) such that $Cl_A(B) \subseteq \bigcup \mathcal{U}$. Since A is open, then $\mathcal{U}$ is a collection of open sets in X. Since A is closed, $Cl_A(B) = Cl_X(B)$. Applying the set star-Lindelöfness property of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $B \subseteq St(\bigcup \mathcal{V}, \mathcal{U})$. Hence A is a set star-Lindelöf. □

Consider the Alexandorff duplicate $A(X) = X \times \{0,1\}$ of a space X. The basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X and each point $\langle x, 1\rangle \in X \times \{1\}$ is an isolated point.

Theorem 3.2. *If X is a T_1-space and $A(X)$ is a set star-Lindelöf space. Then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of

$A(X)$ and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not set star-Lindelöf by Theorem 3.1. □

Theorem 3.3. *Let X be a space such that the Alexandorff duplicate $A(X)$ of X is set star-Lindelöf. Then X is a set star-Lindelöf space.*

Proof. Let B be any nonempty subset of X and $\mathcal{U}$ be an open cover of $\overline{B}$. Let $C = B \times \{0\}$ and

$$A(\mathcal{U}) = \{U \times \{0,1\} : U \in \mathcal{U}\}.$$

Then $A(\mathcal{U})$ is an open cover of $\overline{C}$. Since $A(X)$ is set star-Lindelöf, there is a countable subset $A(\mathcal{V})$ of $A(\mathcal{U})$ such that $C \subseteq \mathrm{St}(\bigcup A(\mathcal{V}), A(\mathcal{U}))$. Let

$$\mathcal{V} = \{U \in \mathcal{U} : U \times \{0,1\} \in A(\mathcal{V})\}.$$

Then $\mathcal{V}$ is a countable subset of $\mathcal{U}$. Now we have to show that

$$B \subseteq \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U}).$$

Let $x \in B$. Then $\langle x, 0\rangle \in \mathrm{St}(\bigcup A(\mathcal{V}), A(\mathcal{U}))$. Choose $U \times \{0,1\} \in A(\mathcal{U})$ such that $\langle x, 0\rangle \in U \times \{0,1\}$ and $U \times \{0,1\} \cap (\bigcup A(\mathcal{V})) \neq \varnothing$, which implies $U \cap (\bigcup \mathcal{V}) \neq \varnothing$ and $x \in U$. Therefore $x \in \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$, which shows that X is set star-Lindelöf space. □

On the images of set star-Lindelöf spaces, we have the following result.

Theorem 3.4. *A continuous image of set star-Lindelöf space is set star-Lindelöf.*

Proof. Let X be a set star-Lindelöf space and $f : X \to Y$ is a continuous mapping from X onto Y. Let B be any subset of Y and $\mathcal{V}$ be an open cover of $\overline{B}$. Let $A = f^{-1}(B)$. Since f is continuous, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is the collection of open sets in X with $\overline{A} = \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \subseteq f^{-1}(\bigcup \mathcal{V}) = \bigcup \mathcal{U}$. As X is set star-Lindelöf, there exists a countable subset $\mathcal{U}'$ of $\mathcal{U}$ such that

$$A \subseteq \mathrm{St}(\bigcup \mathcal{U}', \mathcal{U}).$$

Let $\mathcal{V}' = \{V : f^{-1}(V) \in \mathcal{U}'\}$. Then $\mathcal{V}'$ is a countable subset of $\mathcal{V}$ and $B = f(A) \subseteq f(\mathrm{St}(\bigcup \mathcal{U}', \mathcal{U})) \subseteq \mathrm{St}(\bigcup f(\{f^{-1}(V) : V \in \mathcal{V}'\}), \mathcal{V}) = \mathrm{St}(\bigcup \mathcal{V}', \mathcal{V})$. Thus Y is set star-Lindelöf space. □

Next, we turn to consider preimages of set strongly star-Lindelöf and set star-Lindelöf spaces. We need a new concept called nearly set star-Lindelöf spaces. A space X is said to be nearly set star-Lindelöf in X if for each subset Y of X and each open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subseteq \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$. For the strong version of this property (see [4]).

Theorem 3.5. *If $f : X \to Y$ is an open and perfect continuous mapping and Y is a set star-Lindelöf space, then X is nearly set star-Lindelöf.*

Proof. Let $A \subseteq X$ be any nonempty set and $\mathcal{U}$ be an open cover of X. Then $B = f(A)$ is a subset of Y. Let $y \in \overline{B}$. Then $f^{-1}\{y\}$ is a compact subset of X, thus there is a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subseteq \bigcup \mathcal{U}_y$. Let $U_y = \bigcup \mathcal{U}_y$. Then $V_y = Y \setminus f(X \setminus U_y)$ is a neighborhood of y, since f is closed. Then $\mathcal{V} = \{V_y : y \in \overline{B}\}$ is an open cover of $\overline{B}$. Since Y is set star-Lindelöf, there exists a countable subset $\mathcal{V}'$ of $\mathcal{V}$ such that

$$B \subseteq \mathrm{St}(\bigcup \mathcal{V}', \mathcal{V}).$$

Without loss of generality, we may assume that $\mathcal{V}' = \{V_{y_i} : i \in N' \subseteq \mathbb{N}\}$. Let $\mathcal{W} = \bigcup_{i \in N'} \mathcal{U}_{y_i}$. Since $f^{-1}(V_{y_i}) \subseteq \bigcup\{U : U \in \mathcal{U}_{y_i}\}$ for each $i \in N'$. Then $\mathcal{W}$ is a countable subset of $\mathcal{U}$ and

$$f^{-1}(\bigcup \mathcal{V}') = \bigcup \mathcal{W}.$$

Next, we show that

$$A \subseteq \mathrm{St}(\bigcup \mathcal{W}, \mathcal{U}).$$

Let $x \in A$. Then there exists a $y \in B$ such that

$$f(x) \in V_y \text{ and } V_y \bigcap (\bigcup \mathcal{V}') \neq \varnothing.$$

