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A Kuratowski-Mrówka type characterization of fibrewise compactness

Clara M. Neira U.

Abstract

In this paper a Kuratowski-Mrówka type characterization of fibrewise compact topological spaces is presented.

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1. Introduction

Inspired by the concept that the objective of General Topology is the study of continuous functions, a branch of General Topology, known as Continuous Functions Topology or Fibrewise Topology, was originated. To a great extend, the research in this field has been directed to generalize to fibrewise topological spaces, notions and results classically studied in General Topology. Fibrewise versions of Hausdorffness, compactness, convergence, connectedness, uniform structures, and homotopy theory have been studied using the notions of tied filter, and various other tools (cf. [3, 9, 11]).

The Kuratowski-Mrówka characterization of compact spaces, as those spaces X satisfying the condition that the second projection $\pi_2 : X \times Y \longrightarrow Y$ is a closed map, for each space Y, gave rise to a categorical approach of compactness (cf. [5], [6], [8] and [10], among others).

In its turn, this characterization has become a useful tool in General Topology by giving alternative proofs of classic results on compactness that enhance the understanding of some aspects of General Topology. It is worth mentioning the astounding simplicity of the proof of Tychonoff Theorem obtained, in the finite case, via this characterization.

In this work we generalize to fibrewise topological spaces the Kuratowski-Mrówka characterization of compactness.

2. Preliminaries

In this work we often resort to the following characterization of a closed map.

Let X and Y be two topological spaces. A function $f : X \longrightarrow Y$ is closed, if and only if, for each $y \in Y$ and each open neighborhood O of the fiber $X_y = f^{-1}(y)$ in X, there exists a neighborhood W of y in Y such that $X_W = \{x \in X : f(x) \in W\} \subset O$ (cf. [9], Proposition 1.8, p 7).

A *fibrewise topological space* is by definition a triplet (E, p, T), where E and T are topological spaces and $p : E \longrightarrow T$ is a continuous function. Let (E, p, T) be a fibrewise topological space. A filter $\mathcal{F}$ over E is a *tied filter* to a point $t \in T$ or a *t-filter* if the filter $p(\mathcal{F})$ generated over T by the filter base $\{p(F) : F \in \mathcal{F}\}$ converges to the point t. A *tied ultrafilter* to the point t or a t-ultrafilter, is a maximal t-filter.

Following I. M. James, we adopt the next definition.

Definition 2.1 ([9])**.** A fibrewise topological space (E, p, T) is *fibrewise compact*, if p is a proper map.

Remark 2.2 ([2])**.** Recall that a continuous map $p : E \longrightarrow T$ is proper if for every topological space Z, the map $p \times id_Z : E \times Z \longrightarrow T \times Z$ is closed, or equivalently, if p is a closed map and each fiber is compact.

The next characterization of a fibrewise compact topological space is quite useful in what follows.

Proposition 2.3 ([9])**.** *A fibrewise topological space (E, p, T) is fibrewise compact, if and only if, for each $t \in T$ and each covering $\mathcal{O}$ of E_t by open subsets of E, there exist a neighborhood W of t and a finite subfamily of $\mathcal{O}$ that covers E_W.*

Proposition 2.4. *Let (E, p, T) be a fibrewise topological space. The following assertions are equivalent:*

1. *(E, p, T) is fibrewise compact.*
2. *Each filter over E tied to a point $t \in T$ has a cluster point in E_t (cf. [11]).*
3. *Each ultrafilter over E tied to a point $t \in T$ converges to a point of E_t.*

Proof.

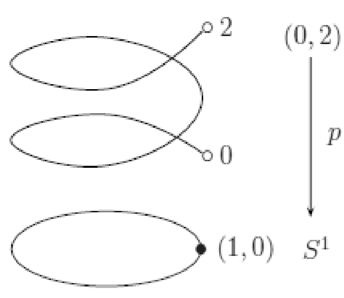
(1) $\Longrightarrow$ (2): Suppose that the t-filter $\mathcal{F}$ over E has no cluster points. Then for each $x \in E_t$, there exist a neighborhood O_x of x in E and an element F_x of $\mathcal{F}$ such that $O_x \cap F_x = \varnothing$. Since E_t is compact, there exist points $x_1, ..., x_n \in E_t$ satisfying $E_t \subset \bigcup_{i=1,\ldots,n} O_{x_i}$ and since p is closed, there exists an open neighborhood W of t in T, such that $E_W \subset \bigcup_{i=1,\ldots,n} O_{x_i}$. Let $F = \bigcap_{i=1,\ldots,n} F_{x_i}$. If $s \in p(F) \cap W$, there exists $a \in F \cap E_W$, such that $p(a) = s$. Then $a \in O_{x_i} \cap F_{x_i}$, for some $i \in \{1, ..., n\}$. This is a contradiction, hence $p(F) \cap W = \varnothing$ and $\mathcal{F}$ is not a filter tied to t. It follows that the t-filter $\mathcal{F}$ has at least one cluster point.

