Applied General Topology
© Universidad Politécnica de Valencia
Volume 13, no. 2, 2012
pp. 151-166

Common fixed points for generalized (ψ, ϕ)-weak contractions in ordered cone metric spaces

Hemant Kumar Nashine and Hassen Aydi

Abstract

The purpose of this paper is to establish coincidence point and common fixed point results for four maps satisfying generalized (ψ, ϕ)-weak contractions in partially ordered cone metric spaces. Also, some illustrative examples are presented.

2010 MSC: *54H25, 47H10.*

Keywords: *Coincidence point, common fixed point, weakly contractive condition, dominating map, dominated map, ordered set, cone metric space.*

1. Introduction

One of the simplest and useful results in the fixed point theory is the Banach–Caccioppoli contraction mapping principle. In the last years, this principal has been generalized in many directions to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received a lot of attention. Fixed point theory in K-metric and K-normed spaces was developed by Perov et al. [24], Mukhamadijev and Stetsenko [16], Vandergraft [33]. For more details on fixed point theory in K-metric and K-normed spaces, we refer the reader to fine survey paper of Zabrejko [34]. The main idea was to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric.
In 2007, Huang and Zhang [13] reintroduced such spaces under the name of cone metric spaces and reintroduced definition of convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. After that, fixed point points in K-metric spaces have been the subject of intensive research (see, e.g.,

[1, 3, 7, 11, 13, 14, 15, 16, 23, 25, 30]). The main motivation for such research is a point raised by Agarwal [4], that the domain of existence of a solution to a system of first-order differential equations may be increased by considering generalized distances.
Recently, Wei-Shih Du [12] used the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in K-metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point theorems for mappings satisfying contractive conditions of a linear type in K-metric spaces can be considered as the corollaries of corresponding theorems in metric spaces. Nevertheless, the fixed point theory in K-metric spaces proceeds to be actual, since the method of scalarization cannot be applied for a wide class of mappings satisfying contractive conditions more general than contractive conditions of a linear type.
On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. One of results in this direction was given by Ran and Reurings [26] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [22] extended the result of Ran and Reurings for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, many authors obtained many fixed point theorems in ordered metric spaces. For more details, see [5, 6, 8, 10, 17, 19, 20, 21, 22, 27, 29, 31] and the references cited therein.
In this paper, an attempt has been made to derive some common fixed point theorems for four maps involving generalized (ψ, ϕ)-weak contractions in ordered cone metric spaces. The presented theorems generalize, extend and improve some recent fixed point results in K-metric spaces.

2. Preliminaries

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

Let E be always a Banach space.

Definition 2.1. A non-empty subset K of E is called a cone if and only if
(i) $\overline{K} = K$, $K \neq 0_E$ where $\overline{K}$ is the closure of K,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K \Rightarrow ax + by \in K$,
(iii) $K \cap (-K) = \{0_E\}$.

A cone K defines a partial ordering $\leq_E$ in E by $x \leq_E y$ if and only if $y - x \in K$. We shall write $x <_E y$ to indicate that $x \leq_E y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(K)$, where $\text{int}(K)$ denotes the interior of K. A cone K is said to be normal if there exists a constant $M \geq 1$ such that $0_E \leq_E x \leq_E y$ implies $\|x\|_E \leq M\|y\|_E$. A cone K is said solid if $\text{int}(K)$ is nonempty. The least positive number M satisfying this inequality is called the normal constant of cone K. For further details on cone theory, one can refer to [28].

Definition 2.2. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0_E \leq_E d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq_E d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3. Let (X, d) be a cone metric space and $\{x_n\}$ is a sequence in X. We say that $\{x_n\}$ is Cauchy if for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n > m > N$. We say that $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > N$. In this case, we denote $x_n \to x$ as $n \to \infty$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.4. Let $f : E \to E$ be a given mapping. We say that f is a monotone non-decreasing mapping with respect to $\leq_E$ if for every $x, y \in E$, $x \leq_E y$ implies $fx \leq_E fy$.

Definition 2.5 ([9]). Let $\psi : K \to K$ be a given function.
(i) We say that ψ is strongly monotone increasing if for $x, y \in K$, we have

$$x \leq_E y \Longleftrightarrow \psi(x) \leq_E \psi(y).$$

(ii) ψ is said to be continuous at $x_0 \in K$ if for any sequence $\{x_n\}$ in K, we have

$$\|x_n - x_0\|_E \to 0 \Longrightarrow \|\psi(x_n) - \psi(x_0)\|_E \to 0.$$

Definition 2.6. Let (X, d) be a cone metric space and $f, g : X \to X$. If $w = fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. If $w = x$, then x is a common fixed point of f and g.
The pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$.

Definition 2.7 ([2]). Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Definition 2.8. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered cone metric space if and only if

(i) (X, d) is a metric space,
(ii) $(X, \preceq)$ is a partial order.

Definition 2.9. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.10 ([2]). Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

Example 2.11 ([2]). Let $X = [0,1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[n]{x}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.

Definition 2.12 ([18]). Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominated if $fx \preceq x$ for each x in X.

Example 2.13 ([18]). Let $X = [0,1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for all $n \geq 1$. Since $fx = x^n \leq x$ for all $x \in X$. Therefore f is a dominated map.

3. Common fixed point results

First, let Ψ be the set of functions $\psi : K \to K$ such that
(i) ψ is continuous;
(ii) $\psi(t) = 0_E$ if and only if $t = 0_E$;
(iii) ψ is strongly monotone increasing.