Since

$$x \in f^{-1}(V_y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$. Thus $U \bigcap f^{-1}(\bigcup \mathcal{V}') \neq \varnothing$. Hence $x \in \mathrm{St}(f^{-1}(\bigcup \mathcal{V}'), \mathcal{U})$. Therefore $x \in \mathrm{St}(\bigcup \mathcal{W}, \mathcal{U})$, which shows that $A \subseteq \mathrm{St}(\bigcup \mathcal{W}, \mathcal{U})$. Thus X is nearly set star-Lindelöf. □

It is known that the product of star-Lindelöf space and compact space is a star-Lindelöf (see [5]).

Problem 3.6. *Does the product of set star-Lindelöf space and a compact space is set star-Lindelöf?*

The following example shows that the product of two countably compact (hence, set star-Lindelöf) spaces need not be set star-Lindelöf.

Example 3.7. There exist two countably compact spaces X and Y such that $X \times Y$ is not set star-Lindelöf.

Proof. Let $D(\mathfrak{c})$ be a discrete space of the cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$ and $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D(\mathfrak{c}))$ which are defined inductively to satisfy the following three conditions:

(1) $E_\alpha \bigcap F_\beta = D(\mathfrak{c})$ *if* $\alpha \neq \beta$;
(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
(3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp, $F_{\alpha+1}$).

Those sets E_α and F_α are well-defined since every infinite closed set in $\beta(D(\mathfrak{c}))$ has the cardinality $2^{\mathfrak{c}}$ (see [20]). Then $X \times Y$ is not set star-Lindelöf, since the diagonal $\{\langle d, d\rangle : d \in D(\mathfrak{c})\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$. □

van Douwen-Reed-Roscoe-Tree [5, Example 3.3.3] gave an example of a countably compact X (hence, set star-Lindelöf) and a Lindelöf space Y such that $X \times Y$ is not strongly star-Lindelöf. Now we use this example to show that $X \times Y$ is not set star-Lindelöf.

Example 3.8. There exists a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not set star-Lindelöf.

Proof. Let $X = [0, \omega_1)$ with the usual order topology. Let $Y = [0, \omega_1]$ with the following topology. Each point $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then, X is countably compact and Y is Lindelöf. It is enough to show that $X \times Y$ is not star-Lindelöf.

For each $\alpha < \omega_1$, $U_\alpha = X \times \{\alpha\}$ is open in $X \times Y$. For each $\beta < \omega_1$, $V_\beta = [0, \beta] \times (0, \omega_1]$ is open in $X \times Y$. Let $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\beta : \beta < \omega_1\}$. Then $\mathcal{U}$ is an open cover of $X \times Y$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$. Since $\mathcal{V}$ is countable, there exists $\alpha' < \omega_1$ such that $U_\alpha \notin \mathcal{V}$ for each $\alpha > \alpha'$. Also, there exists $\alpha'' < \omega_1$ such that $V_\beta \notin \mathcal{V}$ for each $\beta > \alpha''$. Let $\beta = sup\{\alpha', \alpha''\}$. Then $U_\beta \bigcap (\bigcup \mathcal{V}) = \varnothing$ and U_β is the only element containing $\langle \beta, \beta\rangle$. Thus $\langle \beta, \beta\rangle \notin \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$, which shows that X is not star-Lindelöf. □

van Douwen-Reed-Roscoe-Tree [5, Example 3.3.6] gave an example of Hausdorff regular Lindelöf spaces X and Y such that $X \times Y$ is star-Lindelöf. Now we use this example and show that the product of two Lindelöf spaces is not set star-Lindelöf.

Example 3.9. There exists a Hausdorff regular Lindelöf spaces X and Y such that $X \times Y$ is not set star-Lindelöf.

Proof. Let $X = \mathbb{R} \setminus \mathbb{Q}$ have the induced metric topology. Let $Y = \mathbb{R}$ with each point of $\mathbb{R} \setminus \mathbb{Q}$ is isolated and points of $\mathbb{Q}$ having metric neighborhoods. Hence both spaces X and Y are Hausdorff regular Lindelöf spaces and first countable too, so $X \times Y$ Hausdorff regular and first countable. Now we show that $X \times Y$ is not set star-Lindelöf. Let $A = \{(x, x) \in X \times Y : x \in X\}$. Then A is an uncountable closed and discrete set (see [[5], Example 3.3.6]). For $(x, x) \in A$, $U_x = X \times \{x\}$ is the open subset of $X \times Y$. Then $\mathcal{U} = \{U_x : (x, x) \in \overline{A}\}$ is an open cover of $\overline{A}$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$. Then there exists $(a, a) \in A$ such that $(a, a) \notin \bigcup \mathcal{V}$ and thus $(\bigcup \mathcal{V}) \bigcap U_a = \varnothing$. But U_a is the only element of $\mathcal{U}$ containing (a, a). Thus $(a, a) \notin \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$, which completes the proof. □

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