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Zariski topology on the spectrum of graded classical prime submodules

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Abstract

Let R be a G-graded commutative ring with identity and let M be a graded R-module. A proper graded submodule N of M is called graded classical prime if for every $a, b \in h(R)$, $m \in h(M)$, whenever $abm \in N$, then either $am \in N$ or $bm \in N$. The spectrum of graded classical prime submodules of M is denoted by $Cl.Spec_g(M)$. We topologize $Cl.Spec_g(M)$ with the quasi-Zariski topology, which is analogous to that for $Spec_g(R)$.

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1. Introduction

Recently many authors have been interested in equip algebraic structures with Zariski topology (cf. [4, 11, 12]). A grading on a ring and its modules usually aids computations by allowing one to focus on the homogeneous elements, which are presumably simpler or more controllable than random elements. However, for this to work one needs to know that the constructions being studied are graded. One approach to this issue is to redefine the constructions entirely in terms of the category of graded modules and thus avoid any consideration of non-graded modules or non-homogeneous elements; Sharp gives such a treatment of attached primes in [15]. Unfortunately, while such an approach helps to understand the graded modules themselves, it will only help

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to understand the original construction if the graded version of the concept happens to coincide with the original one. Therefore, notably, the study of graded modules is very important.

Our main purpose is to study some new classes of graded submodules of graded modules and endow these classes of submodules with quasi-Zariski topology. Zariski topology on the prime spectrum of a module over a commutative ring have been already studied in [11, 12]. Moreover some topologies on the spectrum of graded prime submodules of a graded module have been studied in [16]. Therefore these results will be used in order to obtain the main aims of this paper.

The organization of this paper is as follows: In section 2 we recollect the results concerning the topologies on the prime spectrum of a module over a commutative ring. Moreover we remind the notation and the elemental properties about graded modules and rings that we will use in this paper. In section 3 we introduce and study the concept of graded classical prime submodules and define the quasi-Zariski topology on the spectrum of all graded classical prime submodules of a graded module.

2. PRELIMINARIES

In this section, we recall some definitions and notations used throughout. Throughout this paper all rings are commutative with a nonzero identity and all modules are considered to be unitary. Prime submodules play an important role in the module theory over commutative rings. Let M be a module over a commutative ring R. A prime (resp. primary) submodule N of M is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a^k \in (N :_R M)$ for some positive integer k). In this case $P = (N :_R M)$ (resp. $P = \sqrt{(N :_R M)}$) is a prime ideal of R and we say that N is a P-prime (resp. P-primary) submodule of M. There are several ways to generalize the notion of prime submodules. We could restrict where am lies or we can restrict where a and/or b lie. We begin by mentioning some examples obtained by restricting where ab lies. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [8]. A proper submodule N of M is weakly prime if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Behboodi and Koohi in [3] defined another class of submodules and called it classical prime. A proper submodule N of M is said to be classical prime when for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N$.

Recently, M. Baziar and M. Behboodi [2] defined a classical primary submodule in the R-module M as a proper submodule Q of M such that if $abm \in Q$, where $a, b \in R$ and $m \in M$, then either $bm \in Q$ or $a^k m \in Q$ for some positive integer k. Clearly, in case $M = R$, classical primary submodules coincide with primary ideals.

Let G be an arbitrary group. A commutative ring R with a non-zero identity is G-graded if it has a direct sum decomposition $R = \bigoplus_{g \in G} R_g$ such that for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. The G-graded ring R is called a graded integral domain provided that $ab = 0$ implies that either $a = 0$ or $b = 0$ where $a, b \in h(R) := \bigcup_{g\in G} R_g$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. For every $g \in G$, an element of R_g or M_g is said to be a homogeneous element. We denote by $h(M)$ the set of all homogeneous elements of M, that is $h(M) = \bigcup_{g\in G} M_g$. Let M be a G-graded R-module. A submodule N of M is called graded (or homogeneous) if $N = \bigoplus_{g\in G}(N \cap M_g)$ or equivalently N is generated by homogeneous elements. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for each $g \in G$. An ideal I of R is called a graded ideal if it is a graded submodule of R and a graded R-module.

Let R be a G-graded ring. A proper graded ideal I of R is said to be a graded prime ideal if whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I , denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$, where $x_{g_i} \in h(M)$ for every $1 \leq i \leq n$. It is clear that a graded module is finitely generated if and only if it is graded finitely generated. For M, consider the subset $T^g(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in h(R)\}$. If R is a graded integral domain, then $T^g(M)$ is a graded submodule of M. M is called graded torsion-free (g-torsion-free for short) if $T^g(M) = 0$, and it is called graded torsion (g-torsion for short) if $T^g(M) = M$. It is clear that if M is torsion-free, then it is g-torsion-free. Moreover, if M is g-torsion, then it is torsion.

