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Hausdorff closed extensions of pre-uniform spaces

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Abstract

The family of densely finite open covers of a Hausdorff space X determines a completable pre-uniformity on X and the canonical completion $\widehat{X}$ is Hausdorff closed. We compare $\widehat{X}$ with the Katetov extension kX of X and give sufficient conditions for the non-equivalence of kX and $\widehat{X}$.

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1. Preliminary results

For the sake of convenience to the reader, we recall some definitions. They also appear in [2].

A filter $\mathcal{T}$ in a pre-uniform space $(X,\mathcal{U})$ is $\mathcal{U}$-*Cauchy* if for every cover $\alpha \in \mathcal{U}$, we have $\mathcal{T} \cap \alpha \neq \varnothing$.

A $\mathcal{U}$-Cauchy filter $\mathcal{T}$ in a pre-uniform space is $\mathcal{U}$-*round* if for every $F_0 \in \mathcal{T}$, there exists a cover $\alpha \in \mathcal{U}$ such that $S^*_T(\mathcal{T}, \alpha) \subset F_0$, where :

$$S^*_T(\mathcal{T}, \alpha) = \bigcup \{A \in \alpha | A \cap F \neq \varnothing \text{ for every } F \in \mathcal{T}\} .$$

$\mathcal{U}$-round filters $\mathcal{T}$ in a Hausdorff pre-uniform space $(X,\mathcal{U})$ satisfy the following conditions: (See [2, Theorem 3.8.4 and 3.8.5]) .

1) For every $p \in X$, $\mathcal{T}$ adheres to p if and only if $\mathcal{T}$ converges to p.
2) Every neighborhood filter is $\mathcal{U}$-round

As a consequence of 1), in Hausdorff pre-uniform spaces, a $\mathcal{U}$-round filter $\mathcal{T}$ is either non-adherent or converges to a unique point.

An *ultrafilter of open sets* in a topological space (X, τ) is a non-empty subfamily $\mathcal{G}$ of $\tau - \{\varnothing\}$ satisfying :

1) If $G_1, G_2 \in \mathcal{G}$, also $G_1 \cap G_2 \in \mathcal{G}$;
2) If $G \in \mathcal{G}$ and $G \subseteq H$, where $H \in \tau$, then $H \in \mathcal{G}$;
3) If $G_0 \in \tau$ and $G_0 \cap G \neq \varnothing$ for every $G \in \mathcal{G}$, then $G_0 \in \mathcal{G}$.

Likewise $\mathcal{U}$-round filters, an ultrafilter of open sets in a Hausdorff space X is either non-adherent or converges to a unique point.

Hausdorff closed spaces are characterized by the property : ([1, p.283])

*) Every ultrafilter of open sets is convergent.

An open cover α of a topological space X is *densely finite* if there exists a finite subfamily $\{A_1, A_2, \dots, A_n\} \subseteq \alpha$ such that $X = A_1^- \cup A_2^- \cup \cdots \cup A_n^-$.

The family $\mathcal{U}$ of densely finite covers of a Hausdorff space (X, τ) constitutes a compatible pre-uniform basis which satisfies the condition :

**) Every $\mathcal{U}$-Cauchy filter contains a $\mathcal{U}$-round filter.

By [5], $(X, \mathcal{U})$ has a canonical completion $(\widehat{X}, \widehat{\mathcal{U}})$ and the topology $\tau_{\widehat{\mathcal{U}}}$ is Hausdorff closed. $\widehat{X}$ consists of all the $\mathcal{U}$-round filters and $\widehat{\mathcal{U}}$ consists of all the extension covers $\widehat{\alpha}$ $(\alpha \in \mathcal{U})$, where $\widehat{\alpha} = \left\{\widehat{A} \mid A \in \alpha\right\}$ and $\widehat{A} = \left\{\xi \in \widehat{X} \mid A \in \xi\right\}$. The canonical embedding $h\colon X \to \widehat{X}$ assigns to each $p \in X$, its neighborhood filter μ_p.

Theorem 2.6 in [3] establishes that a non-adherent filter $\mathcal{T}$ in $(X, \mathcal{U})$ is $\mathcal{U}$-round if and only if $\mathcal{T}$ has as a basis an ultrafilter of open sets.

Besides the completion $(\widehat{X}, \widehat{\mathcal{U}})$, (X, τ) has its Katetov extension kX, where:

$$kX = X \cup \{\mathcal{G} \mid \mathcal{G} \text{ is a non-adherent ultrafilter of open sets}\}$$

If $p \in X$, a neighborhood basis of p is the filter μ_p of τ-neighborhoods of p. If $\mathcal{G} \in kX - X$, a neighborhood basis of $\mathcal{G}$ consists of all the sets $\{\mathcal{G}\} \cup G$, where $G \in \mathcal{G}$.

The resulting topology of kX turns out to be Hausdorff closed and $kX - X$ is a closed discrete subspace without interior points, and hence X is open and dense in kX.

We wonder what is the relation between kX and $\widehat{X}$.

We recall first some definitions :

A subset A of a topological space X is *C-bounded* (or *relatively pseudocompact*) if for every continuous function $\varphi\colon X \to \mathbb{R}$, $\varphi(A)$ is bounded.

$A \subseteq X$ is *C-discrete* (with respect to X) if for each $a \in A$, there exists an open set U_a such that $a \in U_a$ and the family $\{U_a \mid a \in A\}$ is discrete (with respect to X).

The following equivalence is well known (see, for instance [4] : 4.73.3).

A subset A of a Tychonoff space X is C-bounded if and only if every C-discrete subset of X contained in A is finite.

An open set U in a topological space X is *wide* if there exist two open sets W_1, W_2 such that :

1) $W_1 \cup W_2 \subseteq U$;
2) $W_1 \cap W_2 = \varnothing$;
3) W_1^- and W_2^- are non-compact.

