On certain new notion of order Cauchy sequences, continuity in (l)-group

SUDIP KUMAR PAL [a,1] AND SAGAR CHAKRABORTY [b]

[a] Department of Mathematics, Diamond Harbour Women's University, Diamond Harbour-743368, West Bengal, India. (`sudipkmpal@yahoo.co.in`)

[b] Jadavpur University, Kolkata-700032, West Bengal, India. (`sagarchakraborty55@gmail.com`)

Communicated by M. A. Sánchez-Granero

ABSTRACT

In this paper, we introduce the notions of order quasi-Cauchy sequences, downward and upward order quasi-Cauchy sequences, order half Cauchy sequences. Next we consider an associated idea of continuity namely, ward order continuous functions [2] and investigate certain interesting results. The entire investigation is performed in (l)-group setting to extend the recent results in [5, 6].

1. INTRODUCTION

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences especially in computer science, information theory, biological science. In 2010, Burton and Coleman first introduce the term quasi-Cauchy sequence which is weaker than Cauchy sequence but interesting in their own right. They defined the term quasi-Cauchy sequence as: any sequence of real numbers (x_n) is quasi-Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that

[1]Corresponding author.

$n \geq K$ implies $|x_{n+1} - x_n| < \epsilon$. Evidently Cauchy sequences are quasi-Cauchy but the converse is not true in general as the counter example is provided by the sequence of partial sums of the harmonic series. This and several such examples establish the important fact that the class of quasi-Cauchy sequences is much bigger than the class of Cauchy sequences, taking in the process more sequences under the preview. Understandably mathematical consequence are not analogous to the already existing notions bases on Cauchy sequences, like the usual idea of compactness. In a current development of the study of generalized metric space, the term ward continuity comes remembering the definition of continuity in sequential sense. The concepts of ward continuity of real valued function and ward compactness of subsets of $\mathbb{R}$ are introduced by Cakalli [5]. A real valued function f is called ward continuous on E if for every quasi-Cauchy sequence in E, the corresponding f-image sequence is also quasi-Cauchy.

The aim of this paper is to introduce the notion of order quasi-Cauchy sequences and some weaker versions of it [2]. We primarily investigate several features of this new notion. Finally, a new concept, namely the concept of order statistical ward continuity of a function is introduced and investigated[7]. In this investigation we have obtained theorems related to order statistical ward continuity, order statistical ward compactness, compactness, and uniform order continuity. We also introduced and studied some other continuities involving statistical order quasi-Cauchy sequences and order convergent sequences of points in l-group[8].

Throughout $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of all real numbers and natural numbers respectively and our topological terminologies and notations are as in the book [9] from where the notions (undefined inside the article) can be found. All spaces in the sequel are l-group.

2. Preliminaries

First we recall the concept of 'natural density' [9] of a set A of positive integers, which is defined by

$$\delta(A) = \lim_{n \to \infty} \frac{d(k \leq n : k \in A)}{n},$$

Where d denotes the cardinality of the concerned sets.

The notion of statistical convergence which is an extension of the idea of usual convergence, was introduced by H. Fast [10] and I. J. Schoenberg [13]. Any sequence (x_n) in $\mathbb{R}$ is statistical convergent to the number L provided that for each $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{d(k \leq n : |x_k - L| \geq \epsilon)}{n} = 0.$$

Equivalently, $|x_k - L| < \epsilon$ for almost all k. Topological consequences of statistical convergence were studied by Fridy [11] and Šalát [12]. The study of statistical convergence and its numerous extensions and, in particular, of the ideal convergence and its applications has been one of the most active areas of research in the summability theory over the last 15 years.

Now we recall some concepts related to lattice, order convergence and lattice order group. A nonempty set L is said to be a lattice with respect to the partial order $\leq$ if for each pair of elements $x, y \in L$, both the supremum and infimum of the set $\{x, y\}$ exists in L. We shall write $x \vee y = sup\{x, y\}$ and $x \wedge y = inf\{x, y\}$.

Definition 2.1. An abelian group $(L, +)$ is said to be an (l)-group if it is lattice and $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in L$.

From now throughout this paper we will write L for (l)-group $(L, +)$ and θ denotes the identity element of the (l)-group $(L, +)$.

Let $x \in L$ be any element, we define $|x| = x \vee (-x)$ where $-x$ denotes the additive inverse of x. Also we use the notation $a \geq b$ equivalent as $b \leq a$, and $a > b$ as equivalent to $b \leq a$ with $b \neq a$.

