

Upper and lower cl -supercontinuous multifunctions

J. K. KOHLI AND CHAMAN PRAKASH ARYA

ABSTRACT

The notion of cl -supercontinuity (\equiv clopen continuity) of functions is extended to the realm of multifunctions. Basic properties of upper (lower) cl -supercontinuous multifunctions are studied and their place in the hierarchy of strong variants of continuity of multifunctions is discussed. Examples are included to reflect upon the distinctiveness of upper (lower) cl -supercontinuity of multifunctions from that of other strong variants of continuity of multifunctions which already exist in the literature.

2010 MSC: 54C05, 54C10, 54C60, 54D20.

KEYWORDS: upper(lower) cl -supercontinuous multifunction, strongly continuous multifunction, upper(lower) perfectly continuous multifunction, upper(lower) z -supercontinuous multifunction, upper(lower) D -supercontinuous multifunction, upper(lower) supercontinuous multifunction, cl -open set, zero dimensional space, cl -para-Lindelöf space, cl -paracompact space, nonmingled multifunction.

1. INTRODUCTION

Several weak and strong variants of continuity occur in the lore of mathematical literature which have been studied by host of authors. The strong variants of continuity with which we shall be dealing in this paper include strong continuity due to Levine [17], perfect continuity introduced by Noiri [19], clopen continuity (cl -supercontinuity) defined by Reilly and Vamanamurthy [20], and studied by Singh[21], and Kohli and Singh [15], complete continuity initiated by Arya and Gupta [5] and z -supercontinuity introduced by Kohli and Kumar

[14]. Multifunctions arise naturally in many areas of mathematics and applications of mathematics and have wide ranging applications in optimization theory, control theory, game theory, mathematical economics, dynamical systems and differential inclusions. Recently, there has been considerable interest in trying to extend the notions and results of weak and strong variants of continuity of functions to the realm of multifunctions (see [1], [2], [3], [4], [5], [8], [9], [10], [16], [22], [27]). The present paper is written in continuation of the same theme. In this paper we extend the notion of cl-supercontinuity of functions to the framework of multifunctions and introduce the notions of upper and lower cl-supercontinuous multifunctions and elaborate on their place in the hierarchy of strong variants of continuity of multifunctions. In the process we extend certain result of Singh [21] pertaining to cl-supercontinuous functions to the setting of multifunctions. It turns out that class of upper (lower) cl-supercontinuous multifunctions properly includes the class of upper (lower) perfectly continuous multifunctions and so includes all strongly continuous multifunctions [12] and is strictly contained in the class of upper (lower) z-supercontinuous multifunctions [3]. Section 2 is devoted to preliminaries and basic definitions, wherein we introduce the notions of upper and lower cl-supercontinuous multifunctions and discuss the interrelations that exist among them and other strong variants of continuity of multifunctions that already exist in the literature. Examples are included to reflect upon the distinctiveness of the notions so introduced and other strong variants of continuity of multifunctions in the literature. In Section 3 we obtain characterizations and study basic properties of upper cl-supercontinuous multifunctions. It turns out that upper cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, composition of multifunctions, union of multifunctions, restriction to a subspace, and the passage to the graph multifunction. Further, we formulate a sufficient condition for the intersection of two multifunctions to be cl-supercontinuous. Moreover, we prove that the graph of an upper cl-supercontinuous multifunction with closed values into a regular space is cl-closed with respect to X . Furthermore, an upper cl-supercontinuous multifunction maps mildly compact sets to compact sets. Finally it is shown that a closed, open, upper cl-supercontinuous multifunction with paracompact values maps cl-paracompact sets to paracompact sets. Section 4 is devoted to the study of lower cl-supercontinuous multifunctions, wherein characterizations of lower cl-supercontinuity are obtained. It is shown that lower cl-supercontinuity is preserved under the shrinking and expansion of range, union of multifunctions, restriction to a subspace and passage to the graph multifunction. Further, it is shown that a product of multifunctions is lower cl-supercontinuous if and only if each multifunction is lower cl-supercontinuous.

2. PRELIMINARIES AND BASIC DEFINITIONS

Throughout the paper we essentially follow the notations and terminology of L. Górniewicz. Let X and Y be nonempty sets. Then $\varphi : X \multimap Y$ is called a ***multifunction*** from X into Y if for each $x \in X$, $\varphi(x)$ is a nonempty subset of

Y . Let B be a subset of Y . Then the set $\varphi_+^{-1}(B) = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$ is called **large inverse image**[6]¹ of B and the set $\varphi_-^{-1}(B) = \{x \in X : \varphi(x) \subset B\}$ is called **small inverse image** of B . The set $\Gamma_\varphi = \{(x, y) \in X \times Y | y \in \varphi(x)\}$ is called the **graph** of the multifunction. Let A be subset of X . Then $\varphi(A) = \cup\{\varphi(x) : x \in A\}$ is called **image** of A . A multifunction $\varphi : X \multimap Y$ is **upper semicontinuous** (respectively **lower semicontinuous**) if $\varphi_-^{-1}(U)$ (respectively $\varphi_+^{-1}(U)$) is an open set in X for every open set U in Y . A subset U of a topological space X is called a **cl-open** set if it can be expressed as the union of clopen sets. The complement of a cl-open set will be referred to as a **cl-closed** set. A subset A of a space X is called **regular open** if it is the interior of its closure, i.e., $A = \overline{A}^\circ$. A collection β of subsets of a space X is called an **open complementary system** [7] if β consists of open sets such that for each $B \in \beta$, there exist $B_1, B_2, \dots, \in \beta$ with $B = \bigcup\{X \setminus B_i : i \in \mathbb{Z}^+\}$. A subset U of a space X is called **strongly open** F_σ -set [7] if there exists a countable open complementary system $\beta(U)$ with $U \in \beta(U)$. A subset H of a space X is called a **regular G_δ -set** [18] if H is the intersection of a sequence of closed sets whose interiors contain H , i.e., if $H = \bigcap_{i=1}^\infty F_i = \bigcap_{i=1}^\infty F_i^\circ$, where each F_i is a closed subset of X . The complement of a regular G_δ -set is called a **regular F_σ -set**. Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a **θ -adherent point** [25] of A if every closed neighbourhood of x intersects A . Let $cl_\theta A$ denote the set of all θ -adherent point of A . The set A is called **θ -closed** if $A = cl_\theta A$. The complement of a θ -closed set is referred to as a **θ -open set**. A point $x \in X$ is said to be a **cl-adherent point** of A if every clopen set containing x intersects A . Let $[A]_{cl}$ denote the set of all cl-adherent points of A . Then a set A is **cl-closed** if and only if $A = [A]_{cl}$. A subset A of a space X is said to be **cl-closed** if it is the intersection of clopen sets. The complement of a cl-closed set is referred to as a **cl-open set**.

