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On cofree S-spaces and cofree S-flows

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Abstract

Let **S-Tych** *be the category of Tychonoff S-spaces for a topological monoid S. We study the cofree S-spaces and cofree S-flows over topological spaces and we prove that for any topological space X and a topological monoid S, the function space $C(S,X)$ with the compact-open topology and the action $s \cdot f = (t \mapsto f(st))$ is the cofree S-space over X if and only if the compact-open topology is admissible and Tychonoff. Finally we study injective S-spaces and we characterize injective cofree S-spaces, when the compact-open topology is admissible and Tychonoff. As a consequence of this result, we characterize the cofree S-spaces and cofree S-flows, when S is a locally compact topological monoid.*

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1. Introduction and Preliminaries

There are many works about S-spaces or more specially G-spaces and their applications, and some authors study the free and projective S-spaces (G-spaces) and their applications [1, 7, 10, 11, 13, 14, 15, 16, 18]. Also, there are some results about injective and cofree Boolean S-spaces (see [1]).

Recall that, for a monoid S, a set A is a left *S-set* (or *S-act*) if there is, so called, an *action* $\mu : S \times A \to A$ such that, denoting $\mu(s,a) := sa$, $(st)a = s(ta)$ and $1a = a$. The definitions of an *S-subset* A of B and an *S-homomorphism* (also called S-map) between S-sets are clear. In fact S-maps are action-preserving maps: $f : A \to B$ with $f(sa) = sf(a)$, for $s \in S$, $a \in A$. Each monoid S can be considered as an S-set with the action given by its multiplication. Let S be a monoid and A be an S-set. Recall that for $s \in S$, the

S-homomorphism $\lambda_s : A \to A$ is defined by $y \mapsto sy$ for any $y \in A$. Similarly, for $a \in A$, the S-map $\rho_a : S \to A$ is defined by $t \mapsto ta$ for any $t \in S$.

Let $\mathcal{C}$ be a concrete category over $\mathcal{D}$ and $U : \mathcal{C} \to \mathcal{D}$ be the forgetful functor. An object K in $\mathcal{C}$ with a morphism $\psi : K \to D$ in $\mathcal{D}$, where $D \in \mathcal{D}$, is *the cofree object over* D, if for every morphism $f : C \to D$ in $\mathcal{D}$ there exists a unique morphism $\tilde{f} : C \to K$ in $\mathcal{C}$ such that $\psi \circ \tilde{f} = f$ in $\mathcal{D}$.

For any two topological spaces X and Y, we denote the set of all continuous maps from X to Y by $C(X,Y)$. If τ is a topology on the set $C(X,Y)$, then the corresponding space is denoted by $C_\tau(X,Y)$. The category of all Tychonoff spaces is denoted by **Tych**.

Note that all of the spaces in this note are Tychonoff (completely regular and Hausdorff). A monoid S with a Hausdorff topology τ_S such that the multiplication $\cdot : S \times S \to S$ is (jointly) continuous, is called a *topological monoid*. For a topological monoid S, an *S-space* is an S-set A with a topology τ_A such that the action $S \times A \to A$ is (jointly) continuous. The category of all Tychonoff S-spaces with continuous S-maps is denoted by **S-Tych** (see [14, 15, 16, 18]). A compact Hausdorff S-space is called an *S-flow* (see [2, 13]).