A sequence (x_n) in $L($ i.e. a map $: \mathbb{N} \to L)$ is said to be increasing(or decreasing) if $x_1 \leq x_2 \leq ...$ (or $x_1 \geq x_2 \geq ...$) and we write it symbolically as $x_n \uparrow$ (or $x_n \downarrow$).

A sequence (p_n) is called an order sequence if $p_n \downarrow$ and $\inf p_n = \theta$. In this case we write $p_n \downarrow \theta$. Some author use the term monotone sequences instead of order sequence. It is easy to observe that if (a_n) and (b_n) are two order sequences then the sequence $(a_n + b_n)$ is also an order sequence.

A sequence (x_n) in L is said to be order bounded if there exists an order interval $[a, b]$ such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$.

A sequence (a_n) in L is said to be convergent in order(or order convergence)to $a \in L$ if there exists an order sequence (p_n) such that $|a_n - a| \leq p_n$ holds for all n. We write it symbolically as $a_n \xrightarrow{ord} a$.

In the literature, there are two ways to define order convergence. Other than the above way one can define order convergence as: A sequence (a_n) in L is said to be order convergence to $a \in L$ if there exists an order sequence (p_n) such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|a_n - a| \leq p_{n_0}$ for all $n \geq m$.

The later definition is useful for defining order convergence in filter. The first definition is called 1-converging and the second one is called 2-converging. If the lattice is Dedekind complete then the two definitions are equivalent[1] Throughout the paper we use the second type definition of order convergence.

If a sequence (x_n) is order convergent to x_0 then we call the sequence $(x_n - x_0)$ as a null order sequence which converges to θ.

3. Main Results

3.1. **Ward Continuity in (l)-group.** In this section, we introduce the concept of order quasi-Cauchy sequences and ward continuity in (l)-group L, and we study some results related to it.

We know that any sequence of real numbers (x_n) is said to be Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that $m, n \geq K$ implies $|x_m - x_n| < \epsilon$.

In 2010, Burton and Coleman first use the term quasi-Cauchy sequence. They defined the term quasi-Cauchy sequence as: any sequence of real numbers

(x_n) is quasi-Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies $|x_{n+1} - x_n| < \epsilon$. Using this idea we first introduce two definitions.

Definition 3.1. Any sequence (x_n) in a (l)-group L is said to be an order-Cauchy sequence if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_i - x_j| \leq p_{n_0}$ for all $i, j \geq m$.

Definition 3.2. Any sequence (x_n) in a (l)-group L is said to be an order quasi-Cauchy sequence if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_{n+1} - x_n| \leq p_{n_0}$ for all $n \geq m$.

Remark 3.3. It is easy to verify that every order-Cauchy sequence is order quasi-Cauchy but the converse is not true in general. For counter example we take (l)-group $(\mathbb{R}, +)$ and consider the sequence (x_n) where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$. Clearly this sequence is order quasi-Cauchy but not order-Cauchy.

Remark 3.4. Every order convergent sequence is also order quasi-Cauchy.

Remark 3.5. Also every subsequence of order-Cauchy sequence is order-Cauchy. But the analogous property fails for quasi-Cauchy sequences. For instance we take the sequence (x_n) in $\mathbb{R}$, $x_n = \sqrt{n}$. (x_n) is order quasi-Cauchy but the subsequence (x_{n^2}) is not order quasi-Cauchy.

We know that if a function preserves Cauchy sequences, then it is called Cauchy continuous function. Similarly we define quasi-Cauchy continuous function in (l)-group. Some author rename it as 'ward continuous function'. Throughout this paper we use the name ward continuous.

Definition 3.6. Let L be a (l)-group. A function $f : L \to L$ is said to be Ward Continuous on L if the sequence $(f(x_n))$ is order quasi-Cauchy whenever (x_n) is order quasi-Cauchy in L.

Definition 3.7. A subset E of L is said to be ward compact if any sequence in E has an order quasi-Cauchy subsequence.

The following theorems are obvious.

Theorem 3.8.   (1) *Every finite subset of L is ward compact.*
    (2) *Union of any two ward compact subsets of L is ward compact.*
    (3) *Intersection of any family of ward compact sets is ward compact.*
    (4) *Any subset of ward compact set is ward compact.*

We see that for any metric space, continuity can be described by using sequence. Remembering this idea, we introduce continuity in (l)-group L. We call it order continuity.

Definition 3.9. A function $f : L \to L$ is said to be order continuous at x_0 if for any sequence (x_n) in L, which is order convergent to x_0, the corresponding image sequence $(f(x_n))$ is order convergent to $f(x_0)$.

In the next theorem we will investigate the relationship between ward continuity and order continuity [3].