Definition 2.1 ([13]). A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be

- (1) **strongly continuous** if $\varphi_-^{-1}(B)$ is clopen in X for every subset $B \subset Y$.
- (2) **upper perfectly continuous** if $\varphi_-^{-1}(V)$ is clopen in X for every open set $V \subset Y$.
- (3) **lower perfectly continuous** if $\varphi_+^{-1}(V)$ is clopen in X for every open set $V \subset Y$.
- (4) **upper completely continuous** if $\varphi_-^{-1}(V)$ is regular open in X for every open set $V \subset Y$.
- (5) **lower completely continuous** if $\varphi_+^{-1}(V)$ is regular open in X for every open set $V \subset Y$.

¹However, what we call “large inverse image $\varphi_+^{-1}(B)$ ” some authors call it ‘lower inverse image’ and denote it by $\varphi^-(B)$; and similarly they call “small inverse image $\varphi_-^{-1}(B)$ ” as ‘upper inverse image’ and employ the notation $\varphi^+(B)$ for the same.

Definition 2.2. A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be

- (1) **upper z -supercontinuous** [3] if for each $x \in X$ and each open set V containing $\varphi(x)$, there exists a cozero set U containing x such that $\varphi(U) \subset V$.
- (2) **lower z -supercontinuous** [3] if for each $x \in X$ and each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a cozero set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.
- (3) **upper D_δ -supercontinuous** [4] if for each $x \in X$ and each open set V containing $\varphi(x)$, there exists a regular F_σ -set U containing x such that $\varphi(U) \subset V$.
- (4) **lower D_δ -supercontinuous** [4] if for each $x \in X$ and each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a regular F_σ -set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.
- (5) **upper D -supercontinuous** [1] if for each $x \in X$ and each open set V containing $\varphi(x)$, there exists an open F_σ -set U containing x such that $\varphi(U) \subset V$.
- (6) **lower D -supercontinuous** [1] if for each $x \in X$ and each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists an open F_σ -set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.
- (7) **upper D^* -supercontinuous** [12] if for each $x \in X$ and each open set V containing $\varphi(x)$, there exists a strongly open F_σ -set U containing x such that $\varphi(U) \subset V$.
- (8) **lower D^* -supercontinuous** [12] if for each $x \in X$ and each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a strongly open F_σ -set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.
- (9) **upper strongly θ -continuous** [16] if for each $x \in X$ and each open set V containing $\varphi(x)$, there exists a θ -open set U containing x such that $\varphi(U) \subset V$.
- (10) **lower strongly θ -continuous** [16] if for each $x \in X$ and each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a θ -open set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.

Definition 2.3 ([21]). The graph Γ_φ of a multifunction $\varphi : X \multimap Y$ is said to be **cl-closed with respect to X** if for each $(x, y) \notin \Gamma_\varphi$ there exist a clopen set U containing x and an open set V containing y such that $(U \times V) \cap \Gamma_\varphi = \emptyset$.

Definition 2.4 ([26]). A multifunction $\varphi : X \multimap Y$ is said to have **nonmingled** point images provided that for $x, y \in X$ with $x \neq y$, the image sets $\varphi(x)$ and $\varphi(y)$ are either disjoint or identical.

Definition 2.5. A space X is said to be

- (a) **mildly compact** [23] if for every clopen cover of X has a finite subcover. In [24] Sostak calls mildly compact spaces as **clustered spaces**.
- (b) **cl-paracompact (cl-para-Lindelöf)** if every clopen cover of X has locally finite (locally countable) open refinement which covers X .

Definition 2.6. We say that a multifunction $\varphi : X \multimap Y$ is

(a) **upper cl-supercontinuous** at $x \in X$ if for each open set V with $\varphi(x) \subset V$, there exists a clopen set U containing x such that $\varphi(U) \subset V$. The multifunction is said to be upper cl-supercontinuous if it is upper cl-supercontinuous at each $x \in X$.

(b) **lower cl-supercontinuous** at $x \in X$ if for each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a clopen set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$. The multifunction is said to be lower cl-supercontinuous if it is lower cl-supercontinuous at each $x \in X$.

The following diagram well illustrates the interrelations that exist among various strong variants of continuity of multifunctions defined in Definition 2.1, 2.2 and 2.6.

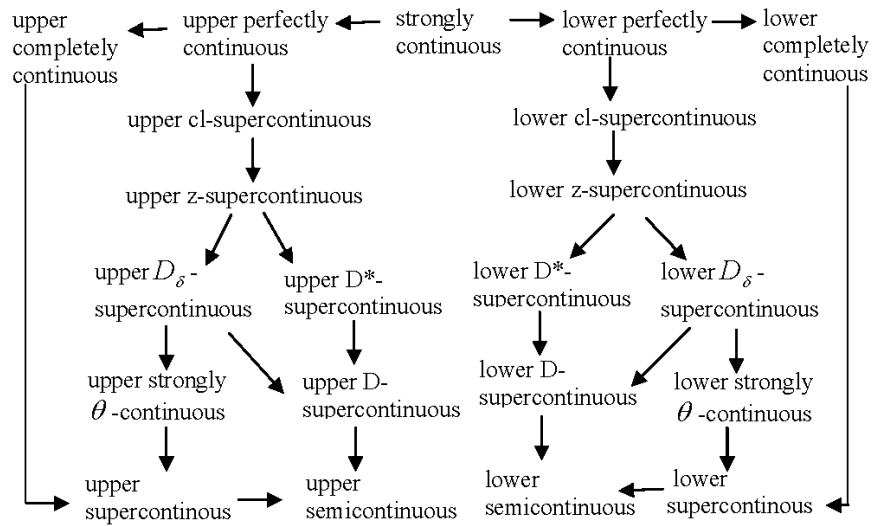


FIGURE 1.

However, none of the above implications is reversible as is well illustrated by the examples in the sequel and the examples in ([1], [2], [3], [4], [12], [16]).

Example 2.7. Let $X = \{a, b, c\}$ with the topology $\mathfrak{S}_X = \{\emptyset, X, \{a\}, \{b, c\}\}$ and let $Y = \{x, y\}$ with the topology $\mathfrak{S}_Y = \{\emptyset, Y, \{y\}\}$. Define a multifunction $\varphi : (X, \mathfrak{S}_X) \multimap (Y, \mathfrak{S}_Y)$ by $\varphi(a) = \{y\}$, $\varphi(b) = \{x, y\}$, $\varphi(c) = \{x\}$. Then the multifunction is upper perfectly continuous but not lower perfectly continuous. Again, for $\{x\} \subset Y$, $\varphi^{-1}(\{x\}) = \{c\}$ is not clopen which implies that the multifunction φ is not strongly continuous.