(2) $\Longrightarrow$ (3): Suppose that $\mathcal{U}$ is a t-ultrafilter over E and that $x \in E_t$ is a cluster point of $\mathcal{U}$. If $O \in \mathcal{V}(x)$, then $O \in \mathcal{U}$, otherwise, $\{O \cap U : U \in \mathcal{U}\}$ would generate a t-filter over E finer than $\mathcal{U}$.

(3) $\Longrightarrow$ (1): Let $t \in T$ and $\mathcal{O}$ be a covering of E_t by open subsets of E. Suppose that for each open neighborhood W of t and each finite sub-collection $\mathcal{A}$ of $\mathcal{O}$ one has that $E_W \setminus \bigcup \mathcal{A} \neq \varnothing$. The collection $\{E_W \setminus \bigcup \mathcal{A} : W \text{ is an open neighborhood of } t, \text{ and } \mathcal{A} \subset \mathcal{O} \text{ is finite}\}$ is a base for a t-filter over E which is contained in a t-ultrafilter $\mathcal{U}$ over E that, by hypothesis, converges to a point $x \in E_t$. Now, there exists $O \in \mathcal{O}$ such that $x \in O$, then $O \in \mathcal{U}$, but also $E \setminus O = E_T \setminus O \in \mathcal{U}$, which is absurd. Then there exist an open neighborhood W of t and a finite sub-collection of $\mathcal{O}$ that covers E_W. This means that (E, p, T) is fibrewise compact.

□

Example 2.5. *The triplet (E, p, S^1), where E is open interval $(0, 2)$ of $\mathbb{R}$ and $p : (0, 2) \to S^1$ is defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a sheaf of sets in which every fiber is a finite set and consequently compact.*



Let $\mathcal{F}$ be the filter over E generated by the collection of intervals $\{(0, \epsilon) : \epsilon > 0\}$. Then $\mathcal{F}$ is a filter tied to the point $(1, 0)$ of S^1 that has no cluster points in the fiber over $(0, 1)$. It follows that (E, p, T) is not fibrewise compact.

3. The Kuratowski-Mrówka characterization of fibrewise compactness

We begin the main section of this paper with the following observation.

Remark 3.1. Every filter $\mathcal{F}$ over a set X determines a topology $\mathfrak{T}_{\mathcal{F}}$ over the set $X \bigcup\{\omega\}$, where $\omega \notin X$, as follows: if $x \neq \omega$, the neighborhood filter of x is $\mathcal{V}(x) = \{V \subset X \bigcup\{\omega\} : x \in V\}$ and the neighborhood filter of ω is $\mathcal{V}(\omega) = \{F \bigcup\{\omega\} : F \in \mathcal{F}\}$. We denote by $X_{\mathcal{F}}$ the topological space $(X \bigcup\{\omega\}, \mathfrak{T}_{\mathcal{F}})$ (cf. [2]).

Let (E, p, T) be a fibrewise topological space and $\mathcal{F}$ be a filter over E tied to a point $t \in T$. The function $p_{\mathcal{F}} : E_{\mathcal{F}} \longrightarrow T$ defined by

$$p_{\mathcal{F}}(x) = \begin{cases} p(x) & \text{if } x \in E \\ t & \text{if } x = \omega \end{cases}$$

is continuous. That is, $(E_{\mathcal{F}}, p_{\mathcal{F}}, T)$ is a fibrewise topological space.

To show this, it suffices to verify the continuity of $p_{\mathcal{F}}$ at ω. Let W be an open neighborhood of t in T. Since $\mathcal{F}$ is a filter tied to t, one has that $W \in p(\mathcal{F})$. Then there exists $F \in \mathcal{F}$ such that $p(F) \subset W$, hence $p_{\mathcal{F}}(F \cup \{\omega\}) \subset W$. This completes the proof.

Let (E, p_E, T) and (F, p_F, T) be two fibrewise topological spaces. The fiber product $E \vee F$ of E with F is the set $E \vee F = \{(x, y) \in E \times F : p(x) = q(y)\}$. Consider $E \vee F$ with the topology induced by the product topology on $E \times F$. The triplet $(E \vee F, p, T)$, where $p : E \vee F \longrightarrow T$ is defined by $p(x, y) = p_E(x)$, is a fibrewise topological space. Furthermore, $(E \vee F, p, T)$ is the product of (E, p_E, T) and (F, p_F, T) in the category of fibrewise topological spaces and fibrewise continuous functions, that is, those continuous functions $\varphi : E \longrightarrow F$ satisfying $p_F \circ \varphi = p_E$.