Theorem 3.10. *Let $f : R \to L$ be a ward continuous function, where $R \subset L$ then it is order continuous on R.*

Proof. Suppose that $f : R \to L$ is ward continuous on $R \subset L$. Let (x_n) be a sequence in R such that $x_n \xrightarrow{ord} x_0$. Now we define a new sequence (y_n) as :

$$y_n = \begin{cases} x_0, & \text{if } n \text{ is even} \\ x_k, & \text{if } n = 2k - 1, \ k \in \mathbb{N}. \end{cases}$$

So,

$$y_n - x_0 = \begin{cases} \theta, & \text{if } n \text{ is even} \\ x_k - x_0, & \text{if } n = 2k - 1, \ k \in \mathbb{N}. \end{cases}$$

As (x_n) is order convergent to x_0 so for any order sequence (p_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_n - x_0| \leq p_{n_0}$ for all $n \geq m$. Now using this (p_n) and $n_0 \in \mathbb{N}$ with some suitable changes of m, from the construction of $y_n - x_0$, we can easily conclude that (y_n) is order convergent. Hence it is an order quasi-Cauchy sequence. As f is ward continuous so it preserves order quasi Cauchy sequence. Hence $(f(y_n))$ is also order quasi-Cauchy, which is given by:

$$f(y_n) = \begin{cases} f(x_0), & \text{if } n \text{ is even} \\ f(x_k), & \text{if } n = 2k - 1, \ k \in \mathbb{N}. \end{cases}$$

Now for any order sequence (q_n) and $n_0' \in \mathbb{N}$, there exists $m' \in \mathbb{N}$ such that $|f(y_{n+1}) - f(y_n)| \leq p_{n_0'}$ for all $n \geq m'$ which implies $|f(x_k) - f(x_0)| \leq p_{n_0'}$, for all $k \geq M$, where $M(\in \mathbb{N})$ depends on m'. So $f(x_n) \xrightarrow{ord} f(x_0)$. Hence f is order continuous. $\square$

Converse of the above theorem is not true in general, which follows from the next example.

Example 3.11. Conisder the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$ and consider the sequence (x_n) given by $x_n = \sqrt{n}$.

Theorem 3.12. *Ward continuous function preserves ward compact set.*

Proof. Let $f : L \to L$ be a ward continuous map and $E \subseteq L$ be ward compact set. Let (x_n) be any sequence in E, as E is ward compact so we get subsequence (y_n) of (x_n) such that (y_n) is order quasi-Cauchy sequence. Now as f is ward continuous function, $(f(y_n))$ is order quasi-Cauchy subsequence of the sequence $(f(x_n))$ in $f(E)$. This completes the proof of the theorem. $\square$

Definition 3.13. A function $f : L \to L$ is said to be uniformly order continuous on a subset E of L if for any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$, such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x - y| < \delta_m$.

Theorem 3.14. *If $f : L \to L$ is an uniform order continuous map on $E \subset L$ then it is ward continuous on E.*

Proof. Let (x_n) be any order quasi-Cauchy sequence of points in E. As f is uniform order continuous so any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$, such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x - y| < \delta_m$. This implies for this δ_n and m, n_0, we get suitably N, depends on δ_n, m, n_0 such that $|x_{n+1} - x_n| < \delta_n$ for all $n > N$. So $|f(x_n) - f(x_{n+1})| < \epsilon_{n_o}$, for all $n > N$. □

From the above theorem we can easily conclude that uniform order continuous functions are also order continuous.

Remark 3.15. Uniform order continuous image of ward compact set is ward compact.

We use the following notations :
$C[L, L]$ = Set of all order continuous functions on L.
$WC[L, L]$ = Set of all ward continuous functions on L.
$UC[L, L]$ = Set of all uniform order continuous functions on L.
Now from the above discussion we can easily conclude that,

Remark 3.16. $UC[L, L] \subseteq WC[L, L] \subseteq C[L, L]$.

We see that, a sequence (x_n) in L is said to be order convergent to $x_0 \in L$ if there exists an order sequence (p_n) such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|x_n - x_0| \leq p_{n_0}$ for all $n \geq m$. In this case choice of m depends on x_0. To deal this type of situation we introduce the concept of uniform order convergence in (l)-group.

Definition 3.17. A sequence (x_n) in $E \subset L$ is said to be uniform order convergent to $x \in E$ if there exists an order sequence (p_n) in L such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|x_n - x| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$.