Also, let Φ be the set of functions $\phi : \mathrm{int}(K) \cup \{0_E\} \to \mathrm{int}(K) \cup \{0_E\}$ such that
(i') ϕ is continuous;
(ii') $\phi(t) = 0_E$ if and only if $t = 0_E$;
(iii') $\phi(t) \ll_E t$ for all $t \in \mathrm{int}(K)$;
(iv') either $\phi(t) \leq_E d(x,y)$ or $d(x,y) \leq_E \phi(t)$ for $t \in \mathrm{int}(K) \cup \{0_E\}$ and $x, y \in X$.

The following Lemma will be useful later.

Lemma 3.1. [30]. Let E be a Banach space, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in E such that $b_n \to b \in E$, $c_n \to c \in E$ as $n \to +\infty$. Suppose also that $a_n \in \{b_n, c_n\}$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\{a_{n(p)}\}$ of $\{a_n\}$ such that $a_{n(p)} \to a \in \{b, c\}$ as $p \to +\infty$.

Our first result is the following.

Theorem 3.2. *Let $(X, d, \preceq)$ be an ordered complete cone metric space over a solid cone K. Let $T, S, I, J : X \to X$ be given mappings satisfying for every pair $(x,y) \in X \times X$ such that x and y are comparable,*

$$\psi(d(Sx, Ty)) \leq_E \psi(\Theta(x,y)) - \phi(\Theta(x,y)), \tag{3.1}$$

where $\Theta(x,y) \in \{d(Ix, Jy), \frac{1}{2}[d(Ix, Sx) + d(Jy, Ty)], \frac{1}{2}[d(Ix, Ty) + d(Jy, Sx)]\}$, $\psi \in \Psi$ *and* $\phi \in \Phi$. *Suppose that*

(i) $TX \subseteq IX$ *and* $SX \subseteq JX$*;*
(ii) I *and* J *are dominating maps and* S *and* T *are dominated maps;*
(iii) *If for a nondecreasing sequence* $\{x_n\}$ *with* $y_n \preceq x_n$ *for all* n *and* $y_n \to u$ *implies that* $u \preceq x_n$.

Also, assume either

(a) $\{S, I\}$ *are compatible,* S *or* I *is continuous and* $\{T, J\}$ *are weakly compatible or*

(b) $\{T, J\}$ *are compatible,* T *or* J *is continuous and* $\{S, I\}$ *are weakly compatible.*

Then S, T, I *and* J *have a common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $TX \subseteq IX$ and $SX \subseteq JX$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \quad y_{2n} = Tx_{2n-1} = Ix_{2n}, \qquad \forall\, n \in \mathbb{N}. \tag{3.2}$$

By given assumptions $x_{2n+1} \preceq Jx_{2n+1} = Sx_{2n} \preceq x_{2n}$ and $x_{2n} \preceq Ix_{2n} = Tx_{2n-1} \preceq x_{2n-1}$. Thus, for all $n \geq 0$, we have

$$x_{n+1} \preceq x_n. \tag{3.3}$$

Putting $x = x_{2n+1}$ and $y = x_{2n}$, from (3.3) and the considered contraction (3.1), we have

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &= \psi(d(Sx_{2n}, Tx_{2n+1})) \\ &\leq_E \psi(\Theta(x_{2n}, x_{2n+1})) - \phi(\Theta(x_{2n}, x_{2n+1})) \\ &\leq_E \psi(\Theta(x_{2n}, x_{2n+1})). \end{aligned} \tag{3.4}$$

The function ψ is strongly increasing, so we get that

$$d(y_{2n+1}, y_{2n+2}) \leq_E \Theta(x_{2n}, x_{2n+1}). \tag{3.5}$$

Note that

$$\begin{aligned} \Theta(x_{2n}, x_{2n+1}) \quad &\in \quad \{d(Ix_{2n}, Jx_{2n+1}), \frac{1}{2}[d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})], \\ &\qquad \frac{1}{2}[d(Ix_{2n}, Tx_{2n+1}) + d(Sx_{2n}, Jx_{2n+1})]\} \\ &= \quad \{d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})], \\ &\qquad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})]\} \\ &= \quad \{d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})], \frac{1}{2}d(y_{2n}, y_{2n+2})\}. \end{aligned}$$

If $\Theta(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1})$, (3.5) becomes

$$d(y_{2n+1}, y_{2n+2}) \leq_E d(y_{2n}, y_{2n+1}).$$

If $\Theta(x_{2n}, x_{2n+1}) = \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]$, then (3.5) becomes

$$d(y_{2n+1}, y_{2n+2}) \leq_E \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})],$$

so $d(y_{2n+1}, y_{2n+2}) \leq_E d(y_{2n}, y_{2n+1})$.

