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p-Compact, p-Bounded and p-Complete

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Abstract

In this paper the nonstandard theory of uniform topological spaces is applied with two main objectives: (1) to give a nonstandard treatment of Bernstein's concept of p-compactness with additional results, (2) to introduce three new concepts (p,q)-compactness, p-totally boundedness and p-completeness. I prove some facts about them and how these three concepts are related with p-compactness. I also give a partial answer to the open question stated in [3]

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1. Introduction

It was in 1966 that Abraham Robinson's book [4] on nonstandard analysis appeared. His methods were based on the theory of models and in particular on the Lowenheim-Skolem theorem. He introduced extension $^*X$ (in a superstructure no standard) of any set X (in a standard superstructure) by looking at nonstandard models of their respectives theories. "Infinitely close" or "infinitely large" were to be found in the enlargement $^*X$. In this way he was able to justify proofs using "infinitesimals" or "monads" and that was not possible before his discovery, more over, he showed that these methods were able to produce original solutions to unsolved mathematical questions as well.

This approach to the nonstandard analysis is based on the axiomatic set theory called ZFC (Zermelo-Fraenkel with Axiom Choice), all theorems of conventional mathematics remain valid. We start with a superstructure S, the

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sets A, B, X , etc. are set of individuals in S. The extensions or enlargements $^*A$, $^*B$, $^*X$, etc. are sets in the superstructure $\hat{S}$.

Nonstandard analysis is a technique rather than a subject. Aside from theorems that tell us that nonstandard notions are equivalent to corresponding standard notions, all the results we obtain can be proved by standard methods. Therefore, the subject can only be claimed to be of importance insofar as it leads to simpler, more accesible expositions, or, more important, to mathematical discoveries. In writing formulas we use conventional symbols such that $\in, =, \forall, \exists, \Rightarrow, \vee, \wedge, \sim, ()$ etc.

2. Some Notions of Nonstandard Theory

The nonstandard enlargements satisfies the following properties:

$A \subset {}^*A$, ${}^*(A \cap B) = {}^*A \cap {}^*B$, ${}^*(A \cup B) = {}^*A \cup {}^*B$, $A \subset B \Rightarrow {}^*A \subset {}^*B$, ${}^*(A \times B) = {}^*A \times {}^*B$, ${}^*(A \setminus B) = {}^*A \setminus {}^*B$,

$A = {}^*A \Leftrightarrow A$ is a finite set. Therefore, ${}^*N \setminus N$, ${}^*R \setminus R$ are non empty sets whose elements are called "infinite large numbers".

In general, for any infinite set X, ${}^*X \setminus X \neq \phi$

Each function $f : A \to B$ has an "extension" ${}^*f : {}^*A \to {}^*B$. For this we observe that $f \subset A \times B$, so that, ${}^*f \subset {}^*A \times {}^*B$

Given a topological space (X, t) and $z \in {}^*X$, the monad of z is

$$\mu(z) = \bigcap \{ {}^*O \,|\, O \in t \text{ } and \text{ } z \in {}^*O \}$$

The space (X, t) is Hausdorff $\Leftrightarrow \forall\, x, y \in X$, $\mu(x) \cap \mu(y) = \phi$

We denote by $\widehat{X} = \bigcup \{ \mu(x) \,|\, x \in X \}$

Let L denote a language in the standard superstructure and by $\hat{L}$ we denote de corresponding language in the nonstandard superstructure. A formula of L in which no variable has a free occurrence is called a sentence of L. For each sentence $\alpha \in L$ we have correspondent sentence ${}^*\alpha \in \hat{L}$.

$\alpha = {}^*\alpha \Leftrightarrow \alpha$ contains only constants $b \in S$

Let α be a sentence of L, we will denote $\models \alpha$ if " α is true in S"; we will denote $* \models {}^*\alpha$ if " ${}^*\alpha$ is true in $\hat{S}$"

Transfer Principle (TP): The admisible proposition $\alpha \in (\hat{S},\, \pounds)$ is true $\Leftrightarrow$ ${}^*\alpha \in ({}^*\hat{S},\, {}^*\pounds)$ is true.

$$\models \alpha \Leftrightarrow * \models *\alpha$$

For instance, $A \cap B \neq \phi \Leftrightarrow {}^*A \cap {}^*B \neq \phi$, $A \subset B \Leftrightarrow {}^*A \subset {}^*B$, etc.