Example 2.8. Let $X = \{a, b, c\}$ with the topology $\mathfrak{S}_X = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$ and let $Y = \{x, y\}$ with the topology $\mathfrak{S}_Y = \{\emptyset, Y, \{y\}\}$. Define a multifunction $\varphi : (X, \mathfrak{S}_X) \multimap (Y, \mathfrak{S}_Y)$ by $\varphi(a) = \{y\}$, $\varphi(b) = \{x, y\}$, $\varphi(c) = \{y\}$. Then clearly φ is lower perfectly continuous. But for $\{y\} \subset Y$ $\varphi^{-1}(\{y\}) = \{a\}$ is not clopen, which implies the multifunction φ is not strongly continuous.

Example 2.9. Let $X = \mathbb{R}$, set of real numbers with upper limit topology \mathfrak{S} and let Y be same as X with usual topology U . Define a multifunction $\varphi : (X, \mathfrak{S}) \multimap (Y, U)$ by $\varphi(x) = \{x\}$ for each $x \in X$. Then clearly φ is upper (lower) cl-supercontinuous. But for $\varphi^{-1}(a, b) = (a, b) = \varphi_+^{-1}(a, b)$ is not clopen in X , which implies that φ is not upper (lower) perfectly continuous.

Example 2.10. Let X be a completely regular space which is not zero dimensional and let Y be same as X . Then the identity mapping $\varphi : X \multimap Y$ defined by $\varphi(x) = \{x\}$ for each $x \in X$, is upper (lower) z -supercontinuous but not upper (lower) cl-supercontinuous.

3. PROPERTIES OF UPPER cl -SUPERCONTINUOUS MULTIFUNCTIONS

Theorem 3.1. For a multifunction $\varphi : X \multimap Y$ from a topological space X into a topological Y the following statements are equivalent:

- (a) φ is upper cl -supercontinuous.
- (b) $\varphi^{-1}(B)$ is a cl -open set in X for every open set B in Y .
- (c) $\varphi_+^{-1}(B)$ is a cl -closed in X for every closed set B in Y .
- (d) $[\varphi_+^{-1}(B)]_{cl} \subset \varphi_+^{-1}(\overline{B})$ for every subset B of Y .

Proof. **(a) \Rightarrow (b).** Let B be an open subset of Y . To show that $\varphi^{-1}(B)$ is cl -open in X , let $x \in \varphi^{-1}(B)$. Then $\varphi(x) \subset B$. Since φ is upper cl -supercontinuous, therefore, there exists a clopen set H containing x such that $\varphi(H) \subset B$. Hence $x \in H \subset \varphi^{-1}(B)$ and so is a cl -open set in X .

(b) \Rightarrow (c). Let B be a closed subset of Y . Then $Y \setminus B$ is an open subset of Y . By (b), $\varphi^{-1}(Y \setminus B)$ is cl -open set in X . Since $\varphi^{-1}(Y \setminus B) = X \setminus \varphi_+^{-1}(B)$, $\varphi_+^{-1}(B)$ is a cl -closed set in X .

(c) \Rightarrow (d). Since \overline{B} is closed, $\varphi_+^{-1}(\overline{B})$ is a cl -closed set containing $\varphi_+^{-1}(B)$. Therefore $[\varphi_+^{-1}(B)]_{cl} \subset \varphi_+^{-1}(\overline{B})$.

(d) \Rightarrow (a). Let $x \in X$ and let V be an open set in Y such that $\varphi(x) \subset V$. Then $\varphi(x) \cap (Y \setminus V) = \emptyset$ and $\overline{(Y \setminus V)} = Y \setminus V$. Hence $[\varphi_+^{-1}(Y \setminus V)]_{cl} \subset \varphi_+^{-1}(Y \setminus V) = X \setminus \varphi_+^{-1}(V)$. Since $\varphi_+^{-1}(Y \setminus V)$ is cl -closed, its complement $\varphi_+^{-1}(V)$ is cl -open set containing x . So there is a clopen set U containing x and contained in $\varphi_+^{-1}(V)$, whence $\varphi(U) \subset V$. Thus φ is upper cl -supercontinuous. \square

Theorem 3.2. If a multifunction $\varphi : X \multimap Y$ is upper cl -supercontinuous and $\varphi(X)$ is endowed with the subspace topology, then, the multifunction $\varphi : X \multimap \varphi(X)$ is upper cl -supercontinuous.

Proof. Since φ is upper cl -supercontinuous for every open set V of Y , $\varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap \varphi^{-1}(\varphi(X)) = \varphi^{-1}(V) \cap X = \varphi^{-1}(V)$ is cl -open and hence $\varphi : X \multimap \varphi(X)$ is cl -supercontinuous. \square

Theorem 3.3. *If $\varphi : X \multimap Y$ is upper cl-supercontinuous and $\psi : Y \multimap Z$ is upper semicontinuous, then $\psi \circ \varphi$ is upper cl-supercontinuous. In particular, composition of upper cl-supercontinuous multifunctions is upper cl-supercontinuous.*

Proof. Let W be an open set in Z . Since ψ is upper semicontinuous, $\psi^{-1}(W)$ is an open set in Y . Again, since φ is upper cl-supercontinuous, $\varphi^{-1}(\psi^{-1}(W)) = (\psi \circ \varphi)^{-1}(W)$ is a cl-open set in X . Thus $\psi \circ \varphi : X \multimap Z$ is upper cl-supercontinuous. \square

In contrast to Theorem 3.2, the following corollary shows that upper cl-supercontinuity of a multifunction remains invariant under extension of its range.

