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A RAFU linear space uniformly dense in $C[a,b]$

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Abstract

In this paper we prove that a RAFU (radical functions) linear space, $\mathtt{C}$, is uniformly dense in $C[a,b]$ by means of a S-separation condition of certain subsets of $[a,b]$ due to Blasco-Moltó. This linear space is not a lattice or an algebra.
Given an arbitrary function $f \in C[a,b]$ we will obtain easily the sequence $(C_n)_n$ of $\mathtt{C}$ that converges uniformly to f and we will show the degree of uniform approximation to f with $(C_n)_n$.

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1. Introduction

Let K be a compact Hausdorff space. The Kakutani-Stone Theorem [10] gives a necessary and sufficient condition for the density of a lattice of $C(K)$ in the topology of the uniform convergence on K. The Stone-Weierstrass Theorem [7] provides a necessary and sufficient condition under which an algebra of $C(K)$ is uniformly dense. Nevertheless, the above conditions are not sufficient to ensure the uniform density of a linear space of $C(K)$. Tietze [5], Jameson [4], Mrowka [11], Blasco-Moltó [6], Garrido-Montalvo [8] and recently Gassó-Hernández-Rojas [9] have studied the uniform approximation for linear spaces.

In the Section 2 we will construct a RAFU (Radical functions) linear space, $\mathtt{C}$, in $C[a,b]$ and we will prove that $\mathtt{C}$ is uniformly dense in $C[a,b]$ by using a S-separation condition according to Blasco-Moltó [6]. We will also see that the uniform density of $\mathtt{C}$ in $C[a,b]$ is not a consequence of the results given by Kakutani-Stone, Stone-Weierstrass, Tietze, Jameson, or Mrowka.

It is true that Blasco-Moltó showed an example of a linear space, $\mathcal{F}$, uniformly dense in $C\,[0,1]$ by using the S-separation condition but some questions were not studied: the linear combinations of elements belonging to $\mathcal{F}$ which approximate uniformly every $f \in C\,[0,1]$ and the degree of uniform approximation that $\mathcal{F}$ provides were unknown. In the Section 3 we will solve these problems by using the RAFU linear space $\mathtt{C}$. Moreover, this linear space $\mathtt{C}$ can be used as an example of approximation by series in the work of Gassó-Hernández-Rojas.

2. A RAFU linear space uniformly dense in $C\,[a,b]$

For each $n \in \mathbb{N}$ we consider the partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0$, ..., n and we define in $[a,b]$ the functions

$$C_n(x) = k_1 + \sum_{i=2}^{n} (k_i - k_{i-1}) \cdot F_n(x_{i-1}, x) \tag{2.1}$$

where $\{k_i\}_{i=1}^n$ are a family of real arbitrary numbers and

$$F_n\,(x_{i-1}, x) = \frac{\sqrt[2n+1]{x_{i-1} - x_0} + \sqrt[2n+1]{x - x_{i-1}}}{\sqrt[2n+1]{x_n - x_{i-1}} + \sqrt[2n+1]{x_{i-1} - x_0}}, \quad i = 2, ..., n \tag{2.2}$$

We designate by $\mathtt{C}_n$ the subset of $C\,[a,b]$ formed by the functions C_n and we also denote by $\mathtt{C}$ the set $\mathtt{C} = \cup_{n\in\mathbb{N}} \mathtt{C}_n$.

Proposition 2.1. *The set $\mathtt{C}$ is a linear space included in $C\,[a,b]$.*

Proof. It is clear that $\mathtt{C} \subset C\,[a,b]$. In the first place it is easy to check that $\mathtt{C}_n$ is a linear space n-dimensional because n is fixed and hence the values $\{x_i\}_{i=0}^n$ are the same points. Moreover, a basis of $\mathtt{C}_n$ is $\{1, F_n(x_1, x), ..., F_n(x_{n-1}, x)\}$.