The TP provides one of the basic tools of nonstandard analysis. A mathematical theorem that is equivalent to $\models \alpha$ for some sentence $\alpha \in L$ can be proved by showing instead that $* \models *\alpha$ [1].

A relation $R \subset A \times B$ is called concurrent if given $a_1, a_2, ..., a_n \in dom(R)$ there exists $b \in B$ such that $(a_i, b) \in R$ for all $i = 1, 2, ..., n$

Concurrence Theorem (CT): If R is a concurrent relation then there exists $b \in {}^*B$ such that $({}^*a, b) \in {}^*R$ for all $a \in A$

We only will consider Hausdorff uniform topological spaces (X, t), with uniformity U. Then, on ${}^*X$ we will consider three non-Hausdorff topologies:

a) τ_U will be the topology generated by the sets ${}^*O$ such that

$$O \in t.$$

b) $\hat{\tau}_U$ will be the uniform topology on ${}^*X$ given by the

$$\text{uniformity } \widehat{U} = \{{}^*V \mid V \in U\}.$$

c) τ_i the topology generated by $\{{}^*A \mid A \subset X\}$.

A basis of τ_U is $\{{}^*V(x) \mid x \in X,\ V \in U\}$.

A basis of $\hat{\tau}_U$ is $\{{}^*V(z) \mid z \in {}^*X,\ V \in U\}$.

For any $z \in {}^*X$, $\mu(z)$, $o(z)$ and $i(z)$ will denote the monads of z for the topologies a), b) and c) respectively.

τ_U and $\hat{\tau}_U$ agree on X, thereby $\mu(x) = o(x)\ \forall x \in X$, in general $\tau_U < \hat{\tau}_U$. On the other hand, $i(x) = \{x\}\ \forall x \in X$ and $\tau_U < \tau_i$.

So that, $i(z) \subset \mu(z)$ and $o(z) \subset \mu(z)$. The bigger the topology the smaller the monad.

If $f : (X, t) \to (Y, t')$ is a continuous function then

$${}^*f : ({}^*X, \tau_U) \to ({}^*Y, \tau'_U) \text{ is a continuous function.}$$

Any function ${}^*f : ({}^*X, \tau_i) \to ({}^*Y, \tau'_i)$ is continuous.

The topology on the set of natural numbers N always be the discrete topology.

For any $p \in {}^*N \setminus N$ and any topological space X

$$\beta'_p X = \bigcup \{\mu({}^*f(p)) \mid f : N \to X\}.$$

3. p-Compactness

Definition 3.1. Let $p \in {}^*N \setminus N$ be and $\{z_n \mid n \in N\} \subset {}^*X$. We will say that $z \in {}^*X$ is a p-limit of the sequence $\{z_n\}$ ($z = p - \lim z_n$) if z is an accumulation point of the sequence such that for each $O \in t$, with $z \in {}^*O$,

$$p \in {}^*\{n \in N \mid z_n \in {}^*O\}.$$

Lemma 3.2. *If $f : X \to Y$ is a continuous function and $z \in {}^*X$ is a p-lim of $\{z_n\} \subset {}^*X$ then ${}^*f(z) = p - \lim {}^*f(z_n)$ in ${}^*Y$.*

Proof. Let $W \subset Y$ be an open set such that ${}^*f(z) \in {}^*W$ and let $O \subset X$ be open such that $z \in {}^*O$ and ${}^*f({}^*O) \subset {}^*W$. Hence,

$$p \in {}^*\{n \in N \mid z_n \in {}^*O\} \subset \{n \in N \mid {}^*f(z_n) \in {}^*W\}.$$

□

Theorem 3.3. $z = p - \lim x_n \Leftrightarrow x_p \in \mu(z)$.

Proof. $\Rightarrow$) Let $O \in t$ such that $z \in {}^*O$, by definition $x_p \in {}^*O$.