Corollary 3.4. *Let $\varphi : X \multimap Y$ be upper cl-supercontinuous. If Z is a space containing Y as a subspace, then $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper cl-supercontinuous.*

Proof. Let W be an open set in Z . Then $W \cap Y$ is an open set in Y . Since $\varphi : X \multimap Y$ is upper cl-supercontinuous, $\varphi^{-1}(W \cap Y)$ is cl-open set in X . Now $\psi^{-1}(W) = \{x \in X : \psi(x) \subset W\} = \{x \in X : \varphi(x) \subset W \cap Y\}$. Thus $\psi : X \multimap Z$ is upper cl-supercontinuous. \square

Theorem 3.5. *If $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ are upper cl-supercontinuous multifunctions, then $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is upper cl-supercontinuous.*

Proof. Let U be an open set in Y . Since φ and ψ are upper cl-supercontinuous, $\varphi^{-1}(U)$ and $\psi^{-1}(U)$ are cl-open sets in X . Since $(\varphi \cup \psi)^{-1}(U) = \varphi^{-1}(U) \cap \psi^{-1}(U)$ and since finite intersection of cl-open sets is cl-open, $(\varphi \cup \psi)^{-1}(U)$ is cl-open in X . Thus $\varphi \cup \psi$ is upper cl-supercontinuous. \square

In general intersection of two upper cl-supercontinuous multifunctions need not be upper cl-supercontinuous. However, in the following theorem we formulate a sufficient condition for the intersection of two multifunctions to be upper cl-supercontinuous.

Theorem 3.6. *Let $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ be multifunctions from a space X into a Hausdorff space Y such that $\varphi(x)$ is compact for each $x \in X$ satisfying*

- (1) φ is upper cl-supercontinuous, and
- (2) the graph Γ_ψ of ψ is cl-closed with respect to X . Then the multifunction $\varphi \cap \psi$ defined by $(\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x)$ for each $x \in X$, is upper cl-supercontinuous.

Proof. Let $x_0 \in X$ and V be an open set containing $\varphi(x_0) \cap \psi(x_0)$. It suffices to find a clopen set U containing x_0 such that $(\varphi \cap \psi)(U) \subset V$. If $V \supset \varphi(x_0)$, it follows from upper cl-supercontinuity of φ . If not, then consider the set $K = \varphi(x_0) \setminus V$ which is compact. Now for each $y \in K$, $y \in Y \setminus \psi(x_0)$. This implies that $(x_0, y) \in X \times Y \setminus \Gamma_\psi$. Since the graph of ψ is cl-closed with respect to X , there exist clopen set U_y containing x_0 and an open set V_y containing y

such that $\Gamma_\psi \cap (U_y \times V_y) = \emptyset$. Therefore, for each $x \in U_y$, $\psi(x) \cap V_y = \emptyset$. Since K is compact, there exist finitely many in y_1, y_1, \dots, y_n in K such that $K \subset \cup_{i=1}^n V_{y_i}$. Let $W = \cup_{i=1}^n V_{y_i}$. Then $V \cup W$ is an open set containing $\varphi(x_0)$. Since φ is upper cl-supercontinuous, there exists a clopen set U_0 containing x_0 such that $\varphi(U_0) \subset V \cup W$. Let $U = U_0 \cap (\cap_{i=1}^n U_{y_i})$. Then U is a clopen set containing x_0 . Hence for each $z \in U$, $\varphi(z) \subset V \cup W$ and $\psi(z) \cap W = \emptyset$. Therefore, $(\varphi(z) \cap \psi(z)) \cap W = \emptyset$ for each $z \in U$. This proves that $\varphi \cap \psi$ is upper cl-supercontinuous at x_0 . \square

Corollary 3.7. *Let $\psi : X \multimap Y$ be a multifunction from a space X into a compact Hausdorff space Y such that the graph Γ_ψ of ψ is cl-closed with respect to X . Then ψ is upper cl-supercontinuous.*

Proof. Let the multifunction $\varphi : X \multimap Y$ be defined by $\varphi(x) = Y$ for each $x \in X$. Now an application of Theorem 3.6 yields the desired result. \square

Theorem 3.8. *Let $\varphi : X \multimap Y$ be any multifunction. Then the following statements are true:*

- (a) *If $\varphi : X \multimap Y$ is upper cl-supercontinuous and $A \subset X$, then the restriction $\varphi|_A : A \multimap Y$ is upper cl-supercontinuous.*
- (b) *If $\{U_\alpha : \alpha \in \Delta\}$ is a cl-open cover of X and if for each α , the restriction $\varphi_\alpha = \varphi|_{U_\alpha} : U_\alpha \multimap Y$ is upper cl-supercontinuous, then $\varphi : X \multimap Y$ is upper cl-supercontinuous.*

Proof. (a) Let W be an open set in Y . Since $\varphi : X \multimap Y$ is upper cl-supercontinuous, $\varphi^{-1}(W)$ is a cl-open set in X . Now $\varphi|_A(W) = \{x \in A \mid \varphi(x) \subset W\} = \{x \in A \mid x \in \varphi^{-1}(W)\} = A \cap \varphi^{-1}(W)$, which is cl-open in X and so $\varphi|_A$ is upper cl-supercontinuous.

(b) Let W be an open set in Y . Since $\varphi_\alpha = \varphi|_{U_\alpha} : U_\alpha \multimap Y$ is upper cl-supercontinuous, $(\varphi_\alpha)^{-1}(W)$ is a cl-open set in U_α and consequently cl-open in X . Since $\varphi^{-1}(W) = \cup_{\alpha \in \Delta} (\varphi_\alpha)^{-1}(W)$ and since the union of cl-open set is cl-open, $\varphi^{-1}(W)$ is cl-open set in X . In view of Theorem 3.1, $\varphi : X \multimap Y$ is upper cl-supercontinuous. \square

Theorem 3.9. *Let $\varphi : X \multimap Y$ be a multifunction and let $g : X \multimap X \times Y$ defined by $g(x) = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ for each $x \in X$, be the graph multifunction. If g is upper cl-supercontinuous, then φ is upper cl-supercontinuous and the space X is zero dimensional. Furthermore, if in addition $\varphi(x)$ is compact for each $x \in X$ and X is zero dimensional, then g is upper cl-supercontinuous whenever φ is.*

Proof. Suppose that g is upper cl-supercontinuous. By Theorem 3.3, the multifunction $\varphi = p_y \circ g$ is upper cl-supercontinuous, where $p_y : X \times Y \multimap Y$ denotes the projection mapping. To show that X is zero dimensional, let U be an open set in X and let $x \in U$. Then $U \times Y$ is an open set in $X \times Y$ and $g(x) \subset U \times Y$. Since g is upper cl-supercontinuous, there exists a clopen set W containing x such that $g(W) \subset U \times Y$ and so $W \subset g^{-1}(U \times Y) = U$. Hence $x \in W \subset U$ and

thus X is zero dimensional.