Let Y and Z be two topological spaces. A topology on the set $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $g : X \times Y \to Z$ implies that of the map $\widetilde{g} : X \to C_\tau(Y,Z)$ defined by $\widetilde{g}(x)(y) = g(x,y)$, for every $x \in X$ and $y \in Y$ (this topology is also called *proper* [3, 8] or *weak* [6]). A topology τ on $C(Y,Z)$ is *admissible* if the mapping $\omega(y,f) := f(y)$ from $Y \times C(Y,Z)$ into Z is continuous in y and f. Equivalently, a topology τ on $C(Y,Z)$ is admissible if for every topological space X, the continuity of an $f : X \to C_\tau(Y,Z)$ implies the continuity of $\widehat{f} : X \times Y \to Z$, where $\widehat{f}(x,y) := f(x)(y)$ for every $(x,y) \in X \times Y$ (see [8]) (the latter definition is usually used as the definition of admissible topology, but we use the former). A topological space Y is said to be exponential if for every space X there is an admissible and splitting topology on $C(Y,X)$ (see [6]). For any topological spaces X and Y, we denote the set $C(X,Y)$ with the compact-open topology by $C_{co}(X,Y)$. For any compact subset K of X and an open set U in Y, by (K,U) we mean the set $\{f \in C(X,Y) | f(K) \subseteq U\}$. For topological topics and facts about Stone-Cech compactification, we refer to [3, 9, 17].

In this note, we study the cofree and injective S-spaces and S-flows. Recall that for a set E and a monoid S, E^S, the set of all functions from S to E with the action $sf := (t \mapsto f(st))$, for any function $f : S \to E$ and $s \in S$, is the cofree S-set over E (see [12]). In Section 2, we study the cofree S-spaces and S-flows over topological spaces. As a consequence of these results, we characterize the cofree S-spaces and the cofree S-flows over topological spaces, when the compact-open topology is admissible and Tychonoff (more specially, when S is locally compact). Finally, in Section 3, we study injective cofree S-spaces and S-flows over topological spaces, when the compact-open topology is admissible and Tychonoff. Note that we state and prove our results for topological monoids and plainly all of our results hold for topological groups and G-spaces.

2. The Cofree S-spaces and S-flows over a topological space

One of the main steps in the study of injective objects in a category is the study of cofree objects. These objects can be used for presenting injective cover for objects in a category[1]. In this section, first we study the cofree Tychonoff S-spaces over a Tychonoff space, then we study the cofree S-flow over a compact space[2]. Finally in this section, as a consequence of these results, we will show that if S is a locally compact topological monoid, then the cofree S-space and the cofree S-flow exist and we present them explicitly.

Remark 2.1. It is a known fact that the cofree S-set over a set E, is the set E^S of all functions from S to E with the action defined by $s \cdot f := (t \mapsto f(st))$ for all $f \in E^S$, $s \in S$ and $t \in S$. Let E and D be two sets. Recall that for an S-set A and any function $h : A \to E$, the S-homomorphism $\overline{h} : A \to E^S$ defined by $\overline{h}(a) := h \circ \rho_a$ is the unique S-map such that $\psi \circ \overline{h} = h$, where $\psi : E^S \to E$ is defined by $\psi(f) := f(1)$, for any $f \in E^S$ (see [12]).

Remark 2.2. (i) Let S be a topological monoid and X be a topological space. The S-space $S \times X$ with the product topology and the action $\lambda_1 : S \times (S \times X) \to S \times X$, $t(s, x) = (ts, x)$, is denoted by $L(X)$. For a topological space X, the S-space X with the trivial action, $sx := x$, is denoted by $T(X)$.

(ii) Note that for any topological space X and a non-empty topological space Y, if we define $c_x(y) := x$ for every $y \in Y$, and $C := \{c_x | x \in X\}$, then C as a subspace of $C_{co}(Y, X)$ is homeomorphic to X. So the function $j : X \to C_{co}(Y, X)$ defined by $j(x) := c_x$ is an embedding from X to $C_{co}(Y, X)$. From now on, we denote this embedding by j_X for any topological space X.

By Theorem 2.9 in [6], we have

Remark 2.3. Let X be a topological space. Then the following are equivalent

(a) X is exponential;
(a) For every space Y, there exists a splitting and admissible topology on $C(X, Y)$;
(c) X is core compact.

