Extensions defined using bornologies

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ABSTRACT

Many extensions of a space X such that the remainder Y is closed can be constructed as B-extensions, that is, by defining a topology on the disjoint union $X \cup Y$, provided there exists a map, satisfying some conditions, from a basis of Y into the family of the subsets of X which are "unbounded" with respect to a given bornology in X. We give a first example of a (nonregular) extension with closed remainder which cannot be obtained as B-extension. Extensions with closed discrete remainders and extensions whose remainders are retract are mostly considered. We answer some open questions about separation properties and metrizability of B-extensions.

1. INTRODUCTION.

The construction of the one-point compactification of a locally compact space can be generalized by taking as open neighborhoods of the new point the complements of the closed members of any *boundedness*. A nonempty family $\mathcal{F}_X$ of subsets of a space X is said to be a boundedness if it is closed with respect to subsets and finite unions. $\mathcal{F}_X$ is said to be *closed* (*open*) if every *bounded* set, that is, every member of $\mathcal{F}_X$, is contained in a closed (resp. open) bounded set ([10]). The one-point extension naturally associated to a boundedness $\mathcal{F}_X$ in X is denoted by $o(\mathcal{F}_X)$. In order to obtain T_1 one-point extensions of a T_1 space X we need that all singletons of X are bounded. In this case the boundedness is also called *bornology*.

It turns out that all possible T_1 one-point extensions of X can be defined in this way. In fact, every extension aX of X determines a closed bornology $\mathcal{H}_X(aX)$

in X, namely the family of sets whose closure in X is also closed in aX. If aX is a one-point extension, then aX is *equivalent* to $o(\mathcal{H}_X(aX))$.

If X is Hausdorff, then $o(\mathcal{F}_X)$ will be Hausdorff provided every point has a bounded closed neighborhood.

It is known that every Hausdorff n-point compactification $X \cup \{x_1, \ldots, x_n\}$ can be obtained associating to every x_i an open non-relatively compact subset U_i of X, where $U_i \cap U_j = \varnothing$ and $X \setminus (\bigcup U_i)$ is compact. Using a suitable bornology, an analogous result was proved for Hausdorff n-point extensions ([5]).

It is natural to try a generalization of this kind of construction to other Hausdorff extensions. The question is whether it is possible to obtain every extension aX of X, in which X is open, using a bornology in X and some kind of correspondence between a basis of the remainder $Y = aX \setminus X$ and the family of unbounded open subsets of X.

This idea inspired the construction of the so-called ESH-compactifications, first defined in [4]. The authors used an *essential semilattice homomorphism* π from a basis of a compact space K into the family of nonrelatively compact open subsets of a locally compact space X to obtain a compactification of X whose remainder is homeomorphic to K. Large families of compactifications (in most cases even the Stone-Čech compactification) can be obtained in this way. The word "essential" stands for "up to relatively compact sets".

This construction can also be generalized using different (closed) bornologies. An extension which can be obtained in this way is said to be a B-extension. In ([5]) B-extensions were first introduced and mostly used in order to construct Lindelöf extensions of locally Lindelöf spaces.

Two questions naturally arise:
- which extensions are B-extensions
- which conditions a bornology $\mathcal{F}_X$ must satisfy to produce a B-extension $aX = X \cup Y$ preserving some specific topological property of X and Y.

In this paper we give a first example of a (non-regular) extension which is not a B-extension. We do not know if every regular extension can be obtained as B-extension. The problem is still open even for compactifications.

In section 5 we will show that every regular extension such that the remainder is a retract is a B-extension.

Some results are known about the second question. If $\mathcal{F}_X$ is a closed bornology in a regular (normal) space X, then the extension $o(\mathcal{F}_X)$ is regular (resp. normal) if and only if $\mathcal{F}_X$ is open ([12]). It was proved in ([6]) that $o(\mathcal{F}_X)$ is Tychonoff if and only if X is Tychonoff and $\mathcal{F}_X$ is *functionally open*. This means that for every (closed) $F \in \mathcal{F}_X$ there is an open $W \in \mathcal{F}_X$ such that F and $X \setminus W$ are completely separated. This concept was used in [3] to obtain a topological and bornological immersion of a Tychonoff space into a cube.

It is also known that $o(\mathcal{F}_X)$ is metrizable if and only X is metrizable and $\mathcal{F}_X$ is induced by a metric ([2], [6]).

In [6] the results about regularity, normality and metrizability of one-point extensions were extended to B-extensions with compact remainder. It was also observed that the Moore-Niemytzki plane and the Mrówka space can be

naturally obtained as B-extensions. These examples was used to prove that the results about normality and metrizability do not hold if we remove the compactness hypothesis. The case of a regular extension with regular non-compact remainder was not solved there.

In Section 3 of this paper we mostly study B-extensions with closed discrete remainders. We prove that all regular extensions of this kind are B-extensions. We also show that the condition that the given bornology is open is not in general either necessary or sufficient to obtain a regular B-extension. We give conditions on the bornology $\mathcal{F}_X$ which are equivalent to the B-extension being regular (Tychonoff) when the remainder is discrete and X is regular (resp. Tychonoff).

In section 4 we discuss the weight of a B-extension. We prove by an example (the so-called butterfly space) that the weight of a B-extension aX can be greater than $\max\{w(X), w(Y), \chi(aX)\}$.

In section 5 we study a particular case of B-extensions, when the map from the open subsets of the remainder Y and the unbounded open subsets of X is induced by a map from X to Y. In this case almost all results about separation and metrizability properties of one-point extensions can be generalized.

2. Basic definitions.

We recall that a boundedness (or bornology) $\mathcal{F}_X$ is said to be *local* if every $x \in X$ has a neighborhood in $\mathcal{F}_X$. In this case one can also say that X is *locally bounded* with respect to $\mathcal{F}_X$. An open bornology is obviously local. If aX is a regular extension of X then $\mathcal{H}_X(aX)$ is local.

A boundedness $\mathcal{F}_X$ is *nontrivial* if X is not bounded.

A *basis* for a boundedness is a cofinal subfamily. An open and closed bornology with a countable basis is said to be *of metric type* (or *M-boundedness*), since the usual boundedness induced by a metric has these properties. It was proved in [11] that every M-boundedness in a metrizable space is induced by a compatible metric.