Conversely, suppose that X is zero dimensional, the multifunction φ is upper cl-supercontinuous and $\varphi(x)$ is compact for each $x \in X$. Let W be an open set containing $g(x) = \{x\} \times \varphi(x)$. Then by Wallace theorem [10, p.142] there exist open sets U in X , V in Y and $g(x) \subset U \times V \subset W$. So $x \in U$ and $\varphi(x) \subset V$. Since X is zero dimensional there exists a clopen set containing x such that $x \in G_1 \subset U$. Again since φ is upper cl-supercontinuous, there exists a clopen set G_2 containing x such that $\varphi(G_2) \subset V$. Let $G = G_1 \cap G_2$. Then G is a clopen set containing x and it is easily verified that $g(G) \subset U \times V \subset W$. This proves that g is upper cl-supercontinuous. \square

The following theorem gives sufficient conditions for the graph of a multifunction to be cl-closed with respect to X .

Theorem 3.10. *If $\varphi : X \multimap Y$ is upper cl-supercontinuous, where Y is a regular space and $\varphi(x)$ is closed for each $x \in X$, then the graph Γ_φ of φ is a cl-closed with respect to X .*

Proof. Let $(x, y) \notin \Gamma_\varphi$. Then $y \notin \varphi(x)$. Since Y is regular, there exist disjoint open sets V_y and $V_{\varphi(x)}$ containing y and $\varphi(x)$, respectively. Since φ is upper cl-supercontinuous, there exists a clopen set U_x containing x such that $\varphi(U_x) \subset V_{\varphi(x)}$. We assert that $(U_x \times V_y) \cap \Gamma_\varphi = \emptyset$. For, if $(h, k) \in (U_x \times V_y) \cap \Gamma_\varphi$, then $h \in \varphi^{-1}(V_{\varphi(x)})$, $k \in V_y$ and $k \in \varphi(h)$. Hence $\varphi(h) \subset V_{\varphi(x)}$ and $k \in \varphi(h) \cap V_y$ which contradicts the fact that V_y and $V_{\varphi(x)}$ are disjoint. Thus the graph Γ_φ of φ is a cl-closed with respect to X . \square

The following theorem is a sort of partial converse to Theorem 3.10 and shows that the multifunctions which have cl-closed graph with respect to X have nice properties.

Theorem 3.11. *If $\varphi : X \multimap Y$ is a multifunction with cl-closed graph with respect to X and $K \subset Y$ is compact, then $\varphi_+^{-1}(K)$ is cl-closed in X . Further, if in addition Y is compact, then φ is upper cl-supercontinuous.*

Proof. To prove that $\varphi_+^{-1}(K)$ is cl-closed, we shall show that $X \setminus \varphi_+^{-1}(K)$ is cl-open. To this end, let $x \in X \setminus \varphi_+^{-1}(K)$. Then $\varphi(x) \cap K = \emptyset$. Since Γ_φ is cl-closed with respect to X , for each $y \in K$ there exist clopen set U_y containing x and an open set V_y containing y such that $(U_y \times V_y) \cap \Gamma_\varphi = \emptyset$. The collection $\Omega = \{V_y | y \in K\}$ is an open cover of the compact set K . So there exists a finite subset $\{y_1, \dots, y_n\}$ of K such that $K \subset \cup_{i=1}^n V_{y_i} = V$ (say). Let $U = \cap_{i=1}^n U_{y_i}$. Then U is a clopen set containing x and since $\varphi(U) \cap K = \emptyset$. Thus $U \subset X \setminus \varphi_+^{-1}(K)$ and so $X \setminus \varphi_+^{-1}(K)$ is cl-open as desired. The last assertion is immediate in view of Theorem 3.1 and the fact that a closed subset of a compact space is compact. \square

Corollary 3.12. *If $\varphi : X \multimap Y$ is a multifunction with $\varphi(X) \subset K$, where K is compact and the graph Γ_φ of φ is cl-closed with respect to X , then φ is upper cl-supercontinuous.*

Theorem 3.13. *Let $\varphi : X \rightarrow Y$ be an upper cl-supercontinuous multifunction such that $\varphi(x)$ is compact for each $x \in X$. If A is a mildly compact set in X , then $\varphi(A)$ is compact.*

Proof. Let Ω be an open cover of $\varphi(A)$. Then Ω is also an open cover of $\varphi(a)$ for each $a \in A$. Since each $\varphi(a)$ is compact, there exists a finite subset $\beta_a \subset \Omega$ such that $\varphi(a) \subset \cup_{B \in \beta_a} B = V_a$ (say). Since φ is upper cl-supercontinuous, there exists a clopen set U_a containing a such that $\varphi(U_a) \subset V_a$ and so $U_a \subset \varphi^{-1}(V_a)$. Let $Q = \{U_a | a \in A\}$. Then Q is a clopen covering of A . Since A is mildly compact, there exists a finite subset $\{a_1, \dots, a_n\}$ of A such that $A \subset \cup_{i=1}^n U_{a_i} \subset \cup_{i=1}^n \varphi^{-1}(V_{a_i})$. Therefore $\varphi(A) \subset \varphi(\cup_{i=1}^n \varphi^{-1}(V_{a_i})) = \cup_{i=1}^n \varphi(\varphi^{-1}(V_{a_i})) \subset \cup_{i=1}^n V_{a_i}$, where $V_{a_i} = \cup_{B \in \beta_{a_i}} B$, $i = 1, \dots, n$ and each β_{a_i} is finite. Thus $\varphi(A)$ is compact. \square

We may recall that a space X is called a P -space if every G_δ -set in X is open in X .

Theorem 3.14. *Let $\varphi : X \rightarrow Y$ be a closed, open, and upper cl-supercontinuous, nonmingled multifunction from a space X into a P -space Y such that $\varphi(x)$ is para-Lindelöf for each $x \in X$. If A is a cl-para-Lindelöf set in X , then $\varphi(A)$ is para-Lindelöf set in Y . In particular, if X is cl-para-Lindelöf and φ is onto, then Y is para-Lindelöf.*