Note that for Hausdorff spaces (and more generally for sober spaces) core compactness is the same as local compactness (see [5]). Furthermore, it is a known fact that if X is locally compact, then the compact-open topology on $C(X, Y)$ is admissible and splitting.

[1] Note that the cofree objects in an arbitrary category are not injective in general.

[2] One can easily see that if the cofree S-flow exists over a space X, then X is compact. So this assumption is not an extra assumption.

Theorem 2.4. *Let S be a topological monoid. Then the following are equivalent*

(a) *For every Tychonoff space X, the compact-open topology on $C(S, X)$ is admissible and Tychonoff;*
(b) *For every space X, $C_{co}(S, X)$ is Tychonoff and $C_{co}(S, X)$ with the action defined by $sf = (s' \mapsto f(ss'))$ is the cofree S-space over X.*

Proof. **(a) $\Rightarrow$ (b)** Let X be a topological space. First we show that $C_{co}(S, X)$ with its introduced action is an S-space, then we show that it has the cofree universal property. First, note that since S is a topological monoid, the action is well-defined. Let $f \in C(S, X)$, $s \in S$ and (K, U) be subbasis element for $C_{co}(S, X)$ containing sf. Therefore for any $k \in K$, $f(sk) = (sf)(k) \in U$. Since by the assumption, the compact-open topology on $C(S, X)$ is admissible and since for any $k \in K$, we have $\omega(sk, f) \in U$, there exist open neighborhoods O_f and W_{sk} of f and sk, in $C_{co}(S, X)$ and S, respectively such that $\omega(W_{sk}, O_f) \subseteq U$. On the other hand, since S is a topological monoid, for $sk \in W_{sk}$, there exist open sets W_s^k and W_k in S which contain $\{s\}$ and $\{k\}$, respectively and $W_s^k \cdot W_k \subseteq W_{sk}$. Since K is compact and obviously $\{W_k\}_{k \in K}$ forms an open cover for K, there exist $k_1, \cdots, k_n$ in K such that $K \subseteq \cup_{i=1}^n W_{k_i}$. Define $W_s := \cap_{i=1}^n W_s^{k_i}$. Clearly for $W_s \in \tau_S$ we have

$$\omega(W_s \cdot K, O_f) \subseteq \omega(W_s \cdot (\cup_{i=1}^n W_{k_i}), O_f) \subseteq U \Rightarrow sf \in W_s O_f \subseteq (K, U).$$

Hence $C_{co}(S, X)$ with its introduced action is an S-space.

Now we prove the universal property. First note that the function $\psi : C_{co}(S, X) \to X$ is continuous. Let (A, τ_A) be an S-space and $h : (A, \tau_A) \to X$ be continuous. We show that for any S-space (A, τ_A) and a continuous function $h : (A, \tau_A) \to X$, the function $\overline{h} : (A, \tau_A) \to C_\tau(S, X)$ defined by $\overline{h}(a) = (s \mapsto h(sa))$ is continuous and $\psi \circ \overline{h} = h$. First, note that for every $a \in A$ and $s \in S$, $\overline{h}(a)(s) = h(sa) = h \circ \rho_a(s)$, and since (A, τ_A) is an S-space, for every $a \in A$, $\overline{h}(a) \in C(S, X)$, so $\overline{h}$ is a well defined function. Consider the continuous function

$$\begin{array}{lccccc} h \circ \lambda : & S \times A & \to & A & \to & X \\ & (s, a) & \mapsto & sa & \mapsto & h(sa) \end{array}$$

Since τ is splitting, the function $\widetilde{(h \circ \lambda)} : A \to C_\tau(S, X)$, where $\widetilde{(h \circ \lambda)}(a) := (s \mapsto (h \circ \lambda)(s, a))$, is continuous. Therefore, $\overline{h}$ is continuous. Hence $C_{co}(S, X)$ with its introduced action is the cofree S-space over X.