Theorem 3.2 (Kuratowski-Mrówka characterization)**.** *The fibrewise topological space (E, p, T) is fibrewise compact, if and only if, for each fibrewise topological space (F, q, T) the projection $\pi_2 : E \vee F \longrightarrow F$ is a closed map.*

Proof.

- $\Rightarrow$ Suppose that (E, p, T) is a fibrewise compact fibrewise topological space and that (F, q, T) is an arbitrary fibrewise topological space. Let $b \in F$, $q(b) = t$, and O be an open neighborhood of $\pi_2^{-1}(b) = E_t \times \{b\}$ in $E \vee F$. For each $x \in E_t$ there exist a neighborhood A_x of x in E and a neighborhood M_x of b in F such that $A_x \vee M_x \subset O$. Compactness of E_t guarantees the existence of $x_1, ..., x_n \in E_t$, such that $E_t \subset \bigcup_{i=1}^{n} A_{x_i}$. Since p is closed, there exists an open neighborhood W of t in T such that $p^{-1}(W) \subset \bigcup_{i=1}^{n} A_{x_i}$. Let $M = \left(\bigcap_{i=1}^{n} M_{x_i}\right) \bigcap q^{-1}(W)$. If $(y, a) \in \pi_2^{-1}(M)$, then $p(y) = q(a) \in W$, hence $y \in A_{x_i}$, for some $i \in \{1, ..., n\}$. Then $(y, a) \in A_{x_i} \vee M_{x_i} \subset \pi_2^{-1}(b)$. This proves that π_2 is a closed map.

$\Leftarrow$ Suppose that $\mathcal{F}$ is a filter over E tied to the point $t \in T$ and suppose that $\mathcal{F}$ has no cluster points, then for each $x \in E_t$ there exists an open neighborhood O_x of x in E and an element $F_x \in \mathcal{F}$ such that $O_x \bigcap F_x = \varnothing$. Consider the fibrewise topological space $(E_{\mathcal{F}}, p_{\mathcal{F}}, T)$ and the set $\Delta_0 = \{(x,x) \in E \vee E_{\mathcal{F}} : \ x \in E\}$. For each $x \in E_t$, the set $O_x \vee (F_x \bigcup \{\omega\})$ is a neighborhood of (x,ω) in $E \vee E_{\mathcal{F}}$ such that $O_x \vee (F_x \bigcup \{\omega\}) \bigcap \Delta_0 = \varnothing$, then $(x,\omega) \notin \overline{\Delta_0}$ for each $x \in E_t$. This implies that $\pi_2(\overline{\Delta_0}) = E$ and since E is not a closed subset of $E_{\mathcal{F}}$, because $\omega \in \overline{E}$, it follows that $\pi_2 : E \vee E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}}$ is not a closed map.

□

Example 3.3. *Every topological space X can be identified with the fibrewise topological space (X, p, T), where T consists of a single point and p is the constant map from E to T. The Kuratowski-Mrówka characterization of the fibrewise compact fibrewise topological spaces asserts that (X, p, T) is fibrewise compact if and only if $\pi_2 : X \vee Y \longrightarrow Y$ is closed, for each fibrewise topological space (Y, q, T). Unfolding this assertion one finds that every fibrewise topological space (Y, q, T) can be identified with the topological space Y: the map q is necessarily the constant map from Y to T. Furthermore, $X \vee Y = \{(x,y) \in X \times Y : p(x) = q(y)\} = X \times Y$. Then*

> *"A topological space X is compact if and only if, for each topological space Y, each $y \in Y$ and each open neighborhood O of $X \times \{y\}$ in $X \times Y$, there exists an open neighborhood N of y in Y, such that $X \times N \subset O$."*

This result is known in General Topology as the Tube's Lemma.

The second part of the proof of Theorem 3.2 implies the following result.

Corollary 3.4. *If (E, p, T) is a fibrewise topological space such that the projection $\pi_2 : E \vee E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}}$ is closed for every t-filter $\mathcal{F}$ over E, then (E, p, T) is fibrewise compact.*

Example 3.5. *Let (E, p, T) be a covering space. Since each fiber has the discrete topology, for (E, p, T) to be fibrewise compact it is necessary, for the fibers, to be finite.*

Conversely, suppose that (E, p, T) is a covering space in which every fiber has a finite number of elements. Let $\mathcal{F}$ be a filter tied to the point $t \in T$ and let $E_t = \{x_1, .., \ x_n\}$. Consider an open neighborhood W of t regularly covered by p and let $\{O_i\}_{i=1,...,\ n}$ be a partition in slices of E_W with $x_i \in O_i$, for each $i = 1, ..., \ n$.