Proof. Let Ψ be an open cover of $\varphi(A)$. Then Ψ is also an open covering of $\varphi(x)$ for each $x \in A$. Since $\varphi(x)$ is para-Lindelöf, Ψ has a locally countable open refinement ψ_x such that $\varphi(x) \subset \cup \psi_x = V_x$ (say). Since φ is upper cl-supercontinuous, there exists a clopen set U_x containing x such that $\varphi(U_x) \subset V_x$. Now $\mathbf{u} = \{U_x | x \in A\}$ is a clopen cover of A . Since A is cl-para-Lindelöf, \mathbf{u} has a locally countable open refinement $\Omega = \{W_\alpha | \alpha \in \Lambda\}$ such that $A \subset \cup_{\alpha \in \Lambda} W_\alpha$. So for each $\alpha \in \Lambda$ there exists a $x_\alpha \in A$ such that $W_\alpha \subset U_{x_\alpha}$ and hence $\varphi(W_\alpha) \subset \varphi(U_{x_\alpha}) \subset \cup \psi_{x_\alpha}$. Let $\mathfrak{R}_\alpha = \{\varphi(W_\alpha) \cap V | V \in \psi_{x_\alpha}\}$ and let $\mathfrak{R} = \{R | R \in \mathfrak{R}_\alpha, \alpha \in \Lambda\}$. We shall show that \mathfrak{R} is a locally countable open refinement of Ψ . Since φ is open, $\varphi(W_\alpha)$ is open and so each $R \in \mathfrak{R}$ is open. Let $R \in \mathfrak{R}$. Then $R \in \mathfrak{R}_\alpha$ for some $\alpha \in \Lambda$, i.e. $R = \varphi(W_\alpha) \cap V \subset V \subset U$ for some $U \in \Psi$. This shows that \mathfrak{R} is an open refinement of Ψ . To show that \mathfrak{R} is locally countable, let $y \in \varphi(A)$. Then $y \in \varphi(x)$ for some $x \in A$. Since Ω is locally countable, for each $x \in A$ we can choose an open neighborhood G_x of x which intersects only countably many members $W_{\alpha_1}, W_{\alpha_2}, \dots, W_{\alpha_n}, \dots$ of Ω . Since φ is a nonmingled open multifunction, it follows that $H_0 = \varphi(G_x)$ is an open neighborhood of y which intersects only countably many members $\varphi(W_{\alpha_1}), \varphi(W_{\alpha_2}), \dots, \varphi(W_{\alpha_n}), \dots$ of the family $\{\varphi(W_\alpha) | \alpha \in \Lambda\}$. Furthermore each \mathfrak{R}_{α_k} ($k = 1, \dots, n, \dots$) is locally countable, hence there exists an open neighborhood H_k ($k = 1, \dots, n, \dots$) of y which intersects only countably many members of \mathfrak{R}_{α_k} ($k = 1, \dots, n, \dots$). Finally let $H = \cap_{k=1}^{\infty} H_k$. Since Y is P -space, H is an open neighborhood of y which intersects at most countably many member of \mathfrak{R} , and so \mathfrak{R} is locally countable. Moreover, $\varphi(A) \subset \varphi(\cup_{\alpha \in \Lambda} W_\alpha) =$

$\cup_{\alpha \in \Lambda} \varphi(W_\alpha) \subset \cup_{\alpha \in \Lambda} (\cup \mathfrak{R}_\alpha) = \cup \{R : R \in \mathfrak{R}\}$. Hence \mathfrak{R} is a locally countable open refinement of Ψ that covers $\varphi(A)$. Thus $\varphi(A)$ is para-Lindelöf. \square

Theorem 3.15. *Let $\varphi : X \multimap Y$ be a closed, open, upper cl-supercontinuous nonmingled multifunction from a space X into a space Y such that $\varphi(x)$ is paracompact for each $x \in X$. If A is a cl-paracompact, then $\varphi(A)$ is paracompact. In particular, if X is cl-paracompact space and φ is onto, then Y is paracompact.*

Proof. Proof of Theorem 3.15 is similar (even simpler) to that of Theorem 3.14 and hence omitted. \square

4. PROPERTIES OF LOWER cl-SUPERCONTINUOUS MULTIFUNCTIONS

Theorem 4.1. *For a multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y the following statements are equivalent.*

- (a) φ is lower cl-supercontinuous.
- (b) $\varphi_+^{-1}(B)$ is a cl-open set in X for every open set B in Y .
- (c) $\varphi_-^{-1}(B)$ is a cl-closed in X for every closed set B in Y .
- (d) For each $x \in X$ and for each open set V with $\varphi(x) \cap V \neq \emptyset$ there exists a cl-open set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.

Proof. (a) \Rightarrow (b). Let B be an open subset of Y . To show that $\varphi_+^{-1}(B)$ is cl-open in X , let $x \in \varphi_+^{-1}(B)$. Then $\varphi(x) \cap B \neq \emptyset$. Since φ is lower cl-supercontinuous, there exists a clopen set H containing x such that $\varphi(h) \cap B \neq \emptyset$ for each $h \in H$. Hence $x \in H \subset \varphi_+^{-1}(B)$ and so $\varphi_+^{-1}(B)$ is a cl-open set in X being a union of clopen sets.

(b) \Rightarrow (c). Let B be a closed subset of Y . Then $Y \setminus B$ is an open subset of Y . By (b), $\varphi_+^{-1}(Y \setminus B)$ is a cl-open set in X . Since $\varphi_+^{-1}(Y \setminus B) = X \setminus \varphi_-^{-1}(B)$, $\varphi_-^{-1}(B)$ is a cl-closed set in X .

(c) \Rightarrow (d). Let $x \in X$ and let V be an open set in Y with $\varphi(x) \cap V \neq \emptyset$. Then $Y \setminus V$ is a closed set in Y with $\varphi(x) \not\subseteq (Y \setminus V)$. Therefore, By (c), $\varphi_-^{-1}(Y \setminus V) = X \setminus \varphi_+^{-1}(V)$ is a cl-closed set in X not containing x and so $\varphi_+^{-1}(V)$ is a cl-open set in X containing x . Let $U = \varphi_+^{-1}(V)$. Then U is a cl-open set containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$.

The assertion (d) \Rightarrow (a) is trivial, since every cl-open set is the union of clopen sets. \square

Theorem 4.2. *A multifunction $\varphi : X \multimap Y$ is lower cl-supercontinuous if and only if $\varphi([A]_{cl}) \subset \overline{\varphi(A)}$ for every subset A of X .*

Proof. Suppose that $\varphi : X \multimap Y$ is lower cl-supercontinuous. Let A be subset of X . Then $\overline{\varphi(A)}$ is a closed subset of Y . By Theorem 4.1 $\varphi_-^{-1}(\overline{\varphi(A)})$ is a cl-closed set in X . Since $A \subset \varphi_-^{-1}(\overline{\varphi(A)})$ and since $[A]_{cl} \subset [\varphi_-^{-1}(\overline{\varphi(A)})]_{cl} = \varphi_-^{-1}(\overline{\varphi(A)})$, $\varphi([A]_{cl}) \subset \varphi(\varphi_-^{-1}(\overline{\varphi(A)})) \subset \overline{\varphi(A)}$.