Theorem 3.18. *Let (f_n) be a sequence of uniform order continuous functions on $E \subset L$ and (f_n) is uniformly order convergent to a function f then f is uniform order continuous on E.*

Proof. Suppose that (f_n) is uniformly order convergent to some function f. Then for a given order sequence (p_n) in L and for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$. Now each $f_n : L \to L$ is uniform order continuous on E. Hence for this order sequence (p_n), $n' \in \mathbb{N}$, there exists order sequence q_n and $m' \in \mathbb{N}$ such that $|f_n(x) - f_n(y)| < p_{n'}$, for all $n \geq m'$, whenever $|x - y| \leq q_n$. Now $f(x) - f(y) = f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)$, sum of three null sequences, hence $f(x)$ is uniform order continuous on E. □

Theorem 3.19. *Let (f_n) be a sequence of ward continuous functions defined on $E \subset L$ and (f_n) is uniformly order convergent to a function f then f is ward continuous on E.*

Proof. Let (x_n) be an order quasi-Cauchy sequence of points on E. As (f_n) is uniformly order convergent to f, given order sequence (p_n) in L such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$. Now each f_n is ward continuous on E. So for this (p_n) and $n' \in \mathbb{N}$ there exists $m' \in \mathbb{N}$ with $|f_m(x_{n+1}) - f_m(x_n)| < p_{n'}$ for all $n \geq m'$. Now

$$f(x_{n+1}) - f(x_n) = f(x_{n+1}) - f_m(x_{n+1}) + f_m(x_{n+1}) - f_m(x_n) + f_m(x_n) - f(x_n).$$

This implies $f(x_{n+1}) - f(x_n)$ is sum of three null sequences, so we easily conclude that f is ward continuous on E. $\square$

3.2. **Statistical ward continuity in (l)-group.** Recently, it has been proved that a real-valued function defined on an interval A of the set of real numbers, is uniformly continuous on A if and only if it preserves quasi-Cauchy sequences of points in A. In this section we call a real-valued function order statistically ward continuous if it preserves statistical order quasi-Cauchy sequences. It turns out that any order statistically ward continuous function on a statistically ward order compact subset A of an l-group is uniformly order continuous on A. We prove theorems related to order statistical ward compactness, order statistical compactness, order continuity, statistical order continuity, ward order continuity, and uniform order continuity.

Definition 3.20. We call a sequence (x_n) of points in (l)-group L statistically order quasi-Cauchy if for any order sequence (p_n) and $n_0 \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{d(k \leq n : |x_{k+1} - x_k| \geq p_{n_0})}{n} = 0.$$

Where $d(A)$ denotes the cardinality of the set A.

It is clear that, any order quasi Cauchy sequence is statistically order quasi-Cauchy.

Definition 3.21. A subset E of L is said to be statistically ward compact if any sequence of points in E has a statistically order quasi-Cauchy subsequence.

Definition 3.22. A function $f : E \to L$ is said to be statistically order continuous on $E \subseteq L$ if it preserves statistically order convergent sequences.

Definition 3.23. Let $E \subseteq L$. A function $f : E \to L$ is said to be statistically ward continuous if it preserves statistically order quasi-Cauchy sequences.

Theorem 3.24. *Every statistically ward continuous functions are also statistically order continuous.*

Proof. Let $f : E \to L$ be a statistically ward continuous function and (x_n) be any statistically order convergent sequence which converges to x_0. For any order sequence (p_n), $n_0 \in \mathbb{N}$, $\lim_{n\to\infty} \frac{d(k \leq n : |x_k - x_0| \geq p_{n_0})}{n} = 0$. Hence the sequence $(x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$ is also statistically order convergent to x_0. Hence it is statistically order quasi-Cauchy. As f is statistically

ward continuous so $(f(x_1), f(x_0), f(x_2), f(x_0), f(x_3), ...)$ is also statistically order quasi-Cauchy. As the even terms of the sequence are $f(x_0)$, odd terms are nothing but the sequence $(f(x_n))$, we can easily conclude that $(f(x_n))$ is statistically order convergent to $f(x_0)$. This completes the proof. $\qquad\square$

The converse is not true in general. For counter example we take the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and consider the sequence $(\sqrt{n})$.