$\mathtt{C}$ is a linear space. Let C_p and C_q be two elements belonging to $\mathtt{C}$. Then, $C_p \in \mathtt{C}_{r\cdot p}$, $r \in \mathbb{N}$ and $C_q \in \mathtt{C}_{s\cdot q}$, $s \in \mathbb{N}$ by considering zero the coefficients $(k_i - k_{i-1})$ of the functions $F_n(x_{i-1}, x)$ that do not appear on the expressions of C_p or C_q. In particular, C_p and C_q belong to the linear space $\mathtt{C}_{p\cdot q}$ and, of course, $C_p + C_q \in \mathtt{C}$. Finally, it is inmediate to check that if $C_p \in \mathtt{C}$ and $\lambda \in \mathbb{R}$ then $\lambda \cdot C_p \in \mathtt{C}$. □

Definition 2.2. A RAFU linear space is a linear space whose basis is formed by radical functions of the type (2.2). We will say that $\mathtt{C}$ is a RAFU linear space.

The theorems of uniform approximation in $C\,(K)$ for lattices are known as Kakutani-Stone theorems (the interested reader can see [10], [7], [12]).

The family $\mathtt{C}$ is not a lattice. In fact, in the interval $[-1,1]$ the function $C(x) = \sqrt[3]{x} \in \mathtt{C}$ but $|C(x)| \notin \mathtt{C}$ because at $x = 0$ its side derivatives do not have the same sign. Therefore, the family $\mathtt{C}$ does not satisfy the Kakutani-Stone theorems.

The theorems of uniform approximation in $C\,(K)$ for algebras are known as Stone-Weierstrass theorems (the interested reader can see [7], [12]).

A simple count proves that C is not an algebra, therefore the set C does not verify the Stone-Weierstrass theorems.

Let X be a topological space and let $C^*(X)$ be the set consisting of all bounded continuous functions and let $C(X)$ be the set consisting of all continuous functions.

Definition 2.3. Let $\mathcal{F}$ be a family of $C^*(X)$. We say that

(1) A *zero-set* in X is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$ with $f \in C^*(X)$.
(2) The *Lebesgue-sets* of $f \in C(X)$ are the sets $L_\alpha(f) = \{x \in X : f(x) \le \alpha\}$ and $L^\beta(f) = \{x \in X : f(x) \ge \beta\}$ where α and β are real numbers.
(3) $\mathcal{F}$ S_1- separates the subsets A and B of X when there is $f \in \mathcal{F}$, $0 \le f \le 1$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.
(4) (Blasco-Moltó [**?**]). $\mathcal{F}$ S-separates the subsets A and B of X if for each $\delta > 0$, there is $f \in \mathcal{F}$ such that $0 \le f \le 1$ for every $x \in X$, $f(A) \subset [0, \delta]$ and $f(B) \subset [1 - \delta, 1]$.
(5) (Garrido-Montalvo [**?**]). $\mathcal{F}$ S'-separates the subsets A and B of X if for each $\delta > 0$, there is $f \in \mathcal{F}$ such that $-\delta \le f \le 1 + \delta$ for every $x \in X$, $f(A) \subset [-\delta, \delta]$ and $f(B) \subset [1 - \delta, 1 + \delta]$.
(6) Given a series of continuous functions $\sum_{i \in I} f_i$ on X, the series is *locally convergent*, for every $x \in X$, if there is a neighborhood U of x such that the series converges uniformly on U. For $E \subset C(X)$, $\sum(E)$ is the set of all $f \in C(X)$ such that $f = \sum_{i \in I} f_i$ with $f_i \in E$ for every $i \in I$ and $\sum_{i \in I} f_i$ is a locally convergent series. $\overline{\sum(E)}$ denotes the uniform closure of $\sum(E)$.

Theorem 2.4 (Tietze [5], Mrowka [11])**.** *Let $\mathcal{F}$ be a linear space of $C^*(X)$. $\mathcal{F}$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ S_1- separates every pair of disjoint zero-sets in X.*

Theorem 2.5 (Jameson [4])**.** *Let $\mathcal{F}$ be a linear space of $C^*(X)$. $\mathcal{F}$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ S_1- separates every pair of disjoint closed subsets in X.*

By the properties of the functions of the linear space C it is possible to deduce that we cannot apply to C the results of Tietze, Mrowka or Jameson.

