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Mappings on weakly Lindelöf and weakly regular-Lindelöf spaces

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Abstract

In this paper we study the effect of mappings and some decompositions of continuity on weakly Lindelöf spaces and weakly regular-Lindelöf spaces. We show that some mappings preserve these topological properties. We also show that the image of a weakly Lindelöf space (resp. weakly regular-Lindelöf space) under an almost continuous mapping is weakly Lindelöf (resp. weakly regular-Lindelöf). Moreover, the image of a weakly regular-Lindelöf space under a precontinuous and contra-continuous mapping is Lindelöf.

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1. Introduction

Among the various covering properties of topological spaces a lot of attention has been made to those covers which involve open and regular open sets. In 1959 Frolik [9] introduced the notion of a weakly Lindelöf space that afterward was studied by several authors. In 1982 Balasubramanian [2] introduced and studied the notion of nearly Lindelöf spaces. In 1984 Willard and Dissanayake [18] gave the notion of almost Lindelöf spaces and in 1996 Cammaroto and Santoro [4] introduced the notion of weakly regular-Lindelöf spaces on using regular covers. Some generalizations of Lindelöf spaces have been recently studied by the authors (see [7]) and by Song and Zhang [17].

Moreover, decompositions of continuity have been recently of major interest among general topologist. They have been studied by many authors, including Singal and Singal [16], Mashhour et al. [11], Abd El-Monsef et al. [1], Nasef and Noiri [12], Noiri and Popa [13], Dontchev [5], Dontchev and Przemski [6], Kohli and Singh [10] and many other topologists.

Throughout the present paper, spaces mean topological spaces on which no separation axioms are assumed unless explicitly stated otherwise. The interior and the closure of any subset A of a space X will be denoted by Int(A) and Cl(A) respectively. By regular open cover of X we mean a cover of X by regular open sets in (X, τ).

The purpose of this paper is to study effect of mappings and decompositions of continuity on weakly Lindelöf and weakly regular-Lindelöf spaces. We also show that some mappings preserve these topological properties. We conclude that the image of a weakly Lindelöf space (resp. weakly regular-Lindelöf space) under an almost continuous mapping is weakly Lindelöf (resp. weakly regular-Lindelöf). Moreover, the image of a weakly regular-Lindelöf space under a precontinuous and contra-continuous mapping is Lindelöf.

2. Preliminaries

Recall that a subset $A \subseteq X$ is called regular open (regular closed) if $A = \text{Int}(\text{Cl}(\text{A}))$ ($A = \text{Cl}(\text{Int}(\text{A}))$). A space (X, τ) is said to be semiregular if the regular open sets form a base for the topology. It is called almost regular if for any regular closed set C and any singleton $\{x\}$ disjoint from C, there exist two disjoint open sets U and V such that $C \subseteq U$ and $x \in V$. Note that a space X is regular if and only if it is semiregular and almost regular [14]. Moreover, a space X is said to be submaximal if every dense subset of X is open in X and it is called extremally disconnected if the closure of each open set of X is open in X. A space X is said to be nearly paracompact [15] if every regular open cover of X admits an open locally finite refinement.

Definition 2.1. Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \longrightarrow Y$ is said to be

(1) almost continuous [16] if $f^{-1}(V)$ is open in X for every regular open set V in Y.
(2) precontinuous [11] $\Big($resp. β-continuous [1]$\Big)$ if $f^{-1}(V) \subseteq \text{Int}(\text{Cl}(f^{-1}(V)))$ $\Big($ resp. $f^{-1}(V) \subseteq \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$ $\Big)$ for every open set V in Y.
(3) almost precontinuous $\Big($ resp. almost β-continuous $\Big)$ [12] if for each $x \in X$ and each regular open set V in Y containing $f(x)$, there exists a set U in X containing x with $U \subseteq \text{Int}(\text{Cl}(U))$ $\Big($ resp. $U \subseteq \text{Cl}(\text{Int}(\text{Cl}(U)))\Big)$ such that $f(U) \subseteq V$.
(4) contra-continuous [5] if $f^{-1}(V)$ is closed in X for every open set V in Y.