If $\Theta(x_{2n}, x_{2n+1}) = \frac{1}{2}d(y_{2n}, y_{2n+2})$, by (3.4) and a triangular inequality, we find that

$$d(y_{2n+1}, y_{2n+2}) \leq_E \frac{1}{2}d(y_{2n}, y_{2n+2}) \leq_E \frac{1}{2}d(y_{2n}, y_{2n+1}) + \frac{1}{2}d(y_{2n+1}, y_{2n+2}),$$

so $d(y_{2n+1}, y_{2n+2}) \leq_E d(y_{2n}, y_{2n+1})$. In all cases, we obtained that

$$d(y_{2n+1}, y_{2n+2}) \leq_E \Theta(x_{2n}, x_{2n+1}) \leq_E d(y_{2n}, y_{2n+1}). \tag{3.6}$$

Similarly, we have

$$d(y_{2n+1}, y_{2n}) \leq_E \Theta(x_{2n}, x_{2n-1}) \leq_E d(y_{2n}, y_{2n-1}). \tag{3.7}$$

By (3.6) and (3.7), we get that

$$d(y_{n+1}, y_n) \leq_E d(y_n, y_{n-1}) \quad \text{for all } \ n \geq 1. \tag{3.8}$$

It follows that the sequence $\{d(y_n, y_{n+1})\}$ is monotone non-increasing. Since K is a regular cone and $0_E \leq_E d(y_n, y_{n+1})$ for all $n \geq 0$, there exists $r \geq_E 0_E$ such that

$$d(y_n, y_{n+1}) \to r \ \text{ as } \ n \to +\infty.$$

By (3.6) and (3.7), we have

$$\lim_{n\to+\infty} \Theta(x_{2n}, x_{2n+1}) = \lim_{n\to+\infty} \Theta(x_{2n}, x_{2n-1}) = r.$$

Now, letting $n \to +\infty$ in (3.4) and using the continuity property of ψ and ϕ, we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which yields that $\phi(r) = 0_E$. Since $\phi(t) = 0_E \iff t = 0_E$, then $r = 0_E$. Therefore,

$$\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0. \tag{3.9}$$

Now, we will show that $\{y_n\}$ is a Cauchy sequence in the cone metric space (X, d). We proceed by negation and suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that for all positive integer i,

$$n(i) > m(i) > i, \ d(y_{2m(i)}, y_{2n(i)}) \geq_E \varepsilon, \ d(y_{2m(i)}, y_{2n(i)-2}) <_E \varepsilon. \tag{3.10}$$

From (3.10) and using a triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(y_{2m(i)}, y_{2n(i)}) \\ &\leq d(y_{2m(i)}, y_{2n(i)-2}) + d(y_{2n(i)-2}, y_{2n(i)-1}) + d(y_{2n(i)-1}, y_{2n(i)}) \\ &< \varepsilon + d(y_{2n(i)-2}, y_{2n(i)-1}) + d(y_{2n(i)-1}, y_{2n(i)}). \end{aligned}$$

Letting $i \to +\infty$ in the above inequality and using (3.9), we obtain

$$\lim_{i\to+\infty} d(y_{2m(i)}, y_{2n(i)}) = \varepsilon. \tag{3.11}$$

Again, a triangular inequality gives us

$$d(y_{2n(i)}, y_{2m(i)-1}) \leq_E d(y_{2n(i)}, y_{2m(i)}) + d(y_{2m(i)}, y_{2m(i)-1}),$$

and

$$d(y_{2n(i)}, y_{2m(i)}) \leq_E d(y_{2n(i)}, y_{2m(i)-1}) + d(y_{2m(i)-1}, y_{2m(i)}).$$

Letting $i \to +\infty$ in the above inequalities and using (3.9) and (3.11), we get that

$$\lim_{i\to+\infty} d(y_{2n(i)}, y_{2m(i)-1}) = \varepsilon. \tag{3.12}$$

Similarly, we have

$$\lim_{i\to+\infty} d(y_{2n(i)+1}, y_{2m(i)-1}) = \varepsilon. \tag{3.13}$$

On the other hand, we have

$$d(y_{2n(i)}, y_{2m(i)}) \leq_E d(y_{2n(i)}, y_{2n(i)+1} + d(y_{2n(i)+1}, y_{2m(i)},$$

so since ψ is monotone non-decreasing and continuous, we obtain that

$$\psi(\varepsilon) \leq_E \lim_{i\to+\infty} \psi(d(y_{2n(i)+1}, y_{2m(i)}). \tag{3.14}$$

Now, using (3.1) for $x = x_{2n(i)}$ and $y = x_{2m(i)-1}$, we have

$$\begin{aligned} \psi(d(y_{2n(i)+1}, y_{2m(i)}) &= \psi(Sx_{2n(i)}, Tx_{2m(i)-1}) \\ &\leq \psi(\Theta(x_{2n(i)}, x_{2m(i)-1})) - \phi(\Theta(x_{2n(i)}, x_{2m(i)-1})), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \Theta(x_{2n(i)}, x_{2m(i)-1}) &\in \{d(Ix_{2n(i)}, Jx_{2m(i)-1}), \frac{1}{2}[d(Ix_{2n(i)}, Sx_{2n(i)}) + \\ &\quad d(Jx_{2m(i)-1}, Tx_{2m(i)-1})], \frac{1}{2}[d(Ix_{2n(i)}, Tx_{2m(i)-1}) + d(Jx_{2m(i)-1}, Sx_{2n(i)})]\} \\ &= \{d(y_{2n(i)}, y_{2m(i)-1}), \frac{1}{2}[d(y_{2n(i)}, y_{2n(i)+1}) + d(y_{2m(i)-1}, y_{2m(i)})], \\ &\quad \frac{1}{2}[d(y_{2n(i)}, y_{2m(i)}) + d(y_{2m(i)-1}, y_{2n(i)+1})]\}. \end{aligned}$$