Let X, Y be Hausdorff spaces, $\mathcal{F}_X$ a nontrivial closed local bornology on X and $\mathcal{B}$ an open basis of Y, closed with respect to finite unions. Let us denote by $\mathcal{T}_X$ the topology of X. A map

$$\pi = \pi(\mathcal{B}, \mathcal{F}_X) : \mathcal{B} \to (\mathcal{T}_X \setminus \mathcal{F}_X) \cup \{\varnothing\},$$

is said to be a *B-map* provided that

B0) $U \neq \varnothing$ implies $\pi(U) \neq \varnothing$, for every $U \in \mathcal{B}$;
B1) If $\{U_i\}_{i \in A} \subset \mathcal{B}$ is a cover of Y, then $X \setminus \left[\bigcup_{i \in A} \pi(U_i)\right] \in \mathcal{F}_X$;
B2) If $U, V \in \mathcal{B}$ then $\pi(U \cup V) \Delta [\pi(U) \cup \pi(V)] \in \mathcal{F}_X$;
B3) If $U, V \in \mathcal{B}$ and $Cl_Y(U) \cap Cl_Y(V) = \varnothing$ then $\pi(U) \cap \pi(V) \in \mathcal{F}_X$.

Putting on the disjoint union $X \cup Y$ the topology generated by

$$\mathcal{T}_X \cup \{U \cup (\pi(U) \setminus F) \mid U \in \mathcal{B},\ F = Cl_X(F) \in \mathcal{F}_X\},$$

we obtain a dense extension of X, denoted by $X \cup_\pi Y$. If Y is T_3 the extension is Hausdorff. If Y is T_2 not T_3 we can obtain a Hausdorff extension if we replace B3) by the stronger condition

- If $U, V \in \mathcal{B}$ and $U \cap V = \varnothing$ then $\pi(U) \cap \pi(V) \in \mathcal{F}_X$.

A B-extension of X is any extension of X which can be constructed in this way. As we have already mentioned, the family of B-extensions includes ESH-compactifications and Hausdorff extensions aX with finite remainder. Moreover normal extensions with 0-dimensional remainder are B-extensions ([5]).
If $aX = X \cup_\pi Y$ is a B-extension, where $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$, then π is also a B-map with respect to $\mathcal{H}_X(aX)$ and $X \cup_{\pi(\mathcal{B}, \mathcal{H}_X(aX))} Y$ is equivalent to aX (see [5]).

3. Extensions with discrete remainders.

All spaces will be Hausdorff and the word "extension" will always mean "dense extension".

Theorem 3.1. *Let aX be a regular extension of X such that $Y = aX \setminus X$ is closed in aX. Suppose there is a basis $\mathcal{B}$ of Y consisting of open and closed sets and for every $B \in \mathcal{B}$ there are disjoint open subsets U and V of aX such that $B \subset U$ and $Y \setminus B \subset V$. Then aX is a B-extension (with respect to $\mathcal{H}_X(aX)$).*

The proof is essentially the same as the one of Theorem 1.5 in [5]. In fact, the hypothesis that aX is normal was used only to find disjoint open neighborhood of B and $Y \setminus B$ for every B in the clopen basis.

For a discrete space Y we denote by $\mathcal{B}_0$ the basis consisting of all finite subsets of Y. By the above theorem we easily obtain the following result.

Theorem 3.2. *Every regular extension aX such that $Y = aX \setminus X$ is closed and discrete is a B-extension.*

Proof. It suffices to take as $\mathcal{B}$, in the above theorem, the family $\mathcal{B}_0$. $\square$

If $aX = X \cup_\pi Y$ is a B-extension, where $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$ and Y is discrete, then $\mathcal{B}$ contains $\mathcal{B}_0$, hence $\mathcal{B}$ can be replaced by $\mathcal{B}_0$ and π by its restriction (see [6], Lemma 4.7).

In the previous theorem, the hypothesis that aX is regular cannot be removed. In fact we can give an example of a nonregular Hausdorff extension aX, such that $Y = aX \setminus X$ is closed and discrete but aX cannot be obtained as a B-extension.

Example 3.3. Let $Y = M \cup \{0\} \subset \mathbf{R}$, where $M = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbf{N} \right\}$. Put $X = \mathbf{R} \setminus Y$ and let $aX = (\mathbf{R}, \mathcal{T})$, where $\mathcal{T}$ is generated by the union of the usual topology and the family $\{(-a, a) \setminus M \mid a \in \mathbf{R}^+\}$ ($(\mathbf{R}, \mathcal{T})$ is often used as example of Hausdorff nonregular space). $\mathcal{T}$ induces the Euclidean topology on X and the discrete topology on Y. Suppose aX is a B-extension, that is $aX = X \cup_\pi Y$. We can suppose $\pi = \pi(\mathcal{B}_0, \mathcal{H}_X(aX))$ (see above). By definition, $\{0\} \cup \pi(\{0\})$

is open in aX so it must contain a set of the form $(-a, a) \setminus M$ with $a \in \mathbf{R}^+$. Let $m \in \mathbf{N}$ such that $\frac{1}{m} < a$. The open set $\left\{\frac{1}{m}\right\} \cup \pi\left(\left\{\frac{1}{m}\right\}\right)$ must contain an interval (b, c), with $0 < b < \frac{1}{m} < c$. The intersection $(b, c) \cap (-a, a) \setminus M \subset X$ cannot be in $\mathcal{H}_X(aX)$ because $\frac{1}{m}$ belongs to its closure. This means that π does not satisfy B3), a contradiction.

For any subset A of a space X we denote by $Fr_X(A)$ the boundary of A in X.

Lemma 3.4. Let $aX = X \cup_\pi Y$ be a B-extension, where Y is T_3 and $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$. Then, for every $B \in \mathcal{B}$ we have

(i) $Cl_{aX}(\pi(B)) = Cl_Y(B) \cup Cl_X(\pi(B))$;

(ii) $Fr_{aX}(B \cup \pi(B)) = Fr_Y(B) \cup Fr_X(\pi(B))$.