Theorem 2.6 (Blasco-Moltó [6])**.** *Let X be a topological space. A linear space $\mathcal{F}$ of $C^*(X)$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ S- separates every pair of disjoint zero-sets in X.*

We go to see that we can apply this theorem to prove the uniform density of C in $C\,[a, b]$. Let us consider in $[a, b]$ the step function defined by $f(x) = \begin{cases} k_1 & a \le x \le x_1 \\ k_2 & x_1 < x \le b \end{cases}$, $k_1, k_2 \in \mathbb{R}$. If we calculate, for each $n \in \mathbb{N}$, the expressions of the radical functions $c_n(x) = M_n + N_n \cdot \sqrt[2n+1]{x - x_1}$ that are obtained by the conditions $c_n(a) = k_1$ and $c_n(b) = k_2$, we obtain $N_n = \frac{k_2 - k_1}{\sqrt[2n+1]{b - x_1} + \sqrt[2n+1]{x_1 - a}}$

and $M_n = k_1 + \frac{(k_2-k_1)\cdot \sqrt[2n+1]{x_1-a}}{\sqrt[2n+1]{b-x_1}+\sqrt[2n+1]{x_1-a}}$. In this case, a elementary count shows that the sequence $(c_n)_n$ satisfies $\lim_{n\to+\infty} c_n(x) = \begin{cases} k_1 & a \le x < x_1 \\ \frac{k_1+k_2}{2} & x = x_1 \\ k_2 & x < x \le b \end{cases}$

Now, we will consider an arbitrary step function in $[a, b]$

$$f(x) = k_1 \cdot \chi_{[x_0,\, x_1]} + \sum_{i=2}^{m} k_i \cdot \chi_{(x_{i-1},\, x_i]} \tag{2.3}$$

where $k_i \in \mathbb{R}$, $i = 1, ..., m$ and $x \in [x_0 = a,\ x_m = b]$. By an elementary count we can write (2.3) in the form $f(x) = \sum_{i=1}^{m} f_i(x)$ where $f_1(x) = k_1 \cdot \chi_{[x_0,\, x_m]}$ and $f_p(x) = (k_p - k_{p-1}) \cdot \chi_{(x_{p-1},\, x_m]}$, $p = 2, ..., m$.

For each f_p we construct its sequence of radical functions $(c_{p,n})_n$. For every $n \in \mathbb{N}$, the corresponding sequence for f_1 is $c_{1,n}(x) = k_1$ and for f_p, with $p > 1$, we obtain $c_{p,n}(x) = M_{p,n} + N_{p,n} \cdot \sqrt[2n+1]{x - x_{p-1}}$ where $N_{p,n}$ and $M_{p,n}$ are given by $N_{p,n} = \frac{k_p - k_{p-1}}{\sqrt[2n+1]{x_m - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - x_0}}$ and $M_{p,n} = \frac{(k_p - k_{p-1})\cdot \sqrt[2n+1]{x_{p-1}-x_0}}{\sqrt[2n+1]{x_m - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - x_0}}$. Finally, if we denote by $(C_{m,n})_n$ to the sequence $C_{m,n}(x) = \sum_{i=1}^{m} c_{i,n}(x)$ then $\lim_{n\to+\infty} C_{m,n}(x) = \begin{cases} f(x) & x \in [x_0,\, x_m] - \{x_1,\, x_2,\, ,\, ...,\, x_{m-1}\} \\ \frac{k_p + k_{p+1}}{2} & x = x_p,\ p = 1,\, ...,\, m-1 \end{cases}$ by elementary properties of the limits.

Proposition 2.7. *Let f be the function defined by (2.3). For any $\beta > 0$ such that $(x_i - \beta, x_i + \beta) \cap (x_j - \beta, x_j + \beta) = \varnothing$ where $i \ne j$ and $i, j \in \{1, ..., m-1\}$ the limit $\lim_{n\to+\infty} C_{m,n} = f$ is uniform on $[x_0, x_1 - \beta] \cup [x_1 + \beta, x_2 - \beta] \cup ... \cup [x_{m-1} + \beta, x_m]$.*

Proof. **1st part.** It verifies that $\lim_{n\to\infty} \sqrt[2n+1]{x} = \begin{cases} -1 & x \in \left[-M, -\frac{1}{K}\right] \\ 1 & x \in \left[\frac{1}{K}, M\right] \end{cases}$ where M and K are large positive real numbers. Moreover, the limit becomes uniform.