By (3.9), (3.12), (3.13) and having in mind Lemma 3.1, there exists a subsequence of $\{\Theta(x_{2n(i)}, x_{2m(i)-1})\}$ still denoted $\Theta(x_{2n(i)}, x_{2m(i)-1})$ such that

$$\lim_{i\to+\infty} \Theta(x_{2n(i)}, x_{2m(i)-1}) \in \{0_E, \varepsilon\}. \tag{3.16}$$

If $\lim_{i\to+\infty} \Theta(x_{2n(i)}, x_{2m(i)-1}) = 0_E$, then letting $i \to +\infty$ in (3.15) and using (3.14) and the continuities of ψ and ϕ, we obtain that $\psi(\varepsilon) \leq_E \psi(0_E) - \phi(0_E)$, so $\psi(\varepsilon) = 0_E$, which is a contradiction with $\varepsilon > 0$.

If $\lim_{i\to+\infty} \Theta(x_{2n(i)}, x_{2m(i)-1}) = \varepsilon$, then using similar arguments, we obtain that $\psi(\varepsilon) \leq_E \psi(\varepsilon) - \phi(\varepsilon)$, so $\phi(\varepsilon) = 0_E$, which is a contradiction.
Thus $\{y_{2n}\}$ is a Cauchy sequence in X, so $\{y_n\}$ is also a Cauchy sequence in X.

Finally, we shall prove existence of a common fixed point of the four mappings I, J, S and T.
Since X is complete, there exists a point z in X, such that $\{y_{2n}\}$ converges to z. Therefore,

$$y_{2n+1} = Jx_{2n+1} = Sx_{2n} \to z \text{ as } n \to \infty \tag{3.17}$$

and

$$y_{2n+2} = Ix_{2n+2} = Tx_{2n+1} \to z \text{ as } n \to \infty. \tag{3.18}$$

Assume that (a) holds. Suppose that I is continuous. Since the pair $\{S, I\}$ is compatible, we have

$$\lim_{n\to\infty} SIx_{2n+2} = \lim_{n\to\infty} ISx_{2n+2} = Iz. \tag{3.19}$$

Also, $Ix_{2n+2} = Tx_{2n+1} \preceq x_{2n+1}$. Now, by (3.1)

(3.20)
$$\psi(d(SIx_{2n+2}, Tx_{2n+1})) \leq_E \psi(\Theta(Ix_{2n+2}, x_{2n+1})) - \phi(\Theta(Ix_{2n+2}, x_{2n+1})),$$

where

$$\Theta(Ix_{2n+2}, x_{2n+1})) \in \{d(IIx_{2n+2}, Jx_{2n+1}), \frac{1}{2}[d(IIx_{2n+2}, SIx_{2n+2})+ d(Jx_{2n+1}, Tx_{2n+1})], \frac{1}{2}[d(IIx_{2n+2}, Tx_{2n+1}) + d(SIx_{2n+2}, Jx_{2n+1})]\}.$$

By (3.9), (3.17), (3.18) and (3.19), we get that

$$\lim_{n\to\infty} d(IIx_{2n+2}, Jx_{2n+1}) = \lim_{n\to\infty} \frac{1}{2}[d(IIx_{2n+2}, Tx_{2n+1})+d(SIx_{2n+2}, Jx_{2n+1})] = d(Iz, z),$$

$$\lim_{n\to\infty} \frac{1}{2}[d(IIx_{2n+2}, SIx_{2n+2}) + d(Jx_{2n+1}, Tx_{2n+1})] = 0_E.$$

By Lemma 3.1, there exists a subsequence of $\{\Theta(Ix_{2n+2}, x_{2n+1})\}$ still denoted $\Theta(Ix_{2n+2}, x_{2n+1})$ such that from the above limits

$$\lim_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) \in \{0_E, d(Iz, z)\}. \tag{3.21}$$

If $\lim\limits_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) = 0_E$, then then letting $n \to +\infty$ in (3.20) and using the fact that

$$\lim_{n\to\infty} d(SIx_{2n+2}, Tx_{2n+1}) = d(Iz, z),$$

and the continuities of ψ and ϕ, we obtain

$$\psi(d(Iz, z)) \leq_E \psi(0_E) - \phi(0_E),$$

so $\psi(d(Iz, z)) = 0_E$, which yields that $d(Iz, z) = 0_E$, so $Iz = z$.
If $\lim\limits_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) = d(Iz, z)$, using the similar arguments we get that

$$\psi(d(Iz, z)) - \psi(d(Iz, z)) - \phi(d(Iz, z)),$$

so similarly, $Iz = z$. In each case, we obtained

$$Iz = z. \tag{3.22}$$

Now, $Tx_{2n+1} \preceq x_{2n+1}$ and $Tx_{2n+1} \to z$ as $n \to \infty$, so by assumption [(iii)] we have $z \preceq x_{2n+1}$. From (3.1),

$$\psi(d(Sz, Tx_{2n+1})) \leq_E \psi(d(\Theta(z, x_{2n+1}))) - \phi(d(\Theta(z, x_{2n+1}))), \tag{3.23}$$

where

$$\Theta(z, x_{2n+1}) \in \{d(Iz, Jx_{2n+1}), \frac{1}{2}[d(Iz, Sz) + d(Jx_{2n+1}, Tx_{2n+1})], \frac{1}{2}[d(Iz, Tx_{2n+1}) + d(Sz, Jx_{2n+1})]\}$$
$$= \{d(z, Jx_{2n+1}), \frac{1}{2}[d(z, Sz) + d(Jx_{2n+1}, Tx_{2n+1})], \frac{1}{2}[d(z, Tx_{2n+1}) + d(Sz, Jx_{2n+1})]\}.$$