Proof. (i) We only have to prove that if for $y \in Y$, $y \notin Cl_Y(B)$ then $y \notin Cl_{aX}(\pi(B))$. Let $B' \in \mathcal{B}$ such that $y \in B' \subset Cl_Y(B') \subset Y \setminus Cl_Y(B)$. Then, by B3, $\pi(B) \cap \pi(B') = K \in \mathcal{F}_X$. Therefore $y \in B' \cup (\pi(B') \setminus K)$ which is disjoint from $\pi(B)$. Then the conclusion follows.

(ii) Let $B \in \mathcal{B}$. Clearly we have $Fr_{aX}(B \cup \pi(B)) = (Cl_{aX}(B \cup \pi(B))) \setminus (B \cup \pi(B))$. We note that $Cl_{aX}(B) \setminus (B \cup \pi(B)) \subset Cl_{aX}(\pi(B)) \setminus (B \cup \pi(B))$. In fact if $y \in Cl_{aX}(B) \setminus (B \cup \pi(B))$ then $y \in Y$ and, for every $B' \in \mathcal{B}$ and $F \in \mathcal{F}_X$ such that $y \in B'$, we have $(B' \cup (\pi(B') \setminus F)) \cap \pi(B) \neq \varnothing$ otherwise, for some $B' \in \mathcal{B}$, $(B' \cup \pi(B') \setminus F) \cap (B \cup \pi(B)) = B \cap B'$ would be a non-empty open subset of $aX = X \cup_\pi Y$ contained in Y. It follows that $(Cl_{aX}(B \cup \pi(B))) \setminus (B \cup \pi(B)) = (Cl_{aX}(\pi(B))) \setminus (B \cup \pi(B))$. By (i) we get $(Cl_{aX}(\pi(B))) \setminus (B \cup \pi(B)) = (Cl_Y(B) \cup Cl_X(\pi(B))) \setminus (B \cup \pi(B))$ and finally the conclusion follows by the obvious equalities $(Cl_Y(B) \cup Cl_X(\pi(B))) \setminus (B \cup \pi(B)) = (Cl_Y(B) \setminus B) \cup ((Cl_X(\pi(B)) \setminus \pi(B)) = Fr_Y(B) \cup Fr_X(\pi(B))$. $\square$

Remark 3.5. If $\mathcal{B}$ consists of open and closed sets then (ii) becomes

(ii') $Fr_{aX}(B \cup \pi(B)) = Fr_X(\pi(B))$, hence $Fr_X(\pi(B))$ is closed in aX.

From now on, for a B-extension $X \cup_\pi Y$, where $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$ and Y is discrete, we will always put $\mathcal{B} = \mathcal{B}_0$ and $U_y = \pi(\{y\})$ for every $y \in Y$.

Corollary 3.6. Let $aX = X \cup_\pi Y$ be a B-extension, where Y is discrete. Then, for every $y \in Y$ we have

(i) $Cl_{aX}(U_y) = \{y\} \cup Cl_X(U_y)$;

(ii) $Fr_X(U_y) = Fr_{aX}(\{y\} \cup U_y)$, hence $Fr_X(U_y)$ is closed in aX.

Proposition 3.7. Let $aX = X \cup_\pi Y$ be a B-extension of a regular space X, where $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$ and $\mathcal{B}$ is a basis of Y consisting of open and closed subsets of Y. Then

(i) if aX is regular and B is compact then $Fr_X(\pi(B))$ belongs to $\mathcal{F}_X$;

(ii) if $\mathcal{F}_X$ is open and, for each $B \in \mathcal{B}$, $Fr_X(\pi(B))$ belongs to $\mathcal{F}_X$, then aX is regular.

Proof. (i) Let aX be regular and let $B \in \mathcal{B}$, B compact. By the Remark 3.5 $A = Fr_X(\pi(B))$ is closed in aX. Then for every $y \in B$ there exists

an open subset W_y of aX that contains A and is disjoint from a basic open neighborhood $B_y \cup (\pi(B_y) \setminus K_y)$ of y. We can suppose $W_y \subset X$. From the compactness of B it follows that $B \subset B' = \bigcup_{i=1}^n B_{y_i}$ for some $y_1, \ldots, y_n \in B$. Hence $\pi(B) \setminus \pi(B') = \pi(B \cup B')\Delta[\pi(B) \cup \pi(B')] \in \mathcal{F}_X$ by B2) and clearly $\pi(B) \setminus (\bigcup_{i=1}^n \pi(B_{y_i}))$ also belongs to $\mathcal{F}_X$. If we put $W = \bigcap_{i=1}^n W_{y_i}$ then $W \cap (\bigcup_{i=1}^n \pi(B_{y_i})) \subset \bigcup_{i=1}^n K_{y_i}$ and $W \cap \pi(B)$ is bounded. Now let $x \in A$ and let V be any neighborhood of x in X. Then $V \cap W \cap \pi(B) \neq \varnothing$ and this means that $x \in Cl_X(W \cap \pi(B))$. Therefore $A \subset Cl_X(W \cap \pi(B))$, which is bounded.

(ii) Let $x \in X$. The hypotheses imply that x has a local basis consisting of bounded closed subset of X, which are also closed neighborhoods in aX. Now let $y \in Y$ and let F be a closed subset of aX such that $y \notin F$. Then there are $B \in \mathcal{B}$ and a closed member K of $\mathcal{F}_X$ such that $y \in B \cup (\pi(B) \setminus K)$, which is disjoint from F. Put

$$V = aX \setminus [Cl_{aX}(B \cup \pi(B))] = aX \setminus [B \cup Cl_X(\pi(B))] = aX \setminus [B \cup \pi(B) \cup Fr_X(\pi(B))].$$

Then we have

$$F \setminus V = (F \cap (B \cup \pi(B))) \cup (F \cap Fr_X(\pi(B))) \subset K \cup Fr_X(\pi(B)).$$

Since $K \cup Fr_X(\pi(B))$ is bounded, it is contained in an open member W of $\mathcal{F}_X$. Then $V \cup W$ is an open subset of aX which contains F and is disjoint from $B \cup [\pi(B) \setminus Cl_X(W)]$, which is a basic neighborhood of y. $\square$

Proposition 3.8. *Let $aX = X \cup_\pi Y$ be a B-extension of X, where Y is discrete and $\pi = \pi(\mathcal{B}_0, \mathcal{F}_X)$. Suppose X is regular. Then*

(i) if aX is regular, then for each $y \in Y$, $Fr_X(U_y)$ belongs to $\mathcal{F}_X$;
(ii) if $\mathcal{F}_X$ is open and, for each $y \in Y$, $Fr_X(U_y)$ belongs to $\mathcal{F}_X$, then aX is regular.