The function $h_n(x) = \sqrt[2n+1]{x}$ is strictly increasing on $\mathbb{R}$, therefore $h_n(-M) \le h_n(x) \le h_n(-\frac{1}{K})$ for $x \in \left[-M, -\frac{1}{K}\right]$ and fixed $\epsilon > 0$ it is possible to find $n_{M,K} \in \mathbb{N}$ such that if $n \ge n_{m,K}$ then $-1-\epsilon < h_n(-M) \le h_n(x) \le h_n(-\frac{1}{K}) < -1+\epsilon$. In other words, $|h_n(x) + 1| < \epsilon$. Analogous, we obtain $|h_n(x) - 1| < \epsilon$ on $\left[\frac{1}{K}, M\right]$.

2nd part. Given a partition $P = \{a = x_0, x_1, ..., x_m = b\}$ of $[a, b]$ with $a < x_1 < ... < b$. For each $n \in \mathbb{N}$ and $p = 2, ..., m$ we define in $[a, b]$ the function

$$F_n\left(x_{p-1}, m, x\right) = \frac{\sqrt[2n+1]{x_{p-1} - x_0} + \sqrt[2n+1]{x - x_{p-1}}}{\sqrt[2n+1]{x_m - x_{p-1}} + \sqrt[2n+1]{x_{p-1} - x_0}}$$

Then, it follows that $\lim_{n\to\infty} F_n\left(x_{p-1},m,x\right) = \begin{cases} 0 & x < x_{p-1} \\ \frac{1}{2} & x = x_{p-1} \\ 1 & x > x_{p-1} \end{cases}$, $p = 2,...,m$

and these limits are uniform on $[x_0, x_1-\beta]\cup[x_1+\beta, x_2-\beta]\cup...\cup[x_{m-1}+\beta, x_m]$

The first assertion is consequence of the elementary properties of the limits and the second is obtained by aplying the first part and take into acount that for each $p = 2,...,m$ only a root of $F_n\left(x_{p-1},m,x\right)$ depends upon x.

3rd part. By the second part and the definitions of $C_{m,n}$ and f we obtain the result which we want to prove. □

Proposition 2.8. *Let $\beta > 0$ be such that $(x_i-\beta, x_i+\beta)\cap(x_j-\beta, x_j+\beta) = \varnothing$ where $i \neq j$ and $i,j \in \{1,...,m-1\}$. Then, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ it follows that*

1. $\mid C_{m,n}(x) - f(x) \mid < \mid k_{j+1} - k_j \mid + \varepsilon$
2. $|C_{m,n}(x) - (k_j\cdot(1-\alpha) + k_{j+1}\cdot\alpha)| < \varepsilon$

where $x \in (x_j-\beta, x_j+\beta)$, $j = 1,...,m-1$and $\alpha \in (0,\,1)$.

Proof. **1st Part.** Let $x \in (x_j-\beta,\, x_j+\beta)$ be, $j = 1,...,m-1$. By the Proposition 2.7 the sequence $\left(F_n\right)_n$ converges uniformly to 1 as $p-1 < j$ and to 0 as $p-1 > j$.

Moreover there exists $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ the function $\left(k_{j+1}-k_j\right)F_n\left(x_{p-1},m,x\right)$ transforms the interval $(x_j-\beta, x_j+\beta)$ into the interval $(0,(k_{j+1}-k_j))$. The rest is obtained by the elementary properties of the limits and the definition of $C_{m,n}(x)$.

2nd Part. It is analogous to the 1st part by considering $\forall n > n_0$ the function $\left(k_{j+1}-k_j\right)F_n\left(x_{p-1},m,x\right)$ attains on $(x_j-\beta,\ x_j+\beta)$ the values $\left(k_{j+1}-k_j\right)\cdot \alpha$, $\alpha \in (0,1)$. □

Theorem 2.9. *The RAFU linear space C is uniformly dense in $C\,[a,b]$.*

Proof. Consider the family $\mathcal{L}$ of all sets which are finite unions of disjoint compact intervals. First, we will prove that C S-separates every pair of disjoint sets of $\mathcal{L}$. Clearly , it suffices to prove the following fact: *Given $\delta > 0$ and the intervals $[\alpha_1, \beta_i]$, $1 \le i \le m$, $m \ge 2$, $0 \le \alpha_j < \beta_j < \alpha_{j+1} < 1$, there is a function f in C such that $0 \le f(x) \le 1$ for every $x \in [a,b]$, $f\left([\alpha_i,\beta_i]\right) \subset [0,\delta]$ for i odd and $f\left([\alpha_i,\beta_i]\right) \subset [1-\delta,1]$ for i even, $1 \le i \le m$.*