We know that any continuous function on a compact set is uniformly continuous. Similarly for statistically ward continuous function defined on a statistically ward compact subset of an (l)-group, we have the following result :

Theorem 3.25. *Let E be a statistically ward compact subset of an (l)-group L and $f : E \to L$ be a statistically ward continuous function on E. Then it is uniform order continuous.*

Proof. If possible suppose that f is not uniformly order continuous on E. Then there exists order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ such that for any δ_n and $m \in \mathbb{N}$ with $|x - y| \leq \delta_m$, $|f(x) - f(y)| > \epsilon_{n_0}$. Now for each $n \in \mathbb{N}$ fix $|x_n - y_n| < \delta_n$ and $|f(x_n) - f(y_n)| \geq \epsilon_{n_0}$. Since E is statistically ward compact so (x_n) has a subsequence (x_{n_k}) which is statistically order quasi-Cauchy. Now $y_{n_{k+1}} - y_{n_k} = (y_{n_{k+1}} - x_{n_{k+1}}) + (x_{n_{k+1}} - x_{n_k}) + (x_{n_k} - y_{n_k})$ which is clearly sum of three null sequences hence (y_{n_k}) is statistically order quasi-Cauchy subsequence of (y_n). As $x_{n_{k+1}} - y_{n_k} = (x_{n_{k+1}} - y_{n_{k+1}}) + (y_{n_{k+1}} - y_{n_k})$ so the sequence $(x_{n_{k+1}} - y_{n_k})$ is statistically order convergent to θ. Hence the sequence $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, ..., x_{n_k}, y_{n_k}, ...)$ is statistically order quasi-Cauchy. Which implies that the sequence $(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), ..., f(x_{n_k}), f(y_{n_k}), ...)$ is also statistically order quasi-Cauchy in $f(E)$. But this contradicts the fact that $|f(x) - f(y)| > \epsilon_{n_0}$. Thus f is uniformly order continuous on E. $\qquad\square$

Theorem 3.26. *Statistically ward continuous image of any statistically ward compact subset of L is statistically ward compact.*

Proof. Let $f : A \to L$ be a statistically ward continuous function defined on a subset A of L and E be a statistically ward compact subset of A. We want to show $f(E)$ is statistically ward compact subset of L. Let (y_n) be any sequence of points in $f(E)$. Then clearly $y_n = f(x_n)$ for some sequence (x_n) of points in E. As E is statistically ward compact set so there exists subsequence $(z_k) = (x_{n_k})$ of (x_n) such that (z_k) is statistically ward compact. Now as f is ward continuous function so $f(z_k)$ is statistically order quasi-Cauchy sequence. Thus we get a order quasi-Cauchy subsequence $(f(z_k))$ of the sequence (y_n) of $f(E)$. Hence $f(E)$ is a ward compact set. $\qquad\square$

Theorem 3.27. *If (f_n) is a sequence of statistically ward continuous functions on a subset E of L and (f_n) is uniformly order convergent to a function f then f is statistically order ward continuous on E.*

Proof. Suppose that (f_n) is uniform order convergence to f. For any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon_{n_0}$,

for all $n \geq m$ and for all $x \in E$. Consider any statistically order quasi-Cauchy sequence (x_n) of points in E. As f_m is statistically ward continuous on E so it preserves statistically order quasi-Cauchy sequence. Hence $\lim_{n \to \infty} \frac{d(k \leq n : |f_m(x_{k+1}) - f_m(x_k)| \geq \epsilon_{n_0})}{n} = 0$. Now

$f(x_{k+1}) - f(x_k) = [f(x_{k+1}) - f_m(x_{k+1})] + [f_m(x_{k+1}) - f_m(x_k)] + [f_m(x_k) - f(x_k)]$. Hence using the fact that f_m is statistically ward continuous and f_m is uniform convergent to f we can easily conclude that

$$\lim_{n \to \infty} \frac{d(k \leq n : |f(x_{k+1}) - f(x_k)| \geq \epsilon_{n_0})}{n} = 0.$$

This completes the proof. □

The following result follows immediately:

Theorem 3.28. *The set $SWC[E, L]$, the set of all statistical ward continuous functions is a closed set.*

From the above discussion we see that $UC[L, L] \subseteq WC[L, L] \subseteq C[L, L]$. Now the obvious question is when these sets are equal. In [2], Burton and Coleman gives some partial idea about the equality of $UC[L, L] \subseteq WC[L, L]$.

Theorem 3.29. *Let $I \subseteq \mathbb{R}$ be any interval. Then $UC[I, \mathbb{R}] = WC[I, \mathbb{R}]$.*

Now we introduce another type of convergence in $(l) - group$ called slowly oscillating order convergence.

Definition 3.30. A sequence (x_n) of points in (l)-group L is called slowly oscillating order convergence if for any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_i - x_j| < \epsilon_{n_0}$ for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$.

From definition it is clear that order Cauchy sequences are obviously slowly oscillating and every slowly oscillating sequence is order quasi-Cauchy.

Definition 3.31 ([4, 14]). A function $f : E \to L$ is said to be slowly oscillating continuous if it preserves slowly oscillating order sequences.

By SOC[E, L] we denote set of all slowly oscillating continuous functions defined on E.