Proof. It easily follows by Proposition 3.7, since $Fr_X(U_y) \in \mathcal{F}_X$ for every $y \in Y$ implies that $Fr_X(\pi(B)) \in \mathcal{F}_X$ for every $B \in \mathcal{B}_0$. $\square$

The following example shows that, for a B-extension $aX = X \cup_\pi Y$, the conditions that X, Y are regular and $\mathcal{F}_X$ is open do not ensure that aX is regular, even if Y is discrete.

Example 3.9. Let X be the upper half plane, defined by $\{(x, y) \in \mathbf{R}^2 \mid y > 0\}$, with the usual topology, and Y be the x-axis with the discrete topology. Let

$$\mathcal{F}_X = \{A \subset X \mid d(A, Y) > 0\},$$

where d is the Euclidean metric. Clearly $\mathcal{F}_X$ is an open boundedness and X is locally bounded. For every $z = (a, 0) \in Y$, put

$$U_z = \{(x, y) \in X \mid |x - a| < y < 1\}.$$

U_z is clearly unbounded. Let $\pi : \mathcal{B}_0 \to \mathcal{T}_X \setminus \mathcal{F}_X$ be defined by $B \mapsto \bigcup_{z \in B} U_z$. It is easy to see that π is a B-map. Then we can define the Hausdorff B-extension $aX = X \cup_\pi Y$. Let z be any point of Y. Clearly $E = Fr_X(U_z)$ is unbounded, so by Proposition 3.7, aX is not regular. In fact, E is closed

in aX and $z \notin E$, but no open set containing E can be disjoint from a basic neighborhood $\{z\} \cup (U_z \setminus K)$, with $K \in \mathcal{F}_X$.

Remark 3.10. In the above example, although $\mathcal{F}_X$ is open, $\mathcal{H}_X(aX)$ is not open. By Lemma 3.4(ii) and Proposition 3.8(ii), if $aX = X \cup_{\pi(\mathcal{B}, \mathcal{F}_X)} Y$, where X is regular, Y is discrete and $\mathcal{H}_X(aX)$ is open, then aX is regular. However, as we will see below, the condition that $\mathcal{H}_X(aX)$ is open is not necessary to obtain a regular B-extension.

We notice that, for any extension aX with closed remainder Y, $\mathcal{H}_X(aX)$ is open if and only if Y is separated, by disjoint open subsets of aX, from every closed subset of aX which is contained in X.
There exists a Tychonoff B-extension aX of a space X, with discrete remainder, which cannot be obtained as B-extension with respect to any open boundedness $\mathcal{F}_X$. In particular $\mathcal{H}_X(aX)$ is not open.

Example 3.11. The Tychonoff plank T can be seen as an extension aX of $X = \omega_1 \times (\omega + 1)$ with closed discrete remainder $Y = \{\omega_1\} \times \omega$. By Theorem 3.2, aX is a B-extension. Put $F = \omega_1 \times \{\omega\}$. Clearly $\mathcal{H}_X(aX)$ is not open, otherwise, F and Y would be contained in disjoint open subsets of T.
Now suppose that $\mathcal{F}_X$ is a closed boundedness on X, and $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$ a B-map such that $T = X \cup_\pi Y$. Put $y_n = (\omega_1, n)$ for every n, so that $Y = \{y_n\}_{n \in \mathbf{N}}$. For sake of simplicity, for every $y \in Y$, we will write $\pi(y)$ instead of $\pi(\{y\})$. Since $\{y_n\} \cup \pi(y_n)$ is open in T, $\pi(y_n)$ must contain a set of the form $(\beta_n, \omega_1) \times \{n\}$, where $\beta_n < \omega_1$. Property B1) implies that $G = F \setminus \left(\bigcup_{n \in \mathbf{N}} \pi(y_n) \right)$ belongs to $\mathcal{F}_X$. Now suppose $\mathcal{F}_X$ is open. Then G and Y are contained in disjoint open subsets of T. This implies $G \subset [0, \alpha] \times \{\omega\}$, where $\alpha < \omega_1$, hence $(\alpha, \omega_1) \times \{\omega\} \subset \bigcup_{n \in \mathbf{N}} \pi(y_n)$. Let $m \in \mathbf{N}$ such that $M = \{\gamma > \alpha \mid (\gamma, \omega) \in \pi(y_m)\}$ is uncountable. For every $\gamma \in M$ let $V_\gamma = (\eta_1(\gamma), \eta_2(\gamma)) \times (n_\gamma, \omega]$ be a basic open neighborhood of (γ, ω) contained in $\pi(y_m)$. Let $h \in \mathbf{N}$ such that $H = \{\gamma \in M \mid n_\gamma = h\}$ is uncountable and let $k \in \mathbf{N}$, $k > max(h, m)$. Then $(\beta_k, \omega_1) \times \{k\}$, which is contained in $\pi(y_k)$, contains all the points (γ, k) with $\gamma \in H$. But every (γ, k), with $\gamma \in H$ belongs to $V_\gamma \subset \pi(y_m)$. Then the uncountable set $\{(\gamma, k) \mid \gamma \in H\}$ is contained in both $\pi(y_k)$ and $\pi(y_m)$. This means that $y_k \in Cl_T(\pi(y_k) \cap \pi(y_m))$. Since $k \neq m$, by B3) $\pi(y_k) \cap \pi(y_m)$ belongs to $\mathcal{F}_X$, so no point of Y can be in its closure, contradiction.

The following proposition provides a condition which is equivalent to the regularity of $X \cup_\pi Y$ where Y is discrete.