To guarantee that the function $\pi_2 : E \vee E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}}$ is closed, consider $\zeta \in E_{\mathcal{F}}$ and an open neighborhood O of $\pi_2^{-1}(\zeta)$. If $\zeta \in E$, $V = \{\zeta\}$ is an open neighborhood of ζ such that $\pi_2^{-1}(V) \subset O$.

Suppose that $\zeta = \omega$. *For each* $i = 1, ..., n$, *there exist an open neighborhood* A_i *of* x_i *in* E *and* $F_i \in \mathcal{F}$ *in such a way that* $A_i \vee (F \cup \{\omega\}) \subset O$. *Let* $V = \bigcap_{i=1}^{n} p(A_i)$. *Since* $V \in p(\mathcal{F})$, *there exists* $F' \in \mathcal{F}$ *such that* $p(F') \subset V$. *Consider* $F = F' \cap F_1 \cap ... \cap F_n$. *If* $(x, y) \in \pi_2^{-1}(F \cup \{\omega\})$ *and* $y \neq \omega$, *then* $p(x) = p(y) \in V$, *therefore* $x \in A_i$, *for some* $i = 1, ..., n$. *Hence* $(x, y) \in A_i \vee (F_i \cup \{\omega\}) \subset O$. *This shows that que* $\pi_2 : E \vee E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}}$ *is a closed map.*

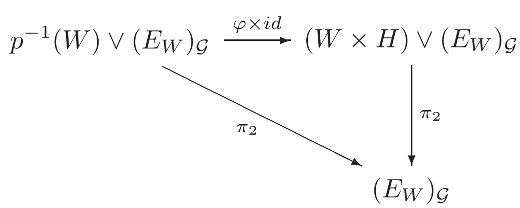
Example 3.6. *If* (E, p, T) *is a fiber bundle with fiber* H *and if* H *is compact, then* (E, p, T) *is fibrewise compact. In fact, let* $\mathcal{F}$ *be a filter tied to the point* $t \in T$. *There exist an open neighborhood* W *of* t *and a homeomorphism* $\varphi : p^{-1}(W) \longrightarrow W \times H$ *such that* $\pi_1 \varphi = p$.

To prove that the function $\pi_2 : E \vee E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}}$ *is closed, consider the following facts.*

(1) *Since* $\mathcal{F}$ *is a filter tied to* t, *then* $p^{-1}(W) \in \mathcal{F}$. *Therefore, the collection* $\mathcal{G} = \{F \in \mathcal{F} : F \subset p^{-1}(W)\}$ *is a filter over* E_W *tied to the point* t. *Here one is considering* $p^{-1}(W)$ *as a fibrewise topological space over* W. *Furthermore,* $(E_W)_{\mathcal{G}}$ *is a subspace of* $E_{\mathcal{F}}$.

(2) *Since* H *is compact, the Kuratowski-Mrówka characterization of compact topological spaces guarantees that* $W \times H$, *seen as a fibrewise topological space over* W, *is fibrewise compact.*

(3) *The commutativity of the diagram*



secures that the second projection from $p^{-1}(W) \vee (E_W)_{\mathcal{G}}$ *to* $(E_W)_{\mathcal{G}}$ *is closed.*

Let O *be an open neighborhood of* $\pi_2^{-1}(\omega)$ *in* $E \vee E_{\mathcal{F}}$. *Then* $O \cap (p^{-1}(W) \vee (E_W)_{\mathcal{G}})$ *is a neighborhood of* $\pi_2^{-1}(\omega)$ *in* $p^{-1}(W) \vee (E_W)_{\mathcal{G}}$, *hence there exists* $G \in \mathcal{G}$ *such that* $\pi_2^{-1}(G \cup \{\omega\}) \subset O \cap (p^{-1}(W) \vee (E_W)_{\mathcal{G}})$. *Since* $G \in \mathcal{F}$, *this completes the proof.*

Remark 3.7. If (E, p, T) is a fibrewise topological space and $\mathcal{U}$ is a t-ultrafilter over E that does not converge, then $\mathcal{U}$ has no cluster points. Again, by the second part of the proof of the previous theorem, it follows that $\pi_2 : E \vee E_{\mathcal{U}} \longrightarrow E_{\mathcal{U}}$ is not a closed map.

The last observation implies the following corollary.

Corollary 3.8. *If* (E, p, T) *is a fibrewise topological space such that the map* $\pi_2 : E \vee E_{\mathcal{U}} \longrightarrow E_{\mathcal{U}}$ *is closed for every* t*-ultrafilter* $\mathcal{U}$ *over* E*, then* (E, p, T) *is fibrewise compact.*

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Clara M. Neira U. (cmneirau@unal.edu.co)
Universidad Nacional de Colombia, Facultad de Ciencias, Departamento de Matemáticas, Carrera 30 # 45–03, Bogotá, Colombia