For instance, every non-empty open set U in a nowhere locally compact regular space is wide.

We also have :

Lemma 1.1. *Every open set U in a Tychonoff space which is not C-bounded, is wide.*

Proof. By hypothesis, there exists an infinite discrete family of open sets $W_1, W_2, \ldots$, where $W_i^- \subseteq U$ for every i. If $S = \bigcup_{i=1}^{\infty} W_{2i-1}$ and $T = \bigcup_{i=1}^{\infty} W_{2i}$, we have $S^- \cup T^- \subseteq U$, $S \cap T = \varnothing$ and none of the sets S^-, T^- is compact. □

2. Main result

We give a sufficient condition on a Tychonoff space X which insures that the extensions $\widehat{X}$ and kX are non-equivalent.

Theorem 2.1. *Let X be a non-compact Tychonoff space where every open set with non-compact closure is wide. Then $\widehat{X} - h(X)$ is dense in itself, where $h\colon (X,\mathcal{U}) \to (\widehat{X},\widehat{\mathcal{U}})$ is the canonical embedding of X into $\widehat{X}$.*

Proof. Let us take any element $\xi \in \widehat{X} - X$ (we identify each point $p \in X$ with its neighborhood filter). Let U be an open set in X such that $\xi \in \widehat{U}$. Therefore, $U \in \xi$. Since the round filter ξ is non-adherent, U^- cannot be compact. By hypothesis, U is wide. Let S, T be open sets such that $S \cup T \subseteq U$, $S \cap T = \varnothing$ and S^-, T^- are both non-compact. By [1, p. 283], S^- and T^- cannot be Hausdorff closed. Hence there exist non-adherent ultrafilters of open sets μ_1, μ_2 in S^-, T^-, respectively. Hence the restrictions $\mu_1 \,|\, S$ and $\mu_2 \,|\, T$ are non-adherent filterbases consisting of open sets in X. Take ultrafilters of open sets ξ_1, ξ_2 in X containing $\mu_1 \,|\, S$ and $\mu_2 \,|\, T$, respectively. Clearly ξ_1 and ξ_2 are also non-adherent and U belongs to both of them. Therefore, at least one of the round filters ξ_1^+, ξ_2^+ is different from ξ. Therefore, $\widehat{U} \cap (\widehat{X} - X)$ consists of more than one element and $\widehat{X} - X$ is dense in itself. □

Corollary 2.2. *Every normal Hausdorff metacompact space X satisfies the condition in the theorem.*

Proof. Let $U \subseteq X$ be an open set whose closure is non compact. By [4, 4.74.5], the subspace U^- cannot be pseudocompact and hence U cannot be C-bounded. □

Corollary 2.3. *t If X is paracompact and T_2, then $\widehat{X} - X$ is dense in itself, and hence the extensions kX and $\widehat{X}$ are non-equivalent (unless X is compact).*

Lemma 2.4. *Let U be an open set in a regular Hausdorff space X and let $\xi \in \widehat{U} \cap (\widehat{X} - X)$. Then U is wide if and only if $(\widehat{U} - \{\xi\}) \cap (\widehat{X} - X) \neq \varnothing$. Hence, if U is not wide, we have $\{\xi\} = \widehat{U} \cap (\widehat{X} - X)$.*

Proof. Reason as in Theorem 2.1. □

Example 2.5. Let X be the space of countable ordinals with the order topology. Then every uncountable open set in X is wide and hence $\widehat{X} - X$ is dense in itself.

Proof. Let D be the set of non-limit ordinals in X. Then D is open, discrete and dense in X. If $U \subseteq X$ is an uncountable open set in X, then $U \cap D$ is also uncountable (because otherwise $U^- = (U \cap D)^-$ would be compact and hence U would be countable). Clearly, $U \cap D$ is the union of two uncountable disjoint subsets. Hence, U is wide. □

Example 2.6. The half disk $X = \{(p, q) \in \mathbb{R}^2 \,|\, p^2 + q^2 \leq 1, q > 0\}$ has a non-compact Hausdorff closed extension Z whose remainder $Z - X$ is closed and discrete. However Z is not equivalent to $\widehat{X}$ neither to kX.

Proof. Let $Z = X \cup \{(z, 0) \,|\, -1 \leq z \leq 1\}$. For each $(z, 0) \in Z - X$, define μ_z be the set of unions of $\{(z, 0)\}$ with upper half open disks in $\mathbb{R}^2$ centered at $(z, 0)$ and intersected with X. If $z \in X$, μ_z consists of all open disks in $\mathbb{R}^2$ centered at z and intersected with X.

We can now topologize Z with the help of these filter bases μ_z and convert it into a Hausdorff, non-regular, extension of X. To see that Z is Hausdorff closed, we consider a cover of Z consisting of elements of the filterbases μ_z. Since

$$\{(p, q) \in \mathbb{R}^2 \,|\, p^2 + q^2 \leq 1, q \geq 0\}$$

is compact in the usual topology of $\mathbb{R}^2$, we could get a finite subcover for this space if we adjoin to the elements of μ_z ($z \in Z - X$) their radii in the X-axis. Therefore, the original cover has a finite subfamily which covers X (recall X is dense in Z). This argument proves that every open cover of Z is densely finite, and hence Z is Hausdorff closed (see [1]). Clearly the remainder $Z - X$ is closed and discrete. For each point $z \in Z - X$, we can find an infinite family of ultrafilters of open sets in X which have z as a convergence point. This remark proves that Z is not equivalent to $\widehat{X}$ neither to kX, because in these extensions, every point of the remainder is the convergence point of a unique ultrafilter of open sets in X. □

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