Proposition 3.12. *Let $aX = X \cup_\pi Y$ be a B-extension of X, where Y is discrete and $\pi = \pi(\mathcal{B}_0, \mathcal{F}_X)$. Suppose X is regular. Then aX is regular if and only if, for every $y \in Y$ and for every closed subset F of X such that $F \cap U_y$ is bounded, there is an open subset W of X containing $F \cap Cl_X(U_y)$ such that $W \cap U_y$ is bounded.*

Proof. Suppose aX is regular. Let $y \in Y$ and $F \cap U_y \in \mathcal{F}_X$, where F is closed in X. This is clearly equivalent to $y \notin Cl_{aX}(F)$. Then there exists an

open subset W_1 of aX that contains $Cl_{aX}(F)$ and is disjoint from a basic open neighborhood $\{y\} \cup (U_y \setminus K)$ of y. Put $W = W_1 \cap X$. Then $F \cap Cl_X(U_y) \subset W$ and $W \cap U_y \subset K$, so that $W \cap U_y$ is bounded.

Conversely, let $y \in aX$ and let G be a closed subset of aX such that $y \notin G$. We can suppose $y \in Y$ (see the proof of Proposition 3.7(ii)). Put $F = G \cap X$. Then $y \notin Cl_{aX}(F)$, hence $F \cap U_y$ is bounded. By hypothesis, there exists an open subset W of X such that $F \cap Cl_X(U_y) \subset W$ and $W \cap U_y$ is bounded. We know that $V = aX \setminus Cl_{aX}(U_y)$ contains all points of Y except y (see Lemma 3.4). Since $y \notin G$, we have $G \setminus V = F \setminus V \subset W$. Therefore we have $G \subset W \cup V$. Clearly $[\{y\} \cup (U_y \setminus W)] \cap (V \cup W) = \varnothing$. We have $\{y\} \cup (U_y \setminus W) = \{y\} \cup (U_y \setminus (W \cap U_y)) \supset \{y\} \cup (U_y \setminus Cl_X(W \cap U_y))$, which is a basic open neighborhood of y disjoint from $V \cup W$. $\qquad\square$

We recall that every one-point extension aX of X can be obtained as $o(\mathcal{F}_X)$ for a suitable closed bornology $\mathcal{F}_X$; moreover we have $\mathcal{H}_X(aX) = \mathcal{F}_X$ ([6]). Then, by Theorem 3.3 in [6], a one-point extension aX is Tychonoff if and only if $\mathcal{H}_X(aX)$ is functionally open.

The following proposition provides a sufficient condition for the complete regularity of a B-extension with discrete remainder.

Proposition 3.13. *Let $aX = X \cup_\pi Y$ be a B-extension of X, where Y is discrete and $\pi = \pi(\mathcal{B}_0, \mathcal{F}_X)$. Suppose X is Tychonoff. If $\mathcal{F}_X$ is functionally open and, for each $y \in Y$, $Fr_X(U_y)$ belongs to $\mathcal{F}_X$, then aX is Tychonoff*

Proof. Let $z \in aX$ and let F be a closed subset of aX such that $z \notin F$. First suppose $z \in X$. Let H be a bounded open neighborhood of z in X which is disjoint from F. Since X is Tychonoff, there exists a continuous function $f : X \to \mathbf{I}$ such that $f(z) = 1$ and $f(X \setminus H) = 0$. The map $\hat{f} : aX \to \mathbf{I}$ defined by $\hat{f}|_X = f$, $\hat{f}(Y) = 0$, is continuous because no $y \in Y$ belongs to $Cl_{aX}(H)$. Clearly $\hat{f}$ separates z from F.

Let now $z \in Y$. Put $A = Fr_X(U_z)$, $T = Cl_X(U_z) = U_z \cup A$ and $T^* = Cl_{aX}(T) = \{z\} \cup T$. T^*, with the topology induced by aX, is a one-point extension of T. We want to prove that $\mathcal{H}_T(T^*)$ is functionally open. Let G be a subset of T which is closed in T^*. There exists a neighborhood $\{z\} \cup (U_z \setminus K)$ which is disjoint from G, where K is a closed member of $\mathcal{F}_X$. Since G is closed in T and is contained in $K \cup A$, G is a closed member of $\mathcal{F}_X$. Then there is an open $W \in \mathcal{F}_X$ such that G and $X \setminus W$ are completely separated. Put $V = W \cap T$. Since T is unbounded, $T \setminus V = T \setminus W$ is nonempty and completely separated from G. Moreover, $Cl_{T^*}(V) \subset Cl_{aX}(W)$ which does not meet Y. Then $V \in \mathcal{H}_T(T^*)$. We have proved that $\mathcal{H}_T(T^*)$ is functionally open, that is, T^* is Tychonoff. Let $f : T^* \to \mathbf{I}$ be a continuous function such that $f(z) = 1$, $f((F \cap T) \cup A) = 0$. We define an extension $\hat{f} : aX \to \mathbf{I}$ of f putting $\hat{f}(aX \setminus T^*) = 0$. Since f is equal to 0 on A, which is the boundary of T^* in aX, $\hat{h}$ is continuous. Moreover $\hat{f}$ separates z from F. $\qquad\square$

In the above proposition the hypothesis $Fr_X(U_y) \in \mathcal{F}_X$ for every y is clearly necessary (see proposition 3.7(i)). The Examples 3.9 and 3.11 show that the

condition that $\mathcal{F}_X$ is functionally open is a neither sufficient nor necessary condition. In view of Corollary 3.6(ii), the condition that $\mathcal{H}_X(aX)$ is functionally open is sufficient, but it is not necessary (see Example 3.11 again).

It is easy to see the following

Proposition 3.14. *For every normal extension aX, $\mathcal{H}_X(aX)$ is functionally open.*

Corollary 3.15. *Every normal extension aX such that $Y = aX \setminus X$ is closed and discrete is a B-extension with respect to a functionally open boundedness.*

Proof. By the proof of Theorem 3.1, aX is a B-extension with respect to $\mathcal{H}_X(aX)$. □

However, there exist nonnormal B-extensions aX with discrete remainder where X is normal and $\mathcal{H}_X(aX)$ is functionally open.

Example 3.16. Let Ψ be the Mrówka space. It is known that Ψ is a nonnormal Tychonoff space (see for instance [9]), and is a B-extension of $\mathbf{N}$ with respect to the boundedness $\mathcal{F}_{\mathbf{N}}$ of the finite subsets of $\mathbf{N}$ (see [6], Example 4.10). Clearly $\mathcal{H}_{\mathbf{N}}(M) = \mathcal{F}_X$ is functionally open, since its members are clopen.