Note that almost continuity as well as precontinuity implies almost precontinuity and almost precontinuity as well as β-continuity implies almost β-continuity but the converses, in general, are not true (see [6], [11] and [13]).

Definition 2.2. A topological space X is said to be nearly Lindelöf [2] $\Big($ resp. almost Lindelöf [18] $\Big)$ if, for every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{Int}(\mathrm{Cl}(\mathrm{U}_{\alpha_n}))$ $\Big($ resp. $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}(\mathrm{U}_{\alpha_n})$ $\Big)$.

3. Mappings on Weakly Lindelöf Spaces

Definition 3.1 ([9])**.** A topological space X is said to be weakly Lindelöf if for every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that $X = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{U}_{\alpha_n})$.

It is obvious that every nearly Lindelöf space is almost Lindelöf and every almost Lindelöf space is weakly Lindelöf, but the converses are not true (see [4]). Moreover, it is well known that the continuous image of a Lindelöf space is Lindelöf and in [8] it was shown that a δ-continuous image of a nearly Lindelöf space is nearly Lindelöf. For weakly Lindelöf spaces we give the following theorem.

Theorem 3.2. *Let (X, τ) and (Y, σ) be topological spaces. Let $f : X \longrightarrow Y$ be an almost continuous surjection from X into Y. If X is weakly Lindelöf then Y is weakly Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then $\{\mathrm{Int}(\mathrm{Cl}(U_\alpha)) : \alpha \in \Delta\}$ is a regular open cover of Y. Since f is almost continuous, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha)))$ is an open set in X. Thus $\{f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha))) : \alpha \in \Delta\}$ is an open cover of the weakly Lindelöf space X. So there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that

$$X = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_{\alpha_n})))) \subseteq \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} f^{-1}(\mathrm{Cl}(U_{\alpha_n})))$$

$$= \mathrm{Cl}(f^{-1}(\bigcup_{n\in\mathbb{N}} \mathrm{Cl}(U_{\alpha_n}))) \subseteq \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))).$$

Since $\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$ is regular closed in Y and f is almost continuous, we have $f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))$ is closed in X. So

$$X = \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))) = f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})).$$

Since f is surjective,

$$Y = f(X) = f(f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))) = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}).$$

Which implies that Y is weakly Lindelöf and completes the proof. □

Corollary 3.3. *The almost continuous image of a weakly Lindelöf space is weakly Lindelöf.*

Since every continuous function is almost continuous, (Proposition 3.5, [17]) becomes a direct corollary of Theorem 3.2 above.

Corollary 3.4. *Weakly Lindelöf property is a topological property.*

Note that a weakly Lindelöf, semiregular and nearly paracompact space X is almost Lindelöf (see [4], Theorem 3.8). So depending on Theorem 3.2 above we conclude the following two corollaries.

Corollary 3.5. *Let $f : X \longrightarrow Y$ be an almost continuous surjection from X into Y. If X is weakly Lindelöf and Y is semiregular and nearly paracompact then Y is almost Lindelöf.*

Corollary 3.6. *Let $f : X \longrightarrow Y$ be an almost continuous surjection from X into Y. If X is weakly Lindelöf and Y is regular and nearly paracompact then Y is Lindelöf.*

Proposition 3.7. *Let $f : X \longrightarrow Y$ be an almost β-continuous surjection. If X is submaximal, extremally disconnected and weakly Lindelöf then Y is weakly Lindelöf.*

Proof. This follows immediately from Theorem 3.2 above and ([12], Theorem 4.3). □

Proposition 3.8. *Let $f : X \longrightarrow Y$ be an almost precontinuous surjection. If X is submaximal and weakly Lindelöf then Y is weakly Lindelöf.*

Proof. This follows immediately from Theorem 3.2 above and ([12], Theorem 4.4). □

4. Mappings on Weakly Regular-Lindelöf Spaces

Definition 4.1 ([3])**.** An open cover $\{U_\alpha : \alpha \in \Delta\}$ of a topological space X is called regular cover if, for every $\alpha \in \Delta$, there exists a nonempty regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} \mathrm{Int}(C_\alpha)$.