By (3.9), (3.17), (3.18) and (3.19), we get that

$$\lim_{n\to\infty} \frac{1}{2}[d(z, Sz)+d(Jx_{2n+1}, Tx_{2n+1})] = \frac{1}{2}d(z, Sz) = \lim_{n\to\infty} \frac{1}{2}[d(IIx_{2n+2}, Tx_{2n+1}) + d(SIx_{2n+2}, Jx_{2n+1})]\},$$
$$\lim_{n\to\infty} d(z, Jx_{2n+1}) = 0_E.$$

By Lemma 3.1, there exists a subsequence of $\{\Theta(z, x_{2n+1})\}$ still denoted $\Theta(Ix_{2n+2}, x_{2n+1})$ such that from the above limits

$$\lim_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) \in \{0_E, \frac{1}{2}d(Sz, z)\}. \tag{3.24}$$

If $\lim_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) = 0_E$, then then letting $n \to +\infty$ in (3.24) and using the fact that

$$\lim_{n\to\infty} d(Sz, Tx_{2n+1}) = d(Sz, z),$$

and the continuities of ψ and ϕ, we obtain

$$\psi(d(Sz, z)) \leq_E \psi(0_E) - \phi(0_E),$$

so $\psi(d(Iz, z)) = 0_E$, which yields that $Sz = z$.

If $\lim_{n\to+\infty} \Theta(Ix_{2n+2}, x_{2n+1}) = \frac{1}{2}d(Sz, z)$ and using the similar arguments, we get that

$$\psi(d(Sz, z)) \leq_E \psi(\frac{1}{2}d(Sz, z)) - \phi(\frac{1}{2}d(Sz, z)) \leq_E \psi(\frac{1}{2}d(Sz, z)),$$

so $d(Sz, z) \leq_E \frac{1}{2}d(Sz, z)$, which holds unless $d(Sz, z) = 0_E$, so

$$Sz = z. \tag{3.25}$$

Since $S(X) \subseteq J(X)$, there exists a point $w \in X$ such that $Sz = Jw$. Suppose that $Tw \neq Jw$. Since $w \preceq Jw = Sz \preceq z$ implies $w \preceq z$. From (3.1), we obtain

$$\psi(d(Jw, Tw)) = \psi(d(Sz, Tw)) \leq_E \psi(\Theta(z, w)) - \phi(\Theta(z, w)), \tag{3.26}$$

where

$$\Theta(z, w) \in \{d(Iz, Jw), \frac{1}{2}[d(Iz, Sz) + d(Jw, Tw)], \frac{1}{2}[d(Iz, Tw) + d(Sz, Jw)]\}$$
$$= \{0_E, \frac{1}{2}d(Jw, Tw)\}.$$

If $\Theta(z,w) = 0_E$, we easily deduce from (3.26) that $d(Jw, Tw) = 0_E$.
If $\Theta(z,w) = d(Jw, Tw)$, similarly we get that $d(Jw, Tw) = 0_E$. Thus, we obtained

$$Jw = Tw. \tag{3.27}$$

Since T and J are weakly compatible, $Tz = TSz = TJw = JTw = JSz = Jz$. Thus, z is a coincidence point of T and J.

Now, since $Sx_{2n} \preceq x_{2n}$ and $Sx_{2n} \to z$ as $n \to \infty$, so by assumption [(iii)], $z \preceq x_{2n}$. Then, from (3.1)

$$\psi(d(Sx_{2n}, Tz)) \leq_E \psi(\Theta(x_{2n}, z)) - \phi(\Theta(x_{2n}, z)), \tag{3.28}$$

where

$$\begin{aligned}\Theta(x_{2n}, z) &\in \{d(Ix_{2n}, Jz), \frac{1}{2}[d(Ix_{2n}, Sx_{2n}) + d(Jz, Tz)], \frac{1}{2}[d(Ix_{2n+1}, Tz) + d(Sx_{2n}, Jz)]\}\\ &= \{d(Ix_{2n}, Tz), \frac{1}{2}d(Ix_{2n}, Sx_{2n}), \frac{1}{2}[d(Ix_{2n+1}, Tz) + d(Sx_{2n}, Tz)]\}\end{aligned}$$

We have

$$\lim_{n\to\infty} d(Ix_{2n}, Tz) = \lim_{n\to\infty} \frac{1}{2}[d(Ix_{2n+1}, Tz) + d(Sx_{2n}, Tz)] = d(z, Tz),$$

and

$$\lim_{n\to\infty} d(Ix_{2n}, Sx_{2n}) = 0, \quad \lim_{n\to\infty} d(Sx_{2n}, Tz) = d(z, Tz).$$

By Lemma 3.1, there exists a subsequence of $\{\Theta(x_{2n}, z))\}$ still denoted $\Theta(x_{2n}, z)$ such that from the above limits

$$\lim_{n\to+\infty} \Theta(x_{2n}, z) \in \{0_E, d(z, Tz)\}. \tag{3.29}$$

Similarly, letting $n \to \infty$ in (3.28) and having in mind (3.29), we get that

$$z = Tz. \tag{3.30}$$

Therefore $Sz = Tz = Iz = Jz = z$, so z is a common fixed point of I, J, S and T. The proof is similar when S is continuous.
Similarly, the result follows when (b) holds.