Conversely, suppose that $\varphi([A]_{cl}) \subset \overline{\varphi(A)}$ for every subset A of X and let F be a closed set in Y . Then $\varphi_-^{-1}(F)$ is subset of X . By hypothesis, $\varphi([\varphi_-^{-1}(F)]_{cl}) \subset$

$\overline{\varphi(\varphi^{-1}(F))} \subset \overline{F} = F$ and $\varphi^{-1}(\varphi([\varphi^{-1}(F)]_{cl})) \subset \varphi^{-1}(F)$ so which in its turn implies that $[\varphi^{-1}(F)]_{cl} \subset \varphi^{-1}(F)$. Hence $\varphi^{-1}(F) = [\varphi^{-1}(F)]_{cl}$ and so in view of Theorem 4.1 $\varphi : X \multimap Y$ is lower cl-supercontinuous. \square

Theorem 4.3. *A multifunction $\varphi : X \multimap Y$ is lower cl-supercontinuous if and only if $[\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(\overline{B})$ for every subset B of Y .*

Proof. Suppose that $\varphi : X \multimap Y$ is lower cl-supercontinuous. Let $B \subset Y$. Then \overline{B} is a closed subset of Y . By Theorem 4.1, $\varphi^{-1}(\overline{B})$ is a cl-closed subset of X . Since, $\varphi^{-1}(B) \subset \varphi^{-1}(\overline{B})$, $[\varphi^{-1}(B)]_{cl} \subset [\varphi^{-1}(\overline{B})]_{cl} = \varphi^{-1}(\overline{B})$. That is $[\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(\overline{B})$.

Conversely Suppose that $[\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(\overline{B})$ for every $B \subset Y$. Let F be any closed subset of Y . By hypothesis $[\varphi^{-1}(F)]_{cl} \subset \varphi^{-1}(\overline{F}) = \varphi^{-1}(F)$. Hence $[\varphi^{-1}(F)]_{cl} = \varphi^{-1}(F)$ and so in view of Theorem 4.1 φ is lower cl-supercontinuous. \square

The following theorem shows that lower cl-supercontinuity of a multifunction remains invariant under the shrinking of its range.

Theorem 4.4. *If $\varphi : X \multimap Y$ is lower cl-supercontinuous and $\varphi(X)$ is endowed with subspace topology, then $\varphi : X \multimap \varphi(X)$ is lower cl-supercontinuous.*

Theorem 4.5. *If $\varphi : X \multimap Y$ is lower cl-supercontinuous and $\psi : Y \multimap Z$ is lower semicontinuous, then $\psi \circ \varphi$ is lower cl-supercontinuous. In particular, composition of two lower cl-supercontinuous multifunctions is upper cl-supercontinuous.*

Proof. Let W be an open set in Z . Since ψ is upper semi continuous, $\psi_+^{-1}(W)$ is an open set in Y . Again since φ is lower cl-supercontinuous, $\varphi_+^{-1}(\psi_+^{-1}(W))$ is cl-open in X , and so $(\psi \circ \varphi)_+^{-1}(W) = \varphi_+^{-1}(\psi_+^{-1}(W))$ is a cl-open set in X . Thus $\psi \circ \varphi : X \multimap Z$ is lower cl-supercontinuous. \square

In contrast to Theorem 4.4 the following corollary shows that lower cl-supercontinuity of a multifunction is preserved under the expansion of its range.

Corollary 4.6. *Let $\varphi : X \multimap Y$ be lower cl-supercontinuous. If Z is a space containing Y as a subspace, then $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for $x \in X$ is lower cl-supercontinuous.*

Proof. Let W be an open set in Z . Then $W \cap Y$ is an open set in Y . Since $\varphi : X \multimap Y$ is lower cl-supercontinuous, $\varphi_+^{-1}(W \cap Y)$ is cl-open set in X . Now, $\psi_+^{-1}(W) = \{x \in X : \psi(x) \cap W \neq \emptyset\} = \{x \in X : \varphi(x) \cap (W \cap Y) \neq \emptyset\} = \varphi_+^{-1}(W \cap Y)$. Thus $\psi : X \multimap Z$ is lower cl-supercontinuous. \square

Theorem 4.7. *If $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ are lower cl-supercontinuous multifunctions, then the multifunction $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is lower cl-supercontinuous.*

Proof. Let U be an open set in Y . Then $\varphi_+^{-1}(U)$ and $\psi_+^{-1}(U)$ are cl-open sets in X . Since $(\varphi \cup \psi)_+^{-1}(U) = \varphi_+^{-1}(U) \cup \psi_+^{-1}(U)$ and since any union of cl-open sets is cl-open, $(\varphi \cup \psi)_+^{-1}(U)$ is cl-open in X . Thus $\varphi \cup \psi$ is lower cl-supercontinuous. \square

Theorem 4.8. *Let $\varphi : X \multimap Y$ be any multifunction. Then the following statements are true:*

- (a) *If $\varphi : X \multimap Y$ is lower cl-supercontinuous and $A \subset X$, then the restriction $\varphi|_A : A \multimap Y$ is lower cl-supercontinuous.*
(b) *If $\{U_\alpha : \alpha \in \Delta\}$ is a cl-open cover of X and for each α , the restriction $\varphi_\alpha = \varphi|_{U_\alpha} : U_\alpha \multimap Y$ is lower cl-supercontinuous, then $\varphi : X \multimap Y$ is lower cl-supercontinuous.*

Proof. (a) Let W be an open set in Y . Since $\varphi : X \multimap Y$ is lower cl-supercontinuous, $\varphi_+^{-1}(W)$ is a cl-open set in X . Now, $(\varphi|_A)_+^{-1}(W) = \{x \in A \mid \varphi(x) \cap W \neq \emptyset\} = \{x \in A \mid x \in \varphi_+^{-1}(W)\} = A \cap \varphi_+^{-1}(W)$, which is cl-open in X and so $\varphi|_A$ is lower cl-supercontinuous.