(b) $\Rightarrow$ (a) Let λ denote the action of the cofree S-space over X. It is a known fact that the compact-open topology is splitting. On the other hand, since $C_{co}(S, X)$ with λ is an S-space and since $\psi : C_{co}(S, X) \to X$ is continuous, $\omega = \psi \circ \lambda$ is continuous. Therefore the compact-open topology is admissible. Therefore, the compact-open topology is admissible and Tychonoff. □

As a quick consequence of the above theorem, we have

Corollary 2.5. *Let S be a locally compact topological monoid. Then for any Tychonoff space X, $C_{co}(S,X)$ with the action defined by $sf = f \circ \lambda_s$ is the cofree S-space over X.*

Proof. Since S is locally compact, by Remark 2.3, the compact-open topology on $C(S,X)$ is admissible. On the other hand, by [5, Corollary 3.8], $C_{co}(S,X)$ is Tychonoff. So by the above theorem we have the result. □

Theorem 2.6. *Let S be a completely regular topological monoid*[3]. *Then the following are equivalent:*

(a) *for every compact space X, the compact-open topology on $C(S,X)$ is admissible and Tychonoff;*
(b) *for every compact space X, $C_{co}(S,X)$ is completely regular and there exists an action $\widetilde{\lambda} : S \times \beta(C_{co}(S,X)) \to \beta(C_{co}(S,X))$ such that $\widetilde{\lambda}|_{C_{co}(S,X)}$ coincides with the action of $C_{co}(S,X)$ and $\beta(C_{co}(S,X))$ is the cofree S-flow over the space X.*

Proof. (a)⇒ (b) Let S be a topological monoid such that the compact-open topology on $C(S,X)$ is admissible, for every compact space X. Let X be a compact space and let λ denote the action of the cofree S-space $C_{co}(S,X)$. First, we introduce $\widetilde{\lambda}$ and we show that it is continuous, then we prove the universal property.

Since $S \times C_{co}(S,X)$ is Tychonoff, $\beta(S \times C_{co}(S,X))$ exists. By the characteristic of the Stone-Cech compactification, for the continuous action $\lambda : S \times C_{co}(S,X) \to C_{co}(S,X)$, there exists a continuous function $\overline{\lambda} : \beta(S \times C_{co}(S,X)) \to \beta(C_{co}(S,X))$ such that $\overline{\lambda}|_{S\times C_{co}(S,X)} = \lambda$. Fix an arbitrary $t \in S$ and define $k : C_{co}(S,X) \to S \times C_{co}(S,X)$ as follows $k(f) := (t,f)$, for every $f \in C_{co}(S,X)$. Consider the closure of $k(C_{co}(S,X))$ in $\beta(S \times C_{co}(S,X))$. It is obvious that there exists a compact space B such that the closure of $k(C_{co}(S,X))$ in $\beta(S \times C_{co}(S,X))$ is equal to $\{t\} \times B$. Again by the characteristic of the Stone-Cech compactification, there exists a continuous function $h : \beta(C_{co}(S,X)) \to B$ such that $h \circ i = k$, where i is the natural inclusion map from $C_{co}(S,X)$ to $\beta(C_{co}(S,X))$. Define $\lambda' := \overline{\lambda}|_{S\times B}$. Now we define $\widetilde{\lambda} := \lambda' \circ (id_S \times h) : S \times \beta(C_{co}(S,X)) \to S \times B \to \beta(C_{co}(S,X))$ and we show that $\widetilde{\lambda}$ is an action. Let $b \in \beta(C_{co}(S,X))$ and $s,s' \in S$. Then since $g \in \beta(C_{co}(S,X))$, there exists a net $(f_\alpha) \subseteq C_{co}(S,X)$ such that $f_\alpha \to g$.