Definition 4.2 ([4])**.** A topological space X is said to be weakly regular-Lindelöf if every regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X admits a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that $X = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$.

Now we prove the following theorem.

Theorem 4.3. *Let (X, τ) and (Y, σ) be topological spaces. Let $f : X \longrightarrow Y$ be an almost continuous surjection from X into Y. If X is weakly regular-Lindelöf then Y is weakly regular-Lindelöf.*

Proof. Let $f : X \longrightarrow Y$ be an almost continuous function from the weakly regular-Lindelöf space X onto Y. Let $\{U_\alpha : \alpha \in \Delta\}$ be a regular cover of Y. (i. e. for every $\alpha \in \Delta$ there exists a regular closed set $C_\alpha \subseteq U_\alpha$ with $Y = \bigcup_{\alpha\in\Delta} \mathrm{Int}(C_\alpha)$.) But

$$\bigcup_{\alpha\in\Delta} \mathrm{Int}(C_\alpha) \subseteq \bigcup_{\alpha\in\Delta} C_\alpha \subseteq \bigcup_{\alpha\in\Delta} U_\alpha \subseteq \bigcup_{\alpha\in\Delta} \mathrm{Int}(\mathrm{Cl}(U_\alpha)).$$

So

$$\bigcup_{\alpha\in\Delta} f^{-1}(\mathrm{Int}(C_\alpha)) \subseteq \bigcup_{\alpha\in\Delta} f^{-1}(C_\alpha) \subseteq \bigcup_{\alpha\in\Delta} f^{-1}(U_\alpha) \subseteq \bigcup_{\alpha\in\Delta} f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha))).$$

Thus

$$X = f^{-1}(Y) = f^{-1}(\bigcup_{\alpha\in\Delta} \mathrm{Int}(C_\alpha)) = \bigcup_{\alpha\in\Delta} f^{-1}(\mathrm{Int}(C_\alpha)).$$

Since C_α is regular closed and f is almost continuous, $f^{-1}(C_\alpha)$ is closed. Thus

$$\mathrm{Cl}(\mathrm{Int}(f^{-1}(C_\alpha))) \subseteq f^{-1}(C_\alpha) \subseteq f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha))).$$

Since $\mathrm{Int}(C_\alpha)$ is regular open in Y and f is almost continuous, $f^{-1}(\mathrm{Int}(C_\alpha))$ is open in X. Thus

$$\bigcup_{\alpha\in\Delta} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(C_\alpha)))) = \bigcup_{\alpha\in\Delta} \mathrm{Int}(f^{-1}(C_\alpha))$$
$$\supseteq \bigcup_{\alpha\in\Delta} \mathrm{Int}(f^{-1}(\mathrm{Int}(C_\alpha))) = \bigcup_{\alpha\in\Delta} f^{-1}(\mathrm{Int}(C_\alpha)) = X.$$

It means that, for every $\alpha \in \Delta$, there exists a regular closed set $\mathrm{Cl}(\mathrm{Int}(f^{-1}(C_\alpha))) \subseteq f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha)))$ such that $X = \bigcup_{\alpha\in\Delta} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(C_\alpha))))$. Thus $\{f^{-1}(\mathrm{Int}(\mathrm{Cl}(U_\alpha))) : \alpha \in \Delta\}$ is a regular cover of the weakly regular-Lindelöf space X. So there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that

$$X = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{f}^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{U}_{\alpha_n})))) \subseteq \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{f}^{-1}(\mathrm{Cl}(\mathrm{U}_{\alpha_n})))$$
$$= \mathrm{Cl}(\mathrm{f}^{-1}(\bigcup_{n\in\mathbb{N}} \mathrm{Cl}(\mathrm{U}_{\alpha_n}))) \subseteq \mathrm{Cl}(\mathrm{f}^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{U}_{\alpha_n}))).$$