□

Now, it is easy to state a corollary of Theorem 3.2 involving a contraction of integral type.

Corollary 3.3. *Let T, S, I and J satisfy the conditions of Theorem 3.2, except that condition (3.1) is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^\varepsilon u(t)dt > 0$ for each $\varepsilon > 0$ and that*

$$\int_0^{\psi(d(Sx,Ty))} u(t)dt \leq \int_0^{\psi(\Theta(x,y))} u(t)dt - \int_0^{\phi(\Theta(x,y))} u(t)dt. \tag{3.31}$$

Then, S, T, I and J have a common fixed point.

Corollary 3.4. *Let $(X, d, \preceq)$ be an ordered complete cone metric space over a solid cone K. Let $T, S, I : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\psi(d(Sx, Ty)) \leq_E \psi(\Theta_1(x, y)) - \phi(\Theta_1(x, y)), \tag{3.32}$$

where $\Theta_1(x, y) \in \{d(Ix, Iy), \frac{1}{2}[d(Ix, Sx) + d(Iy, Ty)], \frac{1}{2}[d(Ix, Ty) + d(Iy, Sx)]\}$, $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that

(i) *$TX \subseteq IX$ and $SX \subseteq IX$;*
(ii) *I is a dominating map and S and T are dominated maps;*
(iii) *If for a nondecreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$.*

Also, assume either

(a) *$\{S, I\}$ are compatible, S or I is continuous and $\{T, I\}$ are weakly compatible or*
(b) *$\{T, I\}$ are compatible, T or I is continuous and $\{S, I\}$ are weakly compatible,*

then S, T and I have a common fixed point.

Proof. It follows by taking $I = J$ in Theorem 3.2. □

Corollary 3.5. *Let $(X, d, \preceq)$ be an ordered complete cone metric space over a solid cone K. Let $S, I : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\psi(d(Sx, Sy)) \leq_E \psi(\Theta_2(x, y)) - \phi(\Theta_2(x, y)), \tag{3.33}$$

where $\Theta_2(x, y) \in \{d(Ix, Iy), \frac{1}{2}[d(Ix, Sx) + d(Iy, Sy)], \frac{1}{2}[d(Ix, Sy) + d(Iy, Sx)]\}$, $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that

(i) *$SX \subseteq IX$;*
(ii) *I is a dominating map and S is dominated maps;*
(iii) *If for a nondecreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$.*

Also, assume $\{S, I\}$ are compatible and S or I is continuous, then S and I have a common fixed point.

Proof. It follows by taking $S = T$ in Corollary 3.4. □

Corollary 3.6. *Let $(X, d, \preceq)$ be an ordered complete cone metric space over a solid cone K. Let $T, S : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\psi(d(Sx, Ty)) \leq_E \psi(\Theta_3(x, y)) - \phi(\Theta_3(x, y)), \tag{3.34}$$

where $\Theta_3(x, y) \in \{d(x, y), \frac{1}{2}[d(x, Sx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$, $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that

(i) *S and T are dominated maps;*
(ii) *If for a nondecreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$.*

Also, assume either S or T is continuous, then S and T have a common fixed point.

Proof. It follows by taking $I = Id_X$, the identity on X, in Corollary 3.4. □

Corollary 3.7. *Let $(X, d, \preceq)$ be an ordered complete cone metric space over a solid cone K. Let $T, S, I, J : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$d(Sx, Ty) \leq_E \Theta(x, y) - \phi(\Theta(x, y)),$$

where $\Theta(x, y) \in \{d(Ix, Jy), \frac{1}{2}[d(Ix, Sx) + d(Jy, Ty)], \frac{1}{2}[d(Ix, Ty) + d(Jy, Sx)]\}$ and $\phi \in \Phi$. Suppose that

(i) *$TX \subseteq IX$ and $SX \subseteq JX$;*
(ii) *I and J are dominating maps and S and T are dominated maps;*
(iii) *If for a nondecreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$.*

Also, assume either

(a) *$\{S, I\}$ are compatible, S or I is continuous and $\{T, J\}$ are weakly compatible or*
(b) *$\{T, J\}$ are compatible, T or J is continuous and $\{S, I\}$ are weakly compatible,*

then S, T, I and J have a common fixed point.

Proof. It suffices to take $\psi(t) = t$ in Theorem 3.2. □

Remark 3.8. Theorem 3.2 extends Theorem 2.1 of Shatanawi and Samet [32] to cone metric spaces.

Now, we state the following illustrative examples.

Example 3.9 (The case of a non-normal cone)**.** Let $X = [0, \frac{1}{4}]$ be equipped with the usual order. Take $E = \mathcal{C}^1_{\mathbb{R}}([0, 1])$ and $K = \{\varphi \in E,\ \varphi(t) \geq 0, t \in [0, 1]\}$. Define $d : X \times X \to E$ by $d(x, y)(t) = |x - y|\varphi$ where $\varphi \in K$ is a fixed function, for example $\varphi(t) = e^t$. Then, (X, d) is a complete cone metric space with a nonnormal solid cone.