Now we introduce the concept of Connectedness in (l)-group L.

Definition 3.32. Let L be a (l)-group. Suppose that $x \in U \subseteq L$. U is called order sequential neighborhood of $x \in L$ if any sequence (x_n) which is order convergent to x then $\{x_n : n \geq m\} \subset U$ for some $m \in \mathbb{N}$.

Definition 3.33. U is said to be an order sequential open subset of L if for each $x \in U$, U is order sequential neighborhood of x.

Definition 3.34. A is said to be an order sequential closed subset of L if $L \setminus A$ is an order sequential open subset of L.

Definition 3.35. Let $A \subseteq L$, by $\bar{A}$ we denote closure of A, defined as intersection of all sequentially closed sets containing A.

Definition 3.36. An (l)-group L is said to be order connected if there do not exists any non empty subsets A, B such that $X = A \cup B$ with $\bar{A} \cap \bar{B} = \phi$.

Now we modify the Lemma 1 in [2] which is given in metric space setting.

Lemma 3.37. *Let $((a_n, b_n))$ be a sequence of ordered pair of points in a connected subset $E \subseteq L$ such that given any ordered sequence (ϵ_n) there exists $n_0, m \in \mathbb{N}$, we have $|a_n - b_n| \leq \epsilon_{n_0}$ for all $n \geq m$. Then there exists an order quasi-Cauchy sequence (t_n) with the property that for any positive integer i there exists a positive integer k such that $(a_i, b_i) = (t_{j-1}, t_j)$.*

It turns out that a function defined on a connected subset E of a metric space is uniformly continuous if and only if it preserves either quasi-Cauchy sequences or slowly oscillating sequences of points in E.

Now we are in the position of most desired result:

Theorem 3.38. *Let E be an order connected subset of a (l)-group L then the three sets $UC[E, L], WC[E, L]$ and $SOC[E, L]$ are equivalent.*

Proof. $\underline{UC[E, L] \subseteq WC[E, L]}$: Let $f : E \to L$ be any uniformly order continuous function on E. Let (x_n) be any order quasi-Cauchy sequence of points in E. As f is uniform order continuous so any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$ such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x - y| < \delta_m$. This implies for this δ_n and m, n_0, we get suitably N, depends on δ_n, m, n_0 such that $|x_{n+1} - x_n| < \delta_n$ for all $n > N$. So, $|f(x_n) - f(x_{n+1})| < \epsilon_{n_0}$, for all $n > N$.

$\underline{UC[E, L] \subseteq SOC[E, L]}$: Let $f : E \to L$ be uniform order continuous. We take slowly oscillating sequence (x_n) of points on E. Let (ϵ_n) be any order sequence. We get another order sequence (δ_n) and $n_0, m_0 \in \mathbb{N}$ such that $|f(x_i) - f(x_j)| \leq \epsilon_{n_0}$ whenever $x_i, x_j \in E$ and $|x_i - x_j| \leq \delta_{m_0}$. As (x_n) is slowly oscillating so $|x_i - x_j| \leq \delta_k$, for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$. Now using uniform continuity of f, $|f(x_i) - f(x_j)| \leq \epsilon_k$, for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$. This implies $(f(x_n))$ is slowly oscillating. Hence $f \in SOC[E, L]$.

$\underline{SOC[E, L] \subseteq UC[E, L]}$: Let $f : E \to L$ be not uniformly continuous on E. Now each $n \in \mathbb{N}$, we fixed $|x_n - y_n| < \delta_n$ then as f is not uniformly order continuous so there exists order sequence ϵ_n such that $|f(x_n) - f(y_n) \geq \epsilon(k)|$. Now As it is given that E is order connected so by the Lemma 3.1, from (x_n) we can construct a slowly oscillating sequence (t_n) but as f is not uniformly continuous, so the transformed sequence $(f(t_n))$ is not slowly oscillating. Hence $f \notin SOC[E, L]$. This implies $SOC[E, L] \subseteq UC[E, L]$.

$\underline{WC[E, L] \subseteq UC[E, L]}$: Suppose f is not uniformly order continuous on E. Since we know that slowly oscillating sequences are also order quasi-Cauchy hence the sequence (t_n) constructed on previous case is also order quasi-Cauchy, but as f is not uniformly order continuous so $f(t_n)$ is not order quasi-Cauchy. So $WC[E, L] \subseteq UC[E, L]$.