(b) Let W be an open set in Y . Since $\varphi_\alpha = \varphi|_{U_\alpha} : U_\alpha \multimap Y$ is lower cl-supercontinuous, $(\varphi_\alpha)_+^{-1}(W)$ is a cl-open set in U_α and consequently cl-open in X . Since $\varphi_+^{-1}(W) = \cup_{\alpha \in \Delta} (\varphi_\alpha)_+^{-1}(W)$ and since any union of cl-open sets is cl-open, $\varphi_+^{-1}(W)$ is cl-open set in X . Thus $\varphi : X \multimap Y$ is lower cl-supercontinuous. \square

Theorem 4.9. *Let $\{\varphi_\alpha : X \multimap X_\alpha \mid \alpha \in \Lambda\}$ be a family of multifunctions and let $\varphi : X \multimap \prod_{\alpha \in \Lambda} X_\alpha$ be defined by $\varphi(x) = \prod_{\alpha \in \Lambda} \varphi_\alpha(x)$. Then φ is lower cl-supercontinuous if and only if each $\varphi_\alpha : X \multimap X_\alpha$ is lower cl-supercontinuous.*

Proof. Let $\varphi : X \multimap \prod_{\alpha \in \Lambda} X_\alpha$ be lower cl-supercontinuous. Let $p_\beta : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow X_\beta$ be the projection map onto X_β . Then p_β being a single valued continuous function is lower semicontinuous. By Theorem 4.5 $\varphi_\beta = p_\beta \circ \varphi$ is lower cl-supercontinuous for each $\beta \in \Lambda$.

Conversely, suppose that $\varphi_\beta : X \multimap X_\beta$ is a lower cl-supercontinuous for each $\beta \in \Lambda$. Since the finite intersections and arbitrary union of cl-open sets is cl-open, therefore, in view of Theorem 4.1 it suffices to prove that $\varphi_+^{-1}(B)$ is a cl-open set for every basic open set B in the product space $\prod_{\alpha \in \Lambda} X_\alpha$. Let $B = U_{\alpha_1} \times U_{\alpha_2} \times \dots \times U_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_N} X_\alpha)$ be a basic open set in $\prod_{\alpha \in \Lambda} X_\alpha$. Now it is easily verified that $\varphi_+^{-1}(U_{\alpha_1} \times \dots \times U_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_N} X_\alpha)) = (\varphi_{\alpha_1})_+^{-1}(U_{\alpha_1}) \cap \dots \cap (\varphi_{\alpha_N})_+^{-1}(U_{\alpha_N})$. Since each φ_{α_i} is cl-supercontinuous, $\varphi_+^{-1}(B)$ is cl-open in X being the finite intersection of cl-open sets. Thus φ is lower cl-supercontinuous. \square

Theorem 4.10. *For each $\alpha \in \Delta$ let $\varphi_\alpha : X_\alpha \multimap Y_\alpha$ be a multifunction and let $\varphi : \prod_{\alpha \in \Delta} X_\alpha \multimap \prod_{\alpha \in \Delta} Y_\alpha$ be a multifunction defined by $\varphi(x) = \prod \varphi_\alpha(x_\alpha)$ for each $x = (x_\alpha) \in \prod_{\alpha \in \Delta} X_\alpha$. Then φ is lower cl-supercontinuous if and only if each φ_α is lower cl-supercontinuous.*

Proof. Suppose that $\varphi : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow \prod_{\alpha \in \Lambda} Y_\alpha$ is lower cl-supercontinuous. Let U_β be an open set in Y_β . Then $U_\beta \times \prod_{\alpha \neq \beta} Y_\alpha$ is a subbasic open set in $\prod_{\alpha \in \Lambda} Y_\alpha$. So in view of Theorem 4.1, $\varphi_+^{-1}(U_\beta \times \prod_{\alpha \neq \beta} Y_\alpha)$ is a cl-open set in $\prod_{\alpha \in \Lambda} X_\alpha$. Now it is easily verified that $\varphi_+^{-1}(U_\beta \times \prod_{\alpha \neq \beta} Y_\alpha) = (\varphi_\beta)_+^{-1}(U_\beta) \times \prod_{\alpha \neq \beta} X_\alpha$, and so $(\varphi_\beta)_+^{-1}(U_\beta)$ is cl-open in X_β . This proves that each φ_β is lower cl-supercontinuous.

Conversely suppose that $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ is lower cl-supercontinuous for each $\alpha \in \Lambda$ and let $B = V_{\alpha_1} \times V_{\alpha_2} \times \dots \times V_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_N} Y_\alpha)$ be a basic open set in $\prod_{\alpha \in \Lambda} Y_\alpha$. Then $\varphi_+^{-1}(V_{\alpha_1} \times \dots \times V_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_N} Y_\alpha)) = (\varphi_{\alpha_1})_+^{-1}(V_{\alpha_1}) \times \dots \times (\varphi_{\alpha_N})_+^{-1}(V_{\alpha_N}) \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \dots, \alpha_N} X_\alpha)$. Since each φ_α is lower cl-supercontinuous, $\varphi_+^{-1}(B)$ is cl-open in $\prod_{\alpha \in \Lambda} X_\alpha$ and so φ is lower cl-supercontinuous. \square

Theorem 4.11. *Let $\varphi : X \rightarrow Y$ be multifunction and let $g : X \rightarrow X \times Y$ defined by $g(x) = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ for each $x \in X$ be the graph multifunction. Then g is lower cl-supercontinuous if and only if φ is lower cl-supercontinuous and the space X is zero dimensional.*

Proof. Suppose that g is lower cl-supercontinuous. By Theorem 4.5 the multifunction $\varphi = p_y \circ g$ is lower cl-supercontinuous. Next we shall show that X is zero dimensional. Let U be an open set in X and let $x \in U$. Then $U \times Y$ is an open set in $X \times Y$ and $g(x) \cap (U \times Y) \neq \emptyset$. Since g is lower cl-supercontinuous, there exists a clopen set W containing x such that $g(z) \cap (U \times Y) \neq \emptyset$ for every $z \in W$ and so $W \subset g_+^{-1}(U \times Y) = U$. Hence $x \in W \subset U$ and X is zero dimensional.

Conversely Suppose that φ is lower cl-supercontinuous. Let $x \in X$ and let W be an open set with $g(x) \cap W \neq \emptyset$. Then there exist open sets U in X and V in Y such that $g(x) \cap (U \times V) \neq \emptyset$ and so $x \in U$ and $\varphi(x) \cap V \neq \emptyset$. Since X is zero dimensional, there exists a clopen set G_1 containing x such that $x \in G_1 \subset U$. Again since φ is lower cl-supercontinuous, there exists a clopen set G_2 containing x such that $\varphi(h) \cap V \neq \emptyset$ for each $h \in G_2$. Let $G = G_1 \cap G_2$. Then G is a clopen set containing x and it is easily verified that $g(h) \cap W \neq \emptyset$ for each $h \in G$. This proves that g is lower cl-supercontinuous. \square

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(Received October 2009 – Accepted April 2010)

J.K. KOHLI (jk.kohli@yahoo.co.in)

Dept.of Mathematics, Hindu College, University of Delhi, Delhi 110007, India.

C. P. ARYA (carya28@gmail.com)

Dept.of Mathematics, University of Delhi, Delhi 110007, India.