4. WEIGHT OF B-EXTENSIONS.

For a boundedness $\mathcal{F}_X$ on X we put

$$\mu(\mathcal{F}_X) = \min\{|\mathcal{C}| \,:\, \mathcal{C} \text{ is a basis of } \mathcal{F}_X\}.$$

Proposition 4.1. *Let $aX = X \cup_\pi Y$ be any B-extension of a space X,where $\pi = \pi(\mathcal{B}, \mathcal{F}_X)$. Then we have*

$$w(aX) \leq \max\{w(X), w(Y), \mu(\mathcal{F}_X)\}.$$

Proof. By [6] Lemma 4.7, we can suppose $|\mathcal{B}| = w(Y)$. Let $\mathcal{B}_1$ be a basis of X with $|\mathcal{B}_1| = w(X)$ and $\mathcal{C}$ a basis of $\mathcal{F}_X$ with $|\mathcal{C}| = \mu(\mathcal{F}_X)$. It is easy to see that

$$\mathcal{B}_1 \cup \{U \cup (\pi(U) \setminus F) \mid U \in \mathcal{B}, F \in \mathcal{C}\}$$

is a basis for the topology of aX whose cardinality is $\max\{w(X), w(Y), \mu(\mathcal{F}_X)\}$. □

The weight of a B-extension $aX = X \cup_\pi Y$ can be greater than

$$\max\{w(X), w(Y), \chi(aX)\}.$$

Example 4.2. Let us consider the so called *butterfly space*, that is the space $Z = (\mathbf{R} \times \mathbf{R}, \mathcal{T})$ where $\mathcal{T}$ can be described as follows. Let us denote the x-axis by Y and put $X = (\mathbf{R} \times \mathbf{R}) \setminus Y$. The points of X have the same open neighborhoods as in the ordinary topology. Let $p_a = (a, 0) \in Y$ and let $r \in \mathbf{R}^+$. We put $R_r(a) = (a - r, a + r) \times (-\frac{1}{r}, \frac{1}{r})$ and we denote by $C_r^1(a)$ and $C_r^2(a)$ the

(closed) circles of radious $\frac{1}{r}$ and center $(a, \frac{1}{r})$, $(a, -\frac{1}{r})$, respectively (so that the circles are tangent to the x-axis in p_a). We also put

$$B_r(a) = [R_r(a) \setminus (C_r^1(a) \cup C_r^2(a))] \cup \{p_a\}.$$

A local basis for p_a will be the family $\{B_r(a)\}_{r \in \mathbf{R}^+}$. If we consider the sub-family $\{B_r(a)\}_{r \in \mathbf{Q}}$, we obtain a countable local basis, hence $\chi(Z) = \omega$. It is known that $w(Z) = c$.

The topology induced on both X and Y are the usual ones. We also observe that the sets of the form $X \cap C_r^i(a) = C_r^i(a) \setminus \{p_a\}$ are closed in Z.

The butterfly space can be considered a dense extension aX of X such that $Y = aX \setminus X$ is naturally homeomorphic to the real line $\mathbf{R}$. We want to prove that aX can be obtained as a B-extension $X \cup_\pi \mathbf{R}$ with respect to the bound-edness $\mathcal{H}_X(aX)$.

Let $\mathcal{B}$ be the family of finite unions of bounded open intervals in $\mathbf{R}$. Given an open interval $(a - r, a + r)$, with $a \in \mathbf{R}, r \in \mathbf{R}^+$, we put $\pi(a - r, a + r) = B_r(a) \cap X$, which is clearly an open unbounded subset of X. For a finite disjoint union $U = \bigcup_i (a_i - r_i, a_i + r_i)$ we put $\pi(U) = \bigcup_i \pi(a_i - r_i, a_i + r_i)$. We want to prove that π is a B-map.

The property B3) is obviously satisfied, since for $U_1, U_2 \in \mathcal{B}$ such that $U_1 \cap U_2 = \varnothing$, we have $\pi(U_1) \cap \pi(U_2) = \varnothing$.

Let $\{U_j\}_{j \in J} \subset \mathcal{B}$ be a cover of $\mathbf{R}$. For $U_j = \bigcup_{i=1}^n (a_i - r_i, a_i + r_i)$, where the union is disjoint, we put $W_j = \bigcup_{i=1}^n B_{r_i}(a_i)$, so that one has $W_j \cap X = \pi(U_j)$. Therefore

$$X \setminus \left(\bigcup_j \pi(U_j) \right) = X \setminus \left(\bigcup_j W_j \right) = aX \setminus \left(\bigcup_j W_j \right) \in \mathcal{H}_X(aX).$$

We have proved that B1) is satisfied.

Now we prove B2) in case U, V are intervals, $U = (a - r, a + r)$, $V = (b - s, b + s)$. The general case will easily follow. If U and V are disjoint the proof is trivial, hence we can suppose that $U \cup V$ is an interval $(c - t, c + t)$. We want to show that

$$E = \pi(c - t, c + t) \triangle [\pi(a - r, a + r) \cup \pi(b - s, b + s)]$$

is bounded. We have $E \subset (c - t, c + t) \times (\mathbf{R} \setminus \{0\})$. Moreover every $x \in E$ must be in $X \setminus \pi(c - t, c + t) = X \setminus B_t(c)$ or in $X \setminus [\pi(a - r, a + r) \cup \pi(b - s, b + s)] = (X \setminus B_r(a)) \cap (X \setminus B_s(b))$. Suppose $x \in E$, and $x \notin B_t(c)$. Then either $x \in (c - t, c + t) \times [\mathbf{R} \setminus (-\frac{1}{t}, \frac{1}{t})]$, which is clearly bounded, or $x \in (C_t^1(c) \cup C_t^2(c)) \cap X$ which is also bounded. Similarly we can prove that, if $x \in E$ and $x \notin B_r(a)$, $x \notin B_s(b)$ then x belongs to a bounded set. Therefore E is contained in a finite union of bounded sets and so it is bounded.

We have proved that π is a B-map.