Also, define $S, T, I, J : X \to X$ by $Sx = Tx = x^2$ and $Ix = Jx = x$. For all comparable $x, y \in X$, we have

$$d(Sx, Ty)(t) = d(Sx, Sy)(t) = |x^2 - y^2|e^t = |x - y||x + y|e^t \leq \frac{1}{2}|x - y|e^t = \frac{1}{2}d(Ix, Jy)(t),$$

that is, (3.1) holds for $\psi(t) = t$ and $\phi(t) = \frac{1}{2}t$.

On the other hand, $x \leq Ix = Jx$ and $Sx = Tx \leq x$ for all $x \in X$. Also, $SX = TX \subseteq IX = JX$ and the pairs $\{S, I\} = \{T, J\}$ are compatible. All hypotheses of Theorem 3.2 are verified and $x = 0$ is a common fixed point of S, T, I and J.

Example 3.10. (The case of a normal cone). Let $X = [0, \infty]$ be equipped with the usual order. Take $E = \mathbb{R}^2$ and $K = \{(x, y), x \geq 0, y \geq 0\}$. Define

$d : X \times X \to E$ by $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ a constant. Then, (X, d) is a complete cone metric space with a normal solid cone.

Also, define $S, T, I, J : X \to X$ by $Sx = Tx = ax$ and $Ix = Jx = bx$ where $0 < a < 1$ and $b > 1$. For all comparable $x, y \in X$, we have

$$d(Sx, Ty) = d(Sx, Sy) = (a|x-y|, a\alpha|x-y|) = (\frac{a}{b}b|x-y|, \frac{a}{b}b\alpha|x-y|) = \frac{a}{b}d(Ix, Jy),$$

that is, (3.1) holds for $\psi(t) = t$ and $\phi(t) = (1 - \frac{a}{b})t$.

Also, it is clear that all other hypotheses of Theorem 3.2 are verified and $x = 0$ is a common fixed point of S, T, I and J.

The following example (which is inspired by [18]) demonstrates the validity of Theorem 3.2.

Example 3.11 (The case of a non-normal cone)**.** Let $X = [0, 1]$ be equipped with the usual order. Take $E = \mathcal{C}^1_{\mathbb{R}}([0, 1])$ and $K = \{\varphi \in E,\ \varphi(t) \geq 0, t \in [0, 1]\}$. Define $d : X \times X \to E$ by $d(x, y)(t) = |x - y|\varphi$ where $\varphi \in K$ is a fixed function, for example $\varphi(t) = e^t$. Then, (X, d) is a complete cone metric space with a nonnormal solid cone.

Define the self maps I, J, S and T on X by

$$S(x) = \begin{cases} 0, \text{ if } x \leq \frac{1}{3} \\ \frac{1}{2}(x - \frac{1}{3}), \text{ if } x \in (\frac{1}{3}, 1] \end{cases}, \quad Tx = \begin{cases} 0, \text{ if } x \leq \frac{1}{3} \\ \frac{1}{3}, \text{ if } x \in (\frac{1}{3}, 1] \end{cases},$$
$$J(x) = \begin{cases} 0, \text{ if } x = 0 \\ x, \text{ if } x \in (0, \frac{1}{3}] \\ 1, \text{ if } x \in (\frac{1}{3}, 1] \end{cases}, \quad Ix = \begin{cases} 0, \text{ if } x = 0 \\ \frac{1}{3}, \text{ if } x \in (0, \frac{1}{3}] \\ 1, \text{ if } x \in (\frac{1}{3}, 1] \end{cases}.$$

Then I and J are dominating maps and S and T are dominated maps with $S(X) \subseteq J(X)$ *and* $T(X) \subseteq I(X)$*,i.e.*

	S *is dominated map*	T *is dominated map*	I *is dominating map*	J *is dominating map*
for each x in X	$Sx \leq x$	$Tx \leq x$	$x \leq Ix$	$x \leq Jx$
$x = 0$	$S(0) = 0$	$T(0) = 0$	$0 = I(0)$	$0 = J(0)$
$x \in (0, \frac{1}{3}]$	$Sx = 0 < x$	$Tx = 0 < x$	$x \leq \frac{1}{3} = I(x)$	$x = J(x)$
$x \in (\frac{1}{3}, 1]$	$Sx = \frac{1}{2}(x - \frac{1}{3}) < x$	$Tx = \frac{1}{3} < x$	$x \leq 1 = I(x)$	$x \leq 1 = J(x)$

Also, $\{S, I\}$ are compatible, S is continuous and $\{T, J\}$ are weakly compatible.

Define $\psi : K \to K$ and $\phi : \mathrm{int}(K) \cup \{0_E\} \to \mathrm{int}(K) \cup \{0_E\}$ by

$$\psi(t) = t \text{ and } \phi(t) = \frac{1}{2}t.$$

The inequality (3.1) holds for all comparable $x, y \in X$. Without loss of generality, take $x \leq y$. We consider the following cases:

(i) If $x = y = 0$, then $d(S0, T0)(t) = 0$ and (3.1) is satisfied.

(ii) For $x = 0$ and $y \in (0, \frac{1}{3}]$, then again $d(Sx, Ty)(t) = 0$ and (3.1) is satisfied.

(iii) For $x = 0$ and $y \in (\frac{1}{3}, 1]$,

$$d(Sx, Ty)(t) = \frac{1}{3}e^t < \frac{1}{2}e^t = \frac{1}{2}d(Ix, Jy)(t).$$

(iv) For $x, y \in (0, \frac{1}{3}]$, then $d(Sx, Ty) = 0$ and hence (3.1) is satisfied.