$\Leftarrow$) Let $O \in t$ be such that $z \in {}^*O$. The following sentence is true in the nonstandard structure $(\exists\, n \in {}^*N)(x_n \in {}^*O)$, by the PT $(\exists\, n \in N)(x_n \in O)$ is true, i.e., $A = \{n \in N \mid x_n \in O\} \neq \varnothing$ y, $p \in {}^*A$. □

Proposition 3.4. *Let $p, q \in {}^*N \setminus N$. If there exists $f : N \to N$ such that $p = q - \lim\{f(n)\}$ then ${}^*f(q) \in \beta'_p N$.*

Proof. $p \in {}^*O \Rightarrow {}^*f(q) \in {}^*O \Rightarrow \mu({}^*f(q)) \subset \mu(p)$. □

Proposition 3.5. *If for $\{x_n\} \subset X$ there exists $\{x_{n_k}\} \to x \in X$ such that $p \in {}^*\{n_k\}$, then $x = p - \lim x_n$.*

Definition 3.6 ([3])**.** A topological space (X, t) is p-compact if every sequence in X has a p-limit in ${}^*X$

$$\widehat{X} = \bigcup\{\mu(x) \mid x \in X\}.$$

$X \subset \widehat{X} \subset \beta'_p X$ taking for each $x \in X\, f : N \to X,\ f(n) = x$

Theorem 3.7. *(X, t) is p-compact $\Leftrightarrow \widehat{X} = \beta'_p X$.*

Proof. $\Rightarrow$) The contention left to proof is $\hat{X} \supset \beta'_p X$

Let $\{x_n\} \subset X$ be such that $q \in \mu(x_p)$. Since $x_p \in \mu(x)$ for some $x \in X$, this implies that $q \in \mu(x_p) \subset \mu(x)$

$\Leftarrow$) Nothing left to prove. □

Corollary 3.8. *(X, t) compact $\Rightarrow$ p-compact for all $p \in {}^*N \setminus N$.*

Proof. $\widehat{X} \subset \beta'_p X \subset {}^*X = \widehat{X}$. □

Definition 3.9. Let t_p be the topology generated by the set $\{O \in t \mid {}^*f(p) \notin {}^*O$ for some $f : N \to X\}$ t_p is closed under intersections.

Lemma 3.10. *X t_p-compact implies p-compact.*

Proposition 3.11. *(X, t) p-compact implies for each numerable t_p-cover of X has a finite sub-cover.*

Corollary 3.12. *If X is p-compact and t_p-Lindelof then it is t_p-compact.*

Definition 3.13 (Comfort Order)**.** $p \leq_C q$ (in $*N \setminus N$)if (X, t) q-compact implies p-compact [2].

Corollary 3.14. *If $\beta'_p X \subset \beta'_q X$ then X q-compact implies p-compact.*

4. (p,q)-Compactness

Definition 4.1. $p \otimes q = \{A \subset N \times N \mid (p,q) \in {}^*A\}$.

Definition 4.2. A topological space (X,t) is (p,q)-compact if for each $f : N \times N \to X$ there exists $x \in X$ such that for every $x \in O \in t$,

$$^*f(p,q) \in \mu(x) \text{ y } (p,q) \in {}^*\{(m,n) \mid f(m,n) \in O\}.$$

Theorem 4.3. *(X,t) is (p,q)-compact if and only if it is p and q compact.*

Proof. $\Rightarrow$) Let us prove that X is p-compact. Let $f : N \to X$ and $\pi_1 : N \times N \to N$ be the projection in the first coordinate, then $f \circ \pi_1 : N \times N \to X$: By hypothesis there exists $x \in X$ such that $^*f(p) = ({}^*f \circ {}^*\pi_1)(p,q) = {}^*(f \circ \pi_1)(p,q) \in \mu(x)$ and for each $x \in O \in t$

$$(p,q) \in {}^*\{(m,n) \mid f(m) = (f \circ \pi_1)(m,n) \in O\} \text{ i.e.,}$$

$$\{(m,n) \mid f(m) = (f \circ \pi_1)(m,n) \in O\} \in p \otimes q\,.$$

This implies that

$$p = {}^*\pi_1(p,q) \in {}^*\pi_1^*(\{(m,n) \mid f(m) = (f \circ \pi_1)(m,n) \in O\}) = \\ =^* \{m \in N \mid f(m) \in O\}.$$

Analogous proof for $(f \circ \pi_2) : N \times N \to X$.
Since X is q-compact, there exists $y \in X$ such that

$$^*f(q) = {}^*(f \circ \pi_2)(p,q) \in \mu(y) \text{ and for each } y \in U \in t\,, \\ (p,q) \in {}^*\{(m,n) \mid (f \circ \pi_2)(m,n) \in U\}\,, \text{ i.e.,} \\ \{(m,n) \mid f(n) = (f \circ \pi_2)(m,n) \in U\} \in p \otimes q\,.$$