$$\widetilde{\lambda}(ss',g) = \widetilde{\lambda}(ss', lim_\alpha f_\alpha) = lim_\alpha \lambda'(ss', k(f_\alpha)) = lim_\alpha \lambda(ss', f_\alpha)$$
$$= lim_\alpha \lambda(s, \lambda(s', f_\alpha)) = \widetilde{\lambda}(s, \widetilde{\lambda}(s',g)).$$

Therefore $\widetilde{\lambda}$ is a continuous action and $\beta(C_{co}(S,X))$ with action $\widetilde{\lambda}$ is an S-flow. Now we prove the universal property. First, let $\widetilde{\psi} : \beta(C_{co}(S,X)) \to X$ be the continuous extension of $\psi : C_{co}(S,X) \to X$ which exists by the characteristic of the Stone-Cech compactification. To prove the universal property, we show that for any S-flow (F,τ_F) and a continuous function $l : (F,\tau_F) \to X$, there

[3]Clearly for a topological group, this assumption is not necessary.

exists a continuous S-map $\overline{l} : (F, \tau_F) \to \beta(C_{co}(S, X))$ such that $\widetilde{\psi} \circ \overline{l} = l$. Let (F, τ_F) be an S-flow and let $l : (F, \tau_F) \to X$ be a continuous function. Since by Theorem 2.4, $C_{co}(S, X)$ with action λ is the cofree S-space over X, there exists a continuous S-map $\overline{l} : (F, \tau_F) \to C_{co}(S, X)$ such that $\psi \circ \overline{l} = l$. Since $\widetilde{\psi}|_{C_{co}(S,X)} = \psi$, we have clearly $\widetilde{\psi} \circ \overline{l} = l$. Therefore $\beta(C_{co}(S, X))$ with action $\widetilde{\lambda}$ is the cofree S-flow over the space X.

(b) $\Leftarrow$ (a) Suppose that there exists a continuous action $\widetilde{\lambda}$ on $\beta(C_{co}(S, X))$ such that $\beta(C_{co}(S, X))$ with this action is an S-flow and $\widetilde{\lambda}|_{C_{co}(S,X)} = \lambda$, where λ is the action of $C_{co}(S, X)$. Therefore $C_{co}(S, X)$ is an S-space. On the other hand, since $\psi : C_{co}(S, X) \to X$ is continuous, $\omega = \psi \circ \lambda : S \times C_{co}(S, X) \to C_{co}(S, X) \to X$ is continuous. Therefore the compact-open topology on $C(S, X)$ is admissible and Tychonoff. □

Recall that for a topological group G, a G-space $(A.\tau_A)$ is called *G-compactificable* or *G-Tychonoff*, if (A, τ_A) is a G-subspace of a G-flow (compact G-space) (see [14, 15]). Since S is locally compact, it is a Tychonoff space. Therefore by the above theorem, we have

Corollary 2.7. *(a) Let S be a locally compact topological monoid. Then for any compact space X, there exists an action $\widetilde{\lambda}$ on $\beta(C_{co}(S, X))$ such that $\beta(C_{co}(S, X))$ with this action is the cofree S-flow over the space X.*

(b) Let G be a locally compact topological group and X be a topological space. Then the cofree G-space over X is G-compactificable.

3. Injective cofree S-spaces and S-flows

Recall that by an embedding of topological spaces (S-spaces) we mean a homeomorphism (homeomorphism S-map) onto a subspace (an S-subset). A topological space (S-space) Z is called injective over an embedding of topological spaces (of S-spaces) $j : X \hookrightarrow Y$ if any continuous map (continuous S-homomorphism) $f : X \to Z$ extends to a continuous map (continuous S-homomorphism) $\overline{f} : Y \hookrightarrow Z$ along j. A space (an S-space) is injective if it is injective over embeddings (see [5]).

Proposition 3.1. *Let S be a completely regular topological monoid and (E, τ) be an S-space. If (E, τ) is an injective S-space, then $(|E|, \tau)$ is injective in* **Tych**.