If we identify $\mathbf{R}$ with the x-axis Y in $Z = (\mathbf{R} \times \mathbf{R}, \mathcal{T})$, then the topology of $X \cup_\pi \mathbf{R}$ is equal to $\mathcal{T}$. In fact, for $a \in \mathbf{R}, r \in \mathbf{R}^+$, we have $B_r(a) = U \cup \pi(U)$, where $U = (a - r, a + r)$, identified with $(a - r, a + r) \times \{0\}$. Conversely, every set of the form $U \cup (\pi(U) \setminus F)$, where $U \in \mathcal{B}$ and F is a closed member of

$\mathcal{H}_X(aX)$, is open in Z, because $U \cup \pi(U)$ is a union of sets of the form $B_r(a)$ and F is closed in Z by the definition of $\mathcal{H}_X(aX)$.

5. EXTENSIONS WHOSE REMAINDERS ARE RETRACTS.

Given a nontrivial local closed bornology on X, a continuous mapping f from X to any space Y is said to be *B-singular* with respect to $\mathcal{F}_X$ if $f^{-1}(U) \notin \mathcal{F}_X$ for every nonempty open subset U of Y. If f is B-singular, then the map

$$\pi : \mathcal{T}_Y \to (\mathcal{T}_X \setminus \mathcal{F}_X) \cup \{\varnothing\}, \qquad \pi(U) = f^{-1}(U),$$

is clearly a B-map. The B-extension induced by π, denoted by $X \cup_f Y$, is said to be *B-singular* ([5], [7]).

Regular extensions are B-singular if and only if the remainder is a retract. In fact we have

Theorem 5.1. *Let $(aX, \mathcal{T}_{aX})$ be a regular extension of X such that there exists a retraction $g : aX \to Y = aX \setminus X$. Then $f = g|_X$ is B-singular with respect to $\mathcal{H}_X(aX)$ and $(aX, \mathcal{T}_{aX}) = X \cup_f Y$.*

Conversely, for every B-singular extension $aX = X \cup_f Y$, the map $\hat{f} : aX \to Y$ defined by $\hat{f}|_X = f$ and $\hat{f}|_Y = 1_Y$ is a continuous extension of f, hence a retraction.

Proof. Let $g : aX \to Y$ be a retraction and let U be a nonempty open subset of Y. It is easy to see that every point of U belongs to $Cl_{aX}(f^{-1}(U))$, hence $f^{-1}(U) \notin \mathcal{H}_X(aX)$. Since aX is T_3, X is locally bounded with respect to $\mathcal{H}_X(aX)$.

Now we will prove that the topology of $X \cup_f Y$ coincides with $\mathcal{T}_{aX}$. First we observe that $\mathcal{T}_X$ is contained in both topologies. Let $U_1 = U \cup [f^{-1}(U) \setminus F]$ where U is open in Y and F is a closed member of $\mathcal{H}_X(aX)$. Then $U_1 = g^{-1}(U) \setminus F \in \mathcal{T}_{aX}$. This proves the first inclusion.

Let us choose any $W \in \mathcal{T}_{aX}$. We only need to prove that, for every $y \in W \cap Y$, there is a basic open set U_1 in $X \cup_f Y$ such that $y \in U_1 \subset W$. Let U be an open neighborhood of y in Y such that $y \in U \subset Cl_Y(U) \subset W \cap Y$. Put $A = g^{-1}(U) \setminus W$ which is a subset of X. We want to prove that no point of Y belongs to $Cl_{aX}(A)$. This is obvious for $z \in W$. If $z \in Y \setminus W$ then $z \notin Cl_Y(U)$. Let V be an open neighborhood of z in Y such that $U \cap V = \varnothing$. Then $g^{-1}(V) \cap g^{-1}(U) = \varnothing$, that is, $g^{-1}(V)$ is a neighborhood of z in $(aX, \mathcal{T}_{aX})$ which is disjoint from A. We have proved that $Cl_X(A)$ is a (closed) member of $\mathcal{H}_X(aX)$. Then $U_1 = g^{-1}(U) \setminus Cl_X(A) = U \cup [f^{-1}(U) \setminus Cl_X(A)]$ is a basic neighborhood of y in $X \cup_f Y$ and, by the definition of A, we have $U_1 \subset W$. We have proved $(aX, \mathcal{T}_{aX}) = X \cup_f Y$.

Let now $aX = X \cup_f Y$ a B-singular extension and let U be open in Y. Then $\hat{f}^{-1}(U) = U \cup f^{-1}(U)$ is a basic open set of $X \cup_f Y$. $\square$

In [6], Theorem 4.11, it was proved that a B-singular extension $X \cup_f Y$ of X, with respect to an open (and closed) bornology $\mathcal{F}_X$, is regular provided X and

Y are both regular. If we replace "open" by "functionally open", we obtain an analogous result for the Tychonoff property.

Theorem 5.2. *Let X, Y be Tychonoff spaces and let $f : X \to Y$ be a B-singular map with respect to a functionally open bornology $\mathcal{F}_X$. Then $aX = X \cup_f Y$ is Tychonoff.*

Proof. First suppose that Y is compact. Let $q : X \cup_f Y \to o(\mathcal{F}_X) = X \cup \{p\}$ be the natural mapping which takes every point of Y to the point p. By [5], Proposition 1.1, q is a quotient map. Let now $z \in aX$ and let A be a closed subset of aX with $z \notin A$.
First suppose $z \in X$ and put $B = A \cup Y$. Then $q(B)$ is a closed subset of $o(\mathcal{F}_X)$ which does not contain z. Since $o(\mathcal{F}_X)$ is Tychonoff ([6], Theorem 3.3), z and $q(B)$ are separated by a continuous function g from $o(\mathcal{F}_X)$ to $\mathbf{I}$, where $\mathbf{I}$ is the unit interval. Clearly $g \circ q$ separates z and A.
Now, let $z \in Y$ and $A \subset X$. Then $A = q(A)$ is closed in $o(\mathcal{F}_X)$. Let $g : o(\mathcal{F}_X) \to \mathbf{I}$ be a function such that $g(p) = 1$ and $g(A) = 0$. Then $g \circ q$ separates z from A. Note that $g \circ q$ maps all of Y onto 1.
Finally let $C = A \cap Y \neq \varnothing$ and $z \in Y$. Take a map $v : Y \to \mathbf{I}$ such that $v(C) = 0$ and $v(z) = 1$. Put $h = v \circ \hat{f} : aX \to \mathbf{I}$ and $U = h^{-1}([0, 1/2))$. Then $A \setminus U$ is a closed subset of aX contained in X. We can take, as before, a function $u : aX \to \mathbf{I}$ such that $u(Y) = 1$ and $u(A \setminus U) = 0$. Then the map $h \wedge u$ is less than $1/2$ in $U \cup A$ and maps z to 1.
We have proved that aX is Tychonoff in case Y is compact. Let now Y be any Tychonoff space and let (K, k) be any compactification of Y, where $k : Y \to K$ is the embedding. Then $f_1 = k \circ f : X \to K$ is B-singular and $X \cup_{f_1} K$ is Tychonoff. It is easy to see that $aX = X \cup_f Y$ is a subspace of $X \cup_{f_1} K$, hence aX is also Tychonoff. This completes the proof. $\square$