This implies that

$$q = {}^*\pi_2(p,q) \in {}^*\pi_2({}^*\{(m,n) \mid f(n) = (f \circ \pi_2)(m,n) \in U\}) \\ \in {}^*\{n \in N \mid f(n) \in U\}.$$

$\Leftarrow$) Let us consider $f : N \times N \to X$. To prove that there exists $x \in X$ such that $^*f(p,q) \in \mu(x)$. For each $n \in N$ we define $f_n : N \to X$ as $f_n(m) = f(m,n)$. By hypothesis, there exists $x_n \in X$ such that $^*f(p,n) = {}^*f_n(p) \in \mu(x_n)$ and for each $x_n \in O \in t$, $p \in {}^*\{m \in N \mid f_n(m) \in O\}$.

Let us define $g_p : N \to X$ as $g_p(n) = x_n \sim {}^*f(p,n)$. Hence ([1, page 81]) $g_p \sim {}^*f(p,-) \in {}^*(\prod_N X)$. The set $\{{}^*W \mid W$ is an open basic set of $\prod_N X\}$ is a basis of the topology on $^*(\prod_N X)$.

By the q-compacity of X, there exists $y \in X$ such that $^*g_p(q) \in \mu(y)$ and for each $y \in U \in t$, $q \in {}^*A = {}^*\{n \in N \mid g_p(n) \in U\}$.

Since $g_p(n) \sim {}^*f(p,n)$, $^*f(p,n) \in {}^*U \; \forall n \in A$, i.e., the following formula is true $(\forall n \in N)(n \in A \Rightarrow {}^*f(p,n) \in {}^*U)$. By the PT, the following formula is true $(\forall n \in {}^*N)(n \in {}^*A \Rightarrow {}^*f(p,n) \in {}^*U)$.

$$q \in {}^*A \Rightarrow {}^*f(p,q) \in {}^*U \Rightarrow {}^*f(p,q) \in \bigcap\{{}^*U \mid y \in U \in t\} = \mu(y).$$

□

5. P-TOTALLY BOUNDED AND P-COMPLETE

Definition 5.1. All uniform space (X,t) is completely regular, let us denote by $\{\rho_j\}_J$ the saturated family of pseudometrics that define the topology t given by the uniformity U. We recall that for each $z \in {}^*X$

$$o(z) = \{y \in {}^*X \mid \rho_j(y,z) \approx 0 \;\forall\, j \in J\} \qquad [1, 5]$$

$$\mu(z) = \bigcap \{{}^*V(x) \mid V \in U,\, x \in X,\, z \in {}^*V(x)\}$$

Every $V \in U$ is $V = \{(x,y) \in X \times X \mid \rho_j(x,y) < \epsilon\}$ for some $\epsilon > 0$ and some pseudometric ρ_j, also denoted by $V = V_j^\epsilon$.

For all $z \in {}^*X$, $\mu(z) \supset o(z)$ and for all $x \in X$, $\mu(x) = o(x)$

For (X,t) we define the set

$$pns\,{}^*X = \{z \in {}^*X \mid \mu(z) = o(z)\}.$$

Proposition 5.2. $z \in pns\,{}^*X \Rightarrow \mu(z) \subset pns\,{}^*X$.

Proof. If $z \in {}^*X$ and $w \in \mu(z)$ then $o(w) \subset \mu(w) \subset \mu(z)$ and $o(w) = o(z)$ If $z \in pns\,{}^*X$ then $\mu(z) = o(z) = o(w) \subset \mu(w) \subset \mu(z)$, i.e., $o(w) = \mu(w)$ thereby $w \in pns\,{}^*X$. □

Proposition 5.3. $z \in pns\,{}^*X \Leftrightarrow$ *given* $\epsilon > 0$ *and any pseudometric* ρ_j *there exists* $x_j \in X$ *such that* ${}^*\rho_j(z, x_j) < \epsilon$.

Proof. $\Rightarrow$) Let $z \in pns\,{}^*X$ be given, then, $o(z) = \mu(z)$. For any $\epsilon > 0$, a pseudometric ρ_j and $V = V_j^\epsilon \in U$. Since $o(z) \times o(z) \subset V$, $\mu(z) \times \mu(z) \subset V$, PC tells us that there exists $D \in {}^*F_{\mu(z)}$ such that $D \subset \mu(z)$, there exists $D \in {}^*F_{\mu(z)}$ such that $D \times D \subset {}^*V$, hence, by PT, there exists $D \in F_{\mu(z)}$ such that $D \times D \subset V$.