Proof. Suppose that we are given the following diagram in **Tych**

$$\begin{array}{ccc} & & i & \\ & X & \hookrightarrow & Y \\ f & \downarrow & & \\ & (E,\tau) & & \end{array}$$

where X and Y are topological spaces and f is a continuous function. Now consider the following diagram

$$\begin{array}{cccc} & & id \times i & \\ & L(X) & \hookrightarrow & L(Y) \\ id \times f & \downarrow & & \\ & S \times E & \swarrow & \\ \lambda & \downarrow & h & \\ & E & & \end{array}$$

Since (E, τ) is an injective S-space, there exists a continuous S-map h from $L(Y)$ to (E, τ) such that $h(id \times i) = \lambda(id \times f)$. (Note that $\lambda(id \times f)$ is a continuous S-homomorphism.)

Note that for any topological space Z, the spaces Z and $Z \times \{1\}$ with the product topology are homeomorphic. Furthermore, we have $(id \times i)|_{\{1\}\times X} : \{1\} \times X \hookrightarrow \{1\} \times Y$, and

$$\lambda \circ (id \times i)(1, x) = h \circ (id \times i)(1, x) = f(x).$$

Now, define $h' := h|_{\{1\}\times Y} : \{1\} \times Y \to E$ and consider the following diagram

$$\begin{array}{ccccccc} & & (\mathrm{id} \times \mathrm{i})|_{\{1\}\times \mathrm{X}} & & & & \\ & \{1\} \times \mathrm{X} & \longrightarrow & \{1\} \times \mathrm{Y} & \subseteq & S \times Y \\ \mathrm{g}_1 & \downarrow & & \downarrow g_2 & & \\ & \mathrm{X} & \longrightarrow & Y & & \\ \mathrm{f} & \downarrow & i & & & \\ & \mathrm{E} & & & & \end{array}$$

where g_1 and g_2 are the following homeomorphisms

$$\begin{array}{ccc} g_1 : \{1\} \times X \to X, & \text{and} & g_2 : \{1\} \times Y \to Y \\ (1, x) \mapsto x & & (1, y) \mapsto y. \end{array}$$

Since g_1 and g_2 are homeomorphisms and, $h'((id\times i)|_{\{1\}\times X}) = \lambda((id\times f)|_{\{1\}\times X}) = fg_1$, we have

$$h' \circ (id \times i)|_{\{1\}\times X} \circ g_1^{-1} = f. \quad (I)$$

On the other hand, since the rectangular in the above diagram is commutative, we have

$$(id \times i)|_{\{1\}\times X} \circ g_1^{-1} = g_2^{-1} \circ i. \quad (II)$$

Now, define $f^\circ := h' \circ g_2^{-1}$. Clearly, by the Relations (I) and (II), f° is a continuous function from Y to E such that

$$f^\circ i = h' g_2^{-1} i = h' \circ (id \times i)|_{\{1\}\times X} \circ g_1^{-1} = f.$$

□

similarly, we can prove that

Proposition 3.2. *Let S be a compact topological monoid and (F, τ_F) be an S-flow. If (F, τ_F) is an injective S-flow, then $(|F|, \tau_F)$ is injective in the category of compact Hausdorff spaces.*

It is known that the cofree S-spaces are not injective in general. In the next proposition, we characterize the injective cofree S-spaces when S is a locally compact topological monoid.

Proposition 3.3. *For a locally compact monoid S and a topological space X, the cofree S-space over X, $C_{co}(S,X)$ is injective in* **S-Tych** *if and only if X is injective in* **Tych**.