This completes the proof of the theorem. □

4. Downward and upward order Continuity in (l)-group

In this section, we introduce and investigate the concepts of down order continuity and down order compactness. A real valued function f on a subset E of the set of real numbers is down continuous if it preserves downward half Cauchy sequences, i.e. the sequence $(f(a_n))$ is downward half Cauchy whenever (a_n) is a downward half Cauchy sequence of points in E. A sequence (a_k) of points in $\mathbb{R}$ is called downward half Cauchy if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $a_m - a_n < \epsilon$ for $m \geq n \geq n_0$. It turns out that the set of all down continuous functions is a proper subset of the set of all continuous functions. First we introduce the following definition:

Definition 4.1. Let (x_n) be a sequence in (l)-group L. Then (x_n) is called downward order quasi-Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x_{n+1} - x_n \leq p_{n_0}$, for all $n \geq m$.

It is clear that every order quasi-Cauchy sequence is also downward order quasi-Cauchy but the converse is not true in general. For example we take the lattice order group $(\mathbb{R}, +)$ and the sequence (x_n), where $x_n = -n$. This sequence is downward order quasi-Cauchy but not order quasi-Cauchy.

Any order Cauchy sequence is obviously order quasi-Cauchy and hence downward order quasi-Cauchy.

Definition 4.2. A sequence (x_n) of points in L is said to be downward order half Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $x_m - x_n \leq p_{n_0}$ where $m, n \in \mathbb{N}$ with $m \geq n > k$.

It is obvious that downward order half Cauchy sequences are also downward order quasi-Cauchy and any subsequence of downward order half Cauchy sequence are same type. But for the downward order quasi Cauchy sequences the situation is different. We take the sequence (x_n) in $(\mathbb{R}, +)$ such that $x_n = \sqrt{n}$. Clearly (x_n) is downward order half Cauchy but one of it's subsequence, namely (x_{n_k}) is not downward order half Cauchy.

In [2], authors proved that a sequence of real numbers is Cauchy if and only if every subsequence is quasi-Cauchy. In the next theorem we present similar type result for downward order half Cauchy sequences in (l)-group.

Theorem 4.3. *A sequence (x_n) in L is downward order half Cauchy if and only if every subsequence of (x_n) is downward order quasi-Cauchy.*

Proof. If (x_n) is downward order half Cauchy then every subsequence of (x_n) is downward half Cauchy so is downward order quasi-Cauchy.

To prove the converse part, we use contrapositive statement. Let (x_n) be not downward order half Cauchy. Then there exists order sequence (p_n) and $n_0 \in \mathbb{N}$ such that for every positive integer m, $x_{n_i} - x_{n_j} > p_{n_0}$, $n_i > n_j \geq m$.

Now for $m = 1$ we get such n_i, n_j, we rename it as k_1, k_2. So $k_2 > k_1 > 1$ and $x_{k_2} - x_{k_1} > p_{n_0}$. Similarly for $m = 2, 3, \ldots$. Inductively we get $x_{k_{n+1}} - x_{k_n} > p_{n_0}$, where $k_{n+1} > k_n > k_{n-1} > \ldots$. Which shows that the subsequence (x_{k_n}) is not downward order quasi-Cauchy. Hence the proof. $\square$

Now we study the sequential compactness like property. First of all we introducing the idea of downward order compact set in a lattice order group as:

Definition 4.4. A subset E of L is called downward order compact if any sequence of points in E has a downward order quasi-Cauchy subsequence.

We know that a real valued function of real variables is continuous if it preserves convergent sequences. In a similar way we already defined order continuity. Now if a function preserves downward order quasi-Cauchy sequences then we get a new type of continuity, we call it downward order continuity.

Definition 4.5. A function $f : E \to L$ is called downward order continuous on a subgroup E of L if it preserves downward order quasi-Cauchy sequences.

Theorem 4.6. *Sum of two downward order continuous functions is downward order continuous.*

Proof. Suppose that $f : E \to L$ and $g : E \to L$ are two downward order continuous functions on E, a subgroup of L. Let (x_n) be a downward order quasi-Cauchy sequence in E. As f, g both are downward order continuous so the sequences $(f(x_n))$ and $(g(x_n))$ both are downward order quasi-Cauchy. We know that sum of two order sequences is also an order sequence. Take any order sequence (p_n) then $(2p_n)$ is also order sequence. So for order sequence $2p_n$ and for $n_0 \in \mathbb{N}$ there exists positive integers m_1 and m_2 such that $f(x_{n+1}) - f(x_n) \leq p_{n_0}$ for all $n \geq m_1$ and $g(x_{n+1}) - g(x_n) \leq p_{n_0}$ for all $n \geq m_2$. Take $m = max\{m_1, m_2\}$. Then $f(x_{n+1}) + g(x_{n+1}) - g(x_n) - f(x_n) \leq 2p_{n_0}$ for all $n \geq m$. This proves the theorem. $\square$

From definition it is quite obvious that every ward order continuous function is downward order continuous. The following theorem make the link between ward order continuity and order continuity.