Lemma 5.3. *Let $aX = X \cup_f Y$ be a B-singular extension of X and suppose $\mathcal{V}$ be a locally finite family in Y. Then the family $\mathcal{V}_1 = \{V \cup f^{-1}(V) \mid V \in \mathcal{V}\}$ is locally finite in aX.*

Proof. Let $y \in Y$. There is an open neighborhood N_y of y in Y such that $N_y \cap V \neq \varnothing$ for only finitely many $V \in \mathcal{V}$. This implies that $N_y \cup f^{-1}(N_y)$ meets only finitely many members of $\mathcal{V}_1$. Let $x \in X$ and $y = f(x)$. Then $f^{-1}(N_y)$ is a neighborhood of x which meets only finitely many members of $\mathcal{V}_1$. $\square$

Theorem 5.4. *Let $aX = X \cup_f Y$ be a B-singular extension of X with respect to an M-boundedness $\mathcal{F}_X$. Suppose X and Y are metrizable. Then aX is metrizable.*

Proof. Since $X \cup_f Y$ is T_3 by [6], Theorem 4.11, we need only to prove that it admits a σ-locally finite basis. By hypothesis, $\mathcal{F}_X$ has a countable basis $\{M_k\}_{k \in \mathbf{N}}$, where M_k is open and $Cl_X(M_k)$ is bounded. We have $\bigcup_{k \in \mathbf{N}} M_k =$

X. Let $\mathcal{C} = \bigcup_{n \in \mathbf{N}} \mathcal{C}_n$ be a basis for $\mathcal{T}_X$, where every $\mathcal{C}_n$ is a locally finite family. Put

$$\mathcal{C}_n^k = \{C \cap M_k \mid C \in \mathcal{C}_n\}, \qquad n, k \in \mathbf{N}.$$

Similarly, let $\mathcal{U} = \bigcup_{n \in \mathbf{N}} \mathcal{U}_n$ be a basis of $\mathcal{T}_Y$, where every $\mathcal{U}_n$ is locally finite. For every $n, k \in \mathbf{N}$, let

$$\mathcal{U}_n^k = \{U \cup [f^{-1}(U) \setminus Cl_X(M_k)] \mid U \in \mathcal{U}_n\}.$$

We claim that

$$\mathcal{S} = \left(\bigcup_{n,k \in \mathbf{N}} \mathcal{C}_n^k \right) \cup \left(\bigcup_{n,k \in \mathbf{N}} \mathcal{U}_n^k \right)$$

is a σ-locally finite basis for aX. Let W be an open subset of aX and let x be in W. If $x \in X$, then $x \in M_k$ for some k and there is $C \in \mathcal{C}_n$, for some n, such that $x \in C \subset W$. Then $x \in C \cap M_k \subset W$, where $C \cap M_k \in \mathcal{C}_n^k$. If $x \in Y$, then there is $U \in \mathcal{U}_n$ for some n and $F = Cl_X(F) \in \mathcal{F}_X$ such that $x \in U \cup (f^{-1}(U) \setminus F) \subset W$. We have $F \subset M_k$ for some k, hence

$$x \in U \cup [f^{-1}(U) \setminus Cl_X(M_k)] \subset (U \cup (f^{-1}(U) \setminus F)) \subset W,$$

where $U \cup [f^{-1}(U) \setminus Cl_X(M_k)] \in \mathcal{U}_n^k$.
Every $\mathcal{C}_n^k$ is locally finite. In fact, for every $x \in X$, there is a neighborhood that meets only finitely many members of $\mathcal{C}_n$, hence of $\mathcal{C}_n^k$. If $x \in Y$, any basic neighborhood of x of the form $V \cup [f^{-1}(V) \setminus Cl_X(M_k)]$ meets no member of $\mathcal{C}_n^k$.
By Lemma 5.3, $\mathcal{U}_n^k$ is locally finite for every n, k. This completes the proof. $\square$

REFERENCES

[1] G. Beer, *On metric boundedness structures*, Set-Valued Anal. **7** (1999), 195–208.
[2] G. Beer, *On convergence to infinity*, Monathsh. Math. **129** (2000), 267–280.
[3] G. Beer, *Embeddings of bornological universes*, Set-Valued Anal. **16**, no. 4 (2008), 477–488.
[4] A. Caterino, G. D. Faulkner, M. C. Vipera, *Construction of compactifications using essential semilattice homomorphisms*, Proc. Amer. Math. Soc. **116** (1992), 851–860.
[5] A. Caterino and S. Guazzone, *Extensions of unbounded topological spaces*, Rend. Sem. Mat. Univ. Padova **100** (1998), 123–135.
[6] A. Caterino, F. Panduri and M. C. Vipera, *Boundedness, one-point extensions and B-extensions*, Math. Slovaca. **58**, no. 1 (2008), 101–114.
[7] R. E. Chandler and G. D. Faulkner, *Singular compactifications: the order structure*, Proc. Amer. Math. Soc. **100** (1987), 377–382.
[8] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
[9] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.
[10] S. T. Hu, *Boundedness in a topological space*, J. Math. Pures Appl. **28** (1949), 287–320.
[11] S. T. Hu, *Introduction to general topology*, Holden-Day inc., San Francisco, 1969.
[12] M. C. Vipera, *Some results on sequentially compact extensions*, Comment. Math. Univ. Carolinae **39** (1998), 819–831.

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