Proof. ($\Rightarrow$) Suppose that we are given the following diagram in **Tych**

$$\begin{array}{ccc} Z & \hookrightarrow & Y \\ {\scriptstyle f}\downarrow & & \\ X & & \end{array}$$

Consider the following diagram in **S-Tych**.

$$\begin{array}{ccc} T(Z) & \hookrightarrow & T(Y) \\ {\scriptstyle j_X \circ f}\downarrow & & \\ C_{co}(S,X) & & \end{array}$$

Since $C_{co}(S,X)$ is injective, there exists a continuous S-homomorphism $h : T(Y) \to C_{co}(S,X)$ such that $h \circ i = j_X \circ f$. Therefore, $f = \psi \circ h \circ i$. Take $k := \psi \circ h$. Hence $f = k \circ i$ and X is injective.

($\Leftarrow$) Suppose that we are given $i : (A, \tau_A) \hookrightarrow (B, \tau_B)$ and $f : (A, \tau_A) \to C_{co}(S,X)$ for two S-spaces (A, τ_A) and (B, τ_B). Consider the following diagram

$$\begin{array}{cccc} & & i & \\ & (A,\tau_A) & \hookrightarrow & (B,\tau_B) \\ f & \downarrow & & \\ & \mathrm{C_{co}(S,X)} & & \\ \psi & \downarrow & & \\ & X & & \end{array}$$

Since X is injective in **Tych**, there exists $g : (B, \tau_B) \to X$ such that $g \circ i = \psi \circ f$. Since $C_{co}(S,X)$ is the cofree S-space over X, there exists $h : (B, \tau_B) \to C_{co}(S,X)$ such that $\psi \circ h = g$. We claim that $h \circ i = f$. Clearly we have $\psi \circ h \circ i = \psi \circ f$. So for every $a \in A$ and $s \in S$, we have $h \circ i(a)(s) = h \circ i(a) \circ \lambda_s(1) = \psi(h \circ i(a) \circ \lambda_s) = \psi(f(a) \circ \lambda_s) = f(a) \circ \lambda_s(1) = f(a)(s)$. Hence, $h \circ i = f$, as we wanted. So $C_{co}(S,X)$ is an injective S-space. Hence $C_{co}(S,X)$ is injective in **Tych**. □

Similarly we have

Proposition 3.4. *For a completely regular monoid S and a compact Hausdorff space X, the cofree S-flow over X, $\beta(C_{co}(S,X))$ is injective in the category of S-flows if and only if X is injective in the category of compact Hausdorff spaces.*

As an immediate result of Propositions 3.1 and 3.3, we have

Proposition 3.5. *Let S be a locally compact monoid. $C_{co}(S,X)$ is injective in* **Tych** *if and only if $C_{co}(S,X)$ is injective in* **S-Tych**.

Proof. ($\Leftarrow$) Since for any space Z, $T(Z)$ is an S-space, the result is obvious.

($\Rightarrow$) Suppose that $C_{co}(S,X)$ is injective in **Tych** and we are given $i : (A,\tau_A) \hookrightarrow (B,\tau_B)$ and $f : (A,\tau_A) \to C_{co}(S,X)$ for two S-spaces (A,τ_A) and (B,τ_B). Since $C_{co}(S,X)$ is injective in **Tych**, there exists a continuous function $g : (B,\tau_B) \to C_{co}(S,X)$ such that $g \circ i = f$.

$$\begin{array}{cccc} & & \mathrm{i} & \\ & (A,\tau_A) & \hookrightarrow & (B,\tau_B) \\ f & \downarrow & \swarrow_g & \\ & C_{co}(S,X) & & \\ \psi & \downarrow & & \\ & \mathrm{X} & & \end{array}$$

Since $C_{co}(S,X)$ is the cofree S-space over X and $\psi \circ g : (B,\tau_B) \to X$ is continuous, there exists a continuous S-homomorphism $h : (B,\tau_B) \to C_{co}(S,X)$ such that $\psi \circ h = \psi \circ g$. Clearly we have $\psi \circ h \circ i = \psi \circ g \circ i$. So, by the same argument as the proof of Proposition 3.3, we have $h \circ i = f$. Hence $C_{co}(S,X)$ is injective in **S-Tych**. □

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