Consider a partition $P = \{x_i\}_0^n$ of $[a,b]$ with $x_j = a + j\cdot\frac{b-a}{n}$, $j = 0,...,n$ such that $\beta_j < x_p < \alpha_{j+1}$ for every j and some x_p. We also consider a step function h defined in $[a,b]$ from the values x_j such that $h(x) = 0$ or $h(x) = 1$ for every $x \in [a,b]$ but verifying that $h\left([x_s,x_t]\right) = 0$ when $[\alpha_i,\beta_i] \subset [x_s,x_t]$ and i is odd, $h\left([x_k,x_l]\right) = 1$ when $[\alpha_i,\beta_i] \subset [x_k,x_l]$ and i is even, $1 \le i \le m$.

Fixed an appropriate value $\beta \le min\left\{\frac{|x_p-\beta_j|}{2}, \frac{|\alpha_{j+1}-x_p|}{2}\right\}$ and given $\delta > 0$ we can choose suitable partitions of $[a,b]$ into $2kn$ intervals, if it was necessary for some $k \in \mathbb{N}$, supporting the previous conditions and, by the propositions 2.7 and 2.8, we can obtain a function $C_{2kn} \in \mathsf{C}$ such that $0 \le C_{2kn}(x) \le$

1 and $|C_{2kn} - h| < \delta$, that is to say, $C_{2kn}([\alpha_i, \beta_i]) \subset [0, \delta]$ for i odd and $C_{2kn}([\alpha_i, \beta_i]) \subset [1-\delta, 1]$ for i even.

Next, we will prove that $\mathbb{C}$ S-separates every pair of disjoint *zero-sets* Z_1 and Z_2 of $[a,b]$. Since $\mathcal{L}$ is a basis for the closed sets of $[a,b]$ we have $Z_1 = \cap \{B \in \mathcal{L} : Z_1 \subset B\}$. As Z_2 is compact the family $\{Z_2\} \cup \{B \in \mathcal{L} : Z_1 \subset B\}$ does not have the finite intersection property. Therefore $Z_2 \cap B_1 \cap ... \cap B_p = \emptyset$, for some $B_i \in \mathcal{L}$, $Z_i \subset B$, $1 \leq i \leq p$. Since $\mathcal{L}$ is closed under finite intersections it follows that $B' = B_1 \cap ... \cap B_p \in \mathcal{L}$, $Z_1 \subset B'$ and $B' \cap Z_2 = \emptyset$. In the same way we find $B'' \in \mathcal{L}$, such that $Z_2 \subset B''$ and $B' \cap B'' = \emptyset$. Since $\mathbb{C}$ S-separates B' and B'', by Blasco-Moltó's Theorem, $\mathbb{C}$ is uniformly dense in $C[a,b]$. □

The S-separation of subsets is equivalent to the S'-separation of subsets in linear spaces containing constant functions (Garrido-Montalvo [8]). Clearly $\mathbb{C}$ contains the constant functions, therefore we can also deduce the uniform density of $\mathbb{C}$ in $C[a,b]$ by using the S'-separation condition of every pair of disjoint *zero-sets* in X.

3. The degree of uniform approximation with the RAFU linear space

Blasco-Moltó [6] proved that the linear subspace $\mathcal{F}$ of $C[0,1]$ generated by the functions

$$\{exp((x+\mu)^n) : \mu \in \mathbb{R}, x \in [0,1], n = 0, 1, 3, ..., 2k+1, ...\}$$

is uniformly dense in $C[0,1]$, but the linear combinations which approximate uniformly a function $f \in C[0,1]$ and the degree of uniform approximation that $\mathcal{F}$ provides were not studied.

The following result has been proved recently in the **XXII CEDYA-XII CMA** [2] and solves these two problems by considering the linear space $\mathbb{C}$