Theorem 4.7. *Every downward order continuous function is order continuous.*

Proof. Let $f : E \to L$ be downward order continuous on E and (x_n) be a sequence which order converges to x_0. Now we construct a sequence

$$(x_1, x_0, x_1, x_0, x_2, x_0, x_2, x_0, ...).$$

As (x_n) is order converges to x_0 so the new sequence is also order converges to x_0. Also from the construction it is clear that the new sequence is also downward order quasi-Cauchy. As f is downward order continuous so

$$(f(x_1), f(x_0), f(x_2), f(x_0), f(x_2), f(x_0), ...)$$

is downward order quasi-Cauchy. From here we can easily conclude that $f(x_n)$ order converges to $f(x_0)$. Hence f is order continuous. $\square$

Theorem 4.8. *Let E be a downward compact subgroup of L and $f : E \to L$ be downward order continuous function. Then $f(E)$ is also downward order compact.*

Proof. Suppose that E is a downward compact subgroup of L. Let us take any sequence (y_n) in $F(E)$. So $y_n = f(x_n)$ where $x_n \in A$ for each n. As E is downward order compact and (x_n) is any sequence in E so (x_n) has a downward order quasi-Cauchy sub-sequence, say, (z_k). As $f : E \to L$ is a downward order continuous function, $f(z_k)$ is a downward order quasi-Cauchy sequence. Hence we get $f(z_k)$ is a downward order quasi-Cauchy sub-sequence of (y_n). This completes the proof. $\square$

Now the question arise that does the downward continuous function preserves uniform limit? The following theorem gives the answer. The technique of the proof is almost same as word continuous function. So we just state the theorem.

Theorem 4.9. *If (f_n) be a sequence of downward continuous functions defined on a subgroup E of L and (f_n) is uniform order convergent to a function f then f is downward order continuous on E.*

If we change the position of x_{n+1} and x_n in the definition of downward order quasi-Cauchy sequence we get a new type of sequence, we call it upward order quasi-Cauchy sequence. The results are similar. We just state the result.

Definition 4.10. Let (x_n) be a sequence in (l)-group L. Then (x_n) is called upward order quasi-Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x_n - x_{n+1} \leq p_{n_0}$.

It is clear that every order quasi-Cauchy sequence is also upward order quasi-Cauchy. But the converse is not true in general. For example we take the lattice order group $(\mathbb{R}, +)$ and the sequence (x_n), where $x_n = n$. This sequence is upward order quasi-Cauchy but not order quasi-Cauchy.

Any order Cauchy sequence is obviously order quasi-Cauchy and hence upward order quasi-Cauchy.

REFERENCES

[1] Y. Abramovich and G. Sirotkin, On order convergence of nets, Positivity 9 (2005), 287–292.

[2] D. Burton and J. Coleman, Quasi-Cauchy sequences, Amer. Math. Monthly 117, no. 4 (2010), 328–333.

[3] I. Canak and D. Mik, New types of continuity, Abstr. Appl. Anal. 2010 (2010), Article ID: 258980.

[4] H. Çakalli, Slowly oscillating continuity, Abstr. Appl. Anal. 2008 (2008), Article ID 485706.

[5] H. Çakalli, Forward Continuity, J. Comput Anal. Appl. 13, no. 2 (2011), 225–230.

[6] H. Çakalli, Statistical ward continuity, Applied Mathematics Letters 24 (2011), 1724–1728.

[7] H. Çakalli, Statistical quasi-Cauchy sequences, Mathematical and Computer Modelling 54 (2011), 1620–1624.

[8] H. Çakalli and B. Haarika, Ideal quasi-Cauchy sequences, J. Inequal. Appl. 234 (2012), 1–11.

[9] R. Engelking, General topology, Sigma Ser. Pure Math., Heldermann, Berlin Vol. 6 2nd Edition (1989).

[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[11] J. A. Fridy, On stastistical convergence, Analysis 5 (1985), 301–313.

[12] P. Kostyrko, T. Šalát and W. Wilczyński, $\mathcal{I}$-convergence, Real Anal. Exchange. 26, no. 2 (2000/2001), 669–685.

[13] I. J. Schoenberg, The integrability methods, Amer. Math. Monthly 66 (1959), 361–375.

[14] R. W. Vallin, Creating slowly oscillating sequences and slowly oscilating continuous functions, Acta. Math. Univ. Comenianae 25 (2011), 71–78.