Since f is almost continuous and $\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n})$ is regular closed, $f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))$ is closed in X. So $X = \mathrm{Cl}(\mathrm{f}^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{U}_{\alpha_n}))) = \mathrm{f}^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} \mathrm{U}_{\alpha_n}))$. Thus

$$Y = f(X) = f(f^{-1}(\mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}))) \subseteq \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}).$$

This implies that Y is weakly regular-Lindelöf and completes the proof. □

Corollary 4.4. *The almost continuous image of a weakly regular-Lindelöf space is weakly regular-Lindelöf.*

Corollary 4.5. *Weakly regular-Lindelöf property is a topological property.*

Note that every regular and weakly regular-Lindelöf space X is weakly Lindelöf and if X, moreover, is nearly paracompact then it is Lindelöf (see [7]). So depending on Theorem 4.3 above we conclude the following two corollaries.

Corollary 4.6. *Let $f : X \longrightarrow Y$ be an almost continuous mapping from X onto Y. If X is weakly regular-Lindelöf and Y is regular then Y is weakly Lindelöf.*

Corollary 4.7. *Let $f : X \longrightarrow Y$ be an almost continuous mapping from X onto Y. If X is weakly regular-Lindelöf and Y is regular and nearly paracompact then Y is Lindelöf.*

Next we prove the following proposition.

Proposition 4.8. *The image of a weakly regular-Lindelöf space under a precontinuous and contra-continuous mapping is Lindelöf.*

Proof. Let $f : X \longrightarrow Y$ be a contra-continuous and precontinuous mapping from the weakly regular-Lindelöf space X into Y. Let $U = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of $f(X)$. For each $x \in X$, let $U_{\alpha_x} \in U$ such that $f(x) \in U_{\alpha_x}$. Since f is contra-continuous, $f^{-1}(U_{\alpha_x})$ is closed in X. Since f is precontinuous, $f^{-1}(U_{\alpha_x}) \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(U_{\alpha_x}))) = \mathrm{Int}(f^{-1}(U_{\alpha_x}))$. So $f^{-1}(U_{\alpha_x}) = \mathrm{Int}(f^{-1}(U_{\alpha_x}))$. It follows that $f^{-1}(U_{\alpha_x})$ is closed and open in X and hence $\{f^{-1}(U_{\alpha_x}) : x \in X\}$ is a regular cover of the weakly regular-Lindelöf space X. Thus there exists a countable subfamily $\{x_n : n \in \mathbb{N}\}$ such that

$$X = \mathrm{Cl}(\bigcup_{n\in\mathbb{N}} f^{-1}(U_{\alpha_{x_n}})) = \mathrm{Cl}(f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})).$$

Since $\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}$ is open in Y and f is contra-continuous, $f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})$ is closed in X. Thus

$$\mathrm{Cl}(f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})) = f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}).$$

So $X = f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})$. Since f is surjective,

$$f(X) = f(f^{-1}(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}})) \subseteq \bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}.$$

This implies that $f(X)$ is Lindelöf and completes the proof. □

As in weakly Lindelöf spaces we give the following propositions.

Proposition 4.9. *Let $f : X \longrightarrow Y$ be an almost β-continuous surjection. If X is submaximal, extremally disconnected and weakly regular-Lindelöf then Y is weakly regular-Lindelöf.*

Proof. The proof follows immediately from ([12], Theorem 4.3) and Theorem 4.3 above. □

Proposition 4.10. *Let $f : X \longrightarrow Y$ be an almost precontinuous surjection. If X is submaximal and weakly regular-Lindelöf then Y is weakly regular-Lindelöf.*

Proof. The proof follows immediately from ([12], Theorem 4.4) and Theorem 4.3 above. □

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