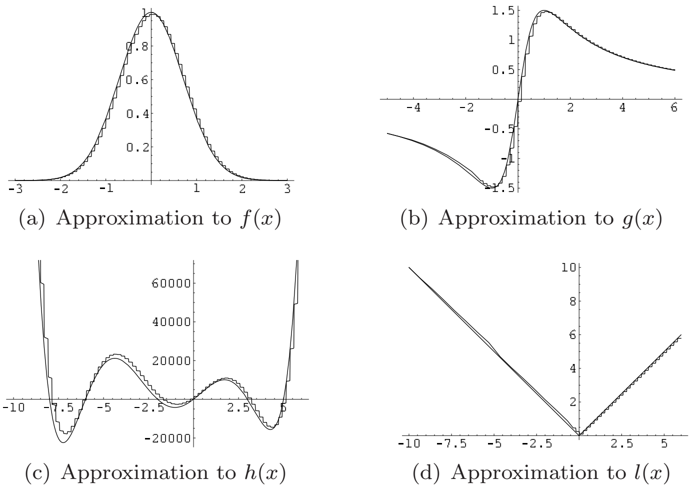
Theorem 3.1. *Let f be a continuous function defined on $[a,b]$ and let $P = \{x_0 = a, x_1, ..., x_n = b\}$ be a partition of $[a,b]$ with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0$, ..., n. Then,*

$$\|C_n - f\| \leq \frac{M-m}{\sqrt{n}} + \omega\left(\frac{b-a}{n}\right), \quad n \geq 2$$

where $\|\|$ denotes the uniform norm, M and m are the maximum and the minimum of f on $[a,b]$ respectively, $\omega(\delta)$ its modulus of continuty and $C_n(x)$ is defined for all $x \in [a,b]$ and $n \in \mathbb{N}$ by $C_n(x) = f(a) + \sum_{j=2}^{n}[f(x_j) - f(x_{j-1})] \cdot F_n(x_{j-1}, x)$

Let us observe that the values $\{k_i\}_{i=1}^{n}$of (2.1) becomes $\{f(x_i)\}_{i=1}^{n}$in this case.

Theorem 3.2 (Gassó-Hernández-Rojas [9])**.** *Let A be a subset of $C(X)$ and E a linear space of $C(X)$ which S-separates Lebesgue-sets of A. Then the sublattice generated by A is contained in $\overline{\sum(E)}$.*



(a) Approximation to $f(x)$
(b) Approximation to $g(x)$
(c) Approximation to $h(x)$
(d) Approximation to $l(x)$

Figure 1. Approximation with the RAFU linear space $\mathtt{C}$

The RAFU linear space $\mathtt{C}$ satifies the Theorem 3.1 when $X = [a,b]$ because every *Lebesgue-set* is also a *zero-set* since $L_\alpha(f) = Z((f-\alpha)\vee 0)$ and $L^\beta(f) = Z((f-\beta)\wedge 0)$ and we have proved that $\mathtt{C}$ S-separates every pair of disjoint *zero-sets* Z_1 and Z_2 of $[a,b]$. In this case, if $A = C\,(X)$ we can say that $C\,(X)$ is contained in$\overline{\sum(\mathtt{C})}$. In fact, given $f \in C\,[a,b]$, we already knew that $f(x) = \sum_{n=1}^{\infty} c_n(x)$ where $c_n \in \mathtt{C}$, $n \in \mathbb{N}$, and the series converges uniformly.

Example 3.3. Given the functions $f(x) = e^{-x^2}$on $[-3,3]$, $g(x) = \frac{3x}{x^2+1}$ on $[-5,6]$, $h(x) = 5(x+8)(x+6)(x+2)x(x-3)(x-5)$ on $[-10,6]$ and $l(x) =\mid x \mid$ on $[-10,6]$, the Figure 1 shows the graphics of these functions together with their approximations by means of its respective radical function C_{75} belonging to the RAFU linear space $\mathtt{C}$.

4. Conclusions

The RAFU method is an original and unknown procedure of uniform approximation on $C\,[a,b]$. This method improves the instability of the polynomial interpolation and it is based in the use of radical functions to approximate any continuous function defined in $[a,b]$. We have constructed a linear Space $\mathtt{C}$ uniformly dense on $C\,[a,b]$ and this linear space is not a lattice or an algebra. At the moment, the proof of this result was direct but in this work we have proved that $\mathtt{C}$ is uniformly dense on $C\,[a,b]$ by using a S-separation condition due to Blasco-Moltó [6] or an equivalent S'-separation condition due to Garrido-Montalvo [8]. We already knew another example of a linear space uniformly dense by using these separation conditions [6] but by considering the set $\mathtt{C}$, we can know easily the linear combinations of elements belonging to

C which approximate uniformly every $f \in C[a,b]$ and the degree of uniform approximation that C provides.

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