$D \in F_{\mu(z)} \Rightarrow \mu(z) \subset {}^*D \Rightarrow \{z\} \times {}^*D \subset {}^*V \Rightarrow (z,x) \in {}^*V$ for all $x \in D \subset X$, i.e., $\rho(z,x) < \epsilon$ for all $x \in D$

$\Leftarrow$) Let $w \in \mu(z)$, $\epsilon > 0$ be given and $(z, x_j) \in {}^*V_j^{\frac{\epsilon}{2}} = \{(x,y) \mid \rho_j(x,y) < \frac{\epsilon}{2}\}$. Since $w \in {}^*V_j^{\frac{\epsilon}{2}}(x_j)$ and ${}^*\rho_j(w,z) \le {}^*\rho_j(w,x_j) + {}^*\rho_j(x_j,z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This implies that $w \in {}^*V_j^\epsilon(z)$ and therefore $w \in o(z)$, i.e., $\mu(z) \subset o(z) \subset \mu(z)$. □

Corollary 5.4. $z \in pns\,{}^*X \Leftrightarrow F_{\mu(z)}$ *is a Cauchy filter*

Remark 5.5. $\hat{X} \subset pns\,{}^*X \subset {}^*X$ $\quad \hat{X} = pns\,{}^*X \Leftrightarrow X$ is complete $\quad pns\,{}^*X = {}^*X \Leftrightarrow X$ is totally bounded

Definition 5.6. (X,t) is p-totally bounded if for each function $f : N \to X$, ${}^*f(p) \in pns\,{}^*X$

Lemma 5.7. *p-compact* $\Rightarrow$ *p-totally bounded.*

Proof. $\widehat{X} \subset pns\,{}^*X \subset {}^*X$. □

Lemma 5.8. X *is p-totally bounded* $\Leftrightarrow$ $\beta'_p X \subset pns\,{}^*X$.

Proof. $\Rightarrow$) X p-totally bounded $\Rightarrow$ ${}^*f(p) \in pns\,{}^*X \Rightarrow \mu({}^*f(p)) \subset pns\,{}^*X$. □

Corollary 5.9. X *is p-totally bounded for all* $p\,{}^*N \setminus N$ $\Leftrightarrow \cup_p \beta'_p X \subset pns\,{}^*X$.

Lemma 5.10. (X,t) *totally bounded* $\Rightarrow$ *p-totally bounded for all* $p\,{}^*N \setminus N$.

Proof. $\Rightarrow$) Totally bounded implies $pns\,{}^*X = {}^*X$. □

Lemma 5.11. *If* (X,t) *is a complete space then* X *p-compact* $\Leftrightarrow$ *p-totally bounded.*

Proof. Complete implies $\widehat{X} = pns\,{}^*X$. □

${}^*f : {}^*N \to {}^*X$ always is a τ_U-continuous function.

Proposition 5.12. X *is p-totally bounded* $\Leftrightarrow$ *every* $f : N \to X$ *is a continuous function in* p *with the* $\hat{\tau}_U$ *topology.*

Proof. $\Rightarrow$) Since ${}^*f(p) \in pns\,{}^*X$, $\mu({}^*f(p)) = o({}^*f(p))$, since ${}^*f$ is a continuous function with τ_U, ${}^*f(o(p)) = {}^*f(\mu(p)) \subset \mu({}^*f(p)) = o({}^*f(p))$ then it also is a continuous function in p with $\hat{\tau}_U$ (Cauchy's Principle, Theorem 8.1.4, [5])

$\Leftarrow$) Let $f : N \to X$ be a continuous function in $p \in N$ with τ_U.

To prove that ${}^*f(p) \in pns\,{}^*X$, that is, $\mu({}^*f(p)) = o({}^*f(p))$, for this only left to prove that $\mu({}^*f(p)) \subset o({}^*f(p))$

We recall that $F_p = \{A \subset N \mid p \in {}^*A\} \subset P(N)$ is a Cauchy filter and so is the filter G generated by the image of $f(F_p)$.

Claim: $\mu(G) = \mu({}^*f(p))$.

Since ${}^*f(p) \in \mu(G)$, $\mu(G) \subset \mu({}^*f(p))$.

On the other hand, ${}^*f(p) \in {}^*S \Rightarrow f^{-1}(S) \in F_p$ thereby $S \supset f(f^{-1}(S)) \in G$ therefore $S \in G$.