(v) For $x = (0, \frac{1}{3}]$ and $y \in (\frac{1}{3}, 1]$,

$$d(Sx, Ty)(t) = \frac{1}{3}e^t < \frac{1}{2}e^t = \frac{1}{2}d(Ix, Jy)(t).$$

(vi) For $x, y \in (\frac{1}{3}, 1]$,

$$d(Sx, Ty)(t) \quad = \quad \frac{1}{2}(1-x)e^t \le \frac{1}{3}e^t \le \frac{1}{2}d(Jy, Ty)(t).$$

All hypotheses of Theorem 3.2 are verified and $x = 0$ is a common fixed point of S, T, I and J.

Acknowledgements. *The authors are grateful to the editor and referee for their valuable remarks for improving this paper.*

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.

[2] M. Abbas, T. Nazir and S. Radenović, *Common fixed point of four maps in partially ordered metric spaces*, Appl. Math. Lett. **24** (2011), 1520–1526.

[3] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), 511–515.

[4] R. P. Agarwal, *Contraction and approximate contraction with an application to multi-point boundary value problems*, J. Comput. Appl. Appl. Math. **9** (1983), 315–325.

[5] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Applicable Anal. **87** (2008), 109–116.

[6] I. Altun and H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. **2010**(2010) Article ID 621492, 17 pages.

[7] H. Aydi, H. K. Nashine, B. Samet and H. Yazidi, *Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations*, Nonlinear Anal. **74**, no. 17 (2011), 6814–6825.

[8] I. Beg and A.R. Butt, *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal. **71** (2009), 3699–3704.

[9] B. S. Choudhury and N. Metiya, *The point of coincidence and common fixed point for a pair of mappings in cone metric spaces*, Comput. Math. Appl. **60** (2010), 1686–1695.

[10] Lj. B. Ćirić, N. Cakić, M. Rajović and J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl. **2008** (2008), Article ID 131294, 11 pages.

[11] Lj. B. Ćirić, B. Samet, N. Cakić and B. Damjanović, *Coincidence and fixed point theorems for generalized (ψ, ϕ)-weak nonlinear contraction in ordered K-metric spaces*, Comput. Math. Appl. **62** (2011), 3305–3316.

[12] W.-S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72** (2010), 2259–2261.

[13] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[14] D. Ilić and V. Rakoèević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.
[15] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59**, no. 12 (2010), 3656–3668.
[16] E.M. Mukhamadiev and V.J. Stetsenko, *Fixed point principle in generalized metric space*, Izvestija AN Tadzh. SSR, fiz.-mat. igeol.-chem. nauki. **10** (4) (1969), 8-19 (in Russian).
[17] H.K. Nashine and I. Altun, *Fixed point theorems for generalized weakly contractive condition in ordered metric spaces*, Fixed Point Theory Appl. **2011** (2011), Article ID 132367, 20 pages.
[18] H. K. Nashine and M. Abbas, *Common fixed point point of mappings satisfying implicit contractive conditions in TVS-valued ordered cone metric spaces*, preprint.
[19] H. K. Nashine and B. Samet, *Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), 2201–2209.
[20] H. K. Nashine, B. Samet and C. Vetro, *Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces*, Math. Comput. Modelling, to appear (`doi:10.1016/j.mcm.2011.03.014`).
[21] H .K. Nashine and W. Shatanawi, *Coupled common fixed point theorems for pair of commuting mappings in partially ordered complete metric spaces*, Comput. Math. Appl. **62** (2011), 1984–1993.
[22] J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordianry differential equations*, Order **22** (2005), 223–239.
[23] J. O. Olaleru, *Some generalizations of fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), Article ID 657914, 10 pages.
[24] A. I. Perov, The Cauchy problem for systems of ordinary differential equations, in: Approximate Methods of Solving Differential Equations, Kiev, Naukova Dumka, 1964, pp. 115–134 (in Russian).
[25] A. I. Perov and A.V. Kibenko, *An approach to studying boundary value problems*, Izvestija AN SSSR, Ser. Math. **30**, no. 2 (1966), 249–264 (in Russian).
[26] A. C. M. Ran and M. C. B. Reurings, *A fixed point thm in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.
[27] D. O'regan and A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. *341* (2008), 241–1252.
[28] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper: cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **345** (2008), 719–724.
[29] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72** (2010), 4508–4517.
[30] B. Samet, *Common fixed point theorems involving two pairs of weakly compatible mappings in K-metric spaces*, Appl. Math. Lett. **24** (2011), 1245–1250.
[31] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, Comput. Math. Appl. **60** (2010), 2508–2515.
[32] W. Shatanawi and B. Samet, *On (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces*, Comput. Math. Appl. **62** (2011), 3204–3214
[33] J. S. Vandergraft, *Newton's method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal. **4** (1967), 406–432.
[34] P. P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math. **48** (1997), 825–859.

(Received December 2011 – Accepted August 2012)

H. Aydi (*hassen.aydi@isima.rnu.tn*)
Université de Sousse, Institut Supérieur d'Informatique et des Technologies de Communication de Hammam Sousse, Route GP1-4011, H. Sousse, Tunisie.

H. K. Nashine (*drhknashine@gmail.com,hemantnashine@rediffmail.com*)
Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, Raipur-492101(Chhattisgarh), India.