Let $V \in U$ and $A \in F_p$ be given such that ${}^*f({}^*A) \subset {}^*V(p)$ ($\hat{\tau}_U$-continuity of ${}^*f$ in p). This tells us that $\mu(G) \subset o({}^*f(p))$, therefore

$$\mu({}^*f(p)) \subset o({}^*f(p)).$$

□

Corollary 5.13. ${}^*f(p) \in pns\,{}^*X \Leftrightarrow {}^*f$ *is* $\hat{\tau}_U$ *-continuous in p.*

Proposition 5.14. $pns\,{}^*X$ *is the biggest subset of* ${}^*X$ *containing* X *such that* τ_U *and* $\hat{\tau}_U$ *agree.*

Theorem 5.15. (X,t) *is totally bounded* $\Leftrightarrow \tau_U = \hat{\tau}_{(*)U}$.

Corollary 5.16. *If* $^*X = \cup_p \beta'_p X$ *and* X *is p-totally bounded for all* $p\, ^*N \setminus N$, *then* X *is totally bounded.*

Definition 5.17. (X,t) is p-complete if any function $f : N \to X$, $^*f(p) \in pns\, ^*X \Rightarrow {}^*f(p) \in \widehat{X}$.

Proposition 5.18. *p-compact* $\Leftrightarrow$ *p-complete and p-totally bounded.*

Definition 5.19. $p \leq_{ta} q$ if q-totally bounded $\Rightarrow$ p-totally bounded.
$p \leq_{cc} q$ if q-complete $\Rightarrow$ p-complete.

Questions 2.20 and 2.21

In [3] appeared Q 2.20 and Q 2.21 (open questions) whose nonstandard versions are:

Q 2.20: Is there $p \in^* N \setminus N$ such that $T_C(p) \cap wP = \varnothing$?

$T_C(p) = \{q \in {}^*N \mid q \leq_C p$ and $p \leq_C q\}$

Q 2.21: Is $\beta' N \cap wP \neq$ for all $p \in^* N \setminus N$? or
Is there $p \in^* N \setminus N$ such that $\beta' N \cap wP = \varnothing$?

I will prove that if Q 2.21 is true then Q 2.20 is true.

Definition 5.20. For $q \in^* N \setminus N$,

$$N_q = \{p \in^* N \mid q \notin \beta'_p N\}.$$

Remark 5.21.

1.- $N_p \subset N_q \Rightarrow p \in \beta'_q N$.

2.- For all $f : N \to N$, $^*f(N_q) \subset N_q$.

Definition 5.22. We will say that N_q is p-compact if each function $f : N \to N_q \hookrightarrow {}^*N$, its extension $\hat{f} : {}^*N \to {}^*N$ satisfies $\hat{f}(p) \in N_q$.

Remark 5.23. N_q p-compact $\Rightarrow p \in N_q$.

Definition 5.24. $p \in^* N \setminus N$ is a weak P-point if $p \notin C \subset^* N \setminus N \Rightarrow p \notin \overline{C}$.

We denote the set of weak P-points by wP.

Theorem 5.25. $q \in wP \Rightarrow N_q$ *p-compact for all* $p \in N_q$.

Theorem 5.26. $p \leq_C q$ *and* $p \in wP \Rightarrow N_p \subset N_q \Rightarrow p \in \beta'_q N$.

Proof. If there is some $z \in N_p \setminus N_q$ then both N_p is z-compact and $q \in \beta'_z N$. Hence, there is $f : N \to N$ such that $q \in {}^*A \Leftrightarrow {}^*f(z) \in {}^*A$ for all $A \subset N$ which implies that $\hat{g}(q) \in N_p$ since $\hat{g}({}^*f(z)) \in N_p$ for all $g : N \to N_p$

Therefore N_p is q-compact and, because $p \leq_C q$, N_p is p-compact then (Theorem 5.25) $p \in N_p$. This contradiction tells us that there is not $z \in N_p \setminus N_q$. □

Now we can restate
Q 2.21: Is there $p \in^* N \setminus N$ such that N_q is p-compact for all $q \in wP$? or $\bigcap_{q \in wP} N_q \neq \varnothing$?

If this last inequality is true, that is, if there is $p \in \bigcap_{q \in wP} N_q$ then $q \not\leq_C p \; \forall\, q \in wP$ thereby $T_C(p) \cap wP = \varnothing$ (Q 2.21)

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