Let R be a G-graded ring and M a graded R-module. We recall from [8] that a proper graded submodule N of M is called graded prime (resp. graded primary) if $rm \in N$, then $m \in N$ or $r \in (N :_R M) = \{r \in R | rM \subseteq N\}$ (resp. $r^k \in (N :_R M)$ for some positive integer k), where $r \in h(R)$, $m \in h(M)$. It is shown in [8, Proposition 2.7] that if N is a graded prime submodule of M, then $P := (N :_R M)$ is a graded prime ideal of R, and N is called graded P-prime submodule. Let N be a graded submodule of M. Then N is a graded prime submodule of M if and only if $P := (N :_R M)$ is a graded prime ideal of R and M/N is a g-torsion-free R/P-module. Note that some graded R-modules M have no graded prime submodules. We call such graded modules g-primeless. A submodule S of M will be called graded semiprime if S is an intersection of graded prime submodules of M. Let $Spec_g(M)$ denote the set of all graded prime submodules of M. Let N be a graded submodule of M. The graded radical of N in M, denoted by $Gr_M(N)$ is defined to be the intersection of all graded prime submodules of M containing N [10]. Hence $Gr_M(N)$ is a graded semiprime submodule. A proper graded submodule N of M is called graded weakly prime if $0 \neq rm \in N$, then $m \in N$ or $r \in (N :_R M)$. Hence every graded prime submodule is graded weakly prime.

From now on, R is a G-graded ring and M is a graded R-module unless otherwise stated.

3. Graded classical prime submodules

A proper graded submodule N of M is called graded classical prime if for every $a, b \in h(R)$, $m \in h(M)$, whenever $abm \in N$, then either $am \in N$ or $bm \in N$. Let N be a graded classical prime submodule of M. Then, it is easy to see that N_g is a classical prime submodule of the R_e-module M_g for every $g \in G$. It is evident that every graded prime submodule is graded classical prime. However the next example shows that a graded classical prime submodule is not necessarily graded prime.

Example 3.1. Assume that R is a graded integral domain and P is a non-zero graded prime ideal of R. In this case the ideal $Q := P \oplus 0$ is a graded classical prime submodule of the graded R-module $R \oplus R$ while it is not graded prime. This example shows also that a graded classical prime submodule need not be classical prime.

We denote by $Cl.Spec_g(M)$, the set of all graded classical prime submodules of M. Obviously, some graded R-modules M have no graded classical prime submodules; such modules are called g-Cl.primeless. For example, the zero module is clearly g-Cl.primeless. A submodule S of M will be called graded classical semiprime if S is an intersection of graded classical prime submodules of M. Let N be a graded submodule of M. The graded classical radical of N in M, denoted by $Cl.Gr_M(N)$, is defined to be the intersection of M and all graded classical prime submodules of M containing N. So if $Cl.Spec_g(M) = \varnothing$, then $Gr^{cl}_M(N) = M$, and if $Cl.Spec_g(M) \neq \varnothing$, then $Gr^{cl}_M(N)$ is a graded classical semiprime submodule. If $N = 0$, then $Gr^{cl}_M(0)$ is called the graded classical nil-radical of M.

We know that $Spec_g(M) \subseteq Cl.Spec_g(M)$. As it is mentioned in example 3.1, it happens sometimes that this containment is strict. We call M a graded compatible R-module if its graded classical prime submodules and graded prime submodules coincide, that is if $Spec_g(M) = Cl.Spec_g(M)$. If R is a G-graded ring, then every graded classical prime ideal of R is a graded prime ideal. So, if we consider R as a graded R-module, it is graded compatible.

The following lemma is obvious.

Lemma 3.2. *Let N be a proper graded submodule of M. Then N is a graded classical prime submodule if and only if for each $x \in h(M) \setminus N$, $(N :_R x)$ is a graded prime ideal of R.*

Proposition 3.3.

(1) *Let N be proper graded submodule of M. Then N is a graded prime submodule of M if and only if N is graded primary and graded classical prime.*

(2) *Assume that N and K are graded submodule of M with $K \subseteq N$. Then N is a graded classical prime submodule of M if and only if N/K is a graded classical prime submodule of the graded R-module M/K.*

Proof. (1) If N is a graded prime submodule of M, then it clearly is both graded primary and graded classical prime. Now assume that N is a graded primary and graded classical prime submodule of M. Let $am \in N$ but $m \notin N$, where $a \in h(R)$ and $m \in h(M)$. Since N is graded primary, there exists a positive integer k such that $a^k \in (N :_R M)$. Therefore, for every $y \in h(M) \setminus N$, $a^k \in (N :_R y)$ and $(N :_R y)$ is a prime ideal of R by Lemma 3.2. Hence $a \in (N :_R y)$. It follows that $a \in (N :_R M)$, i.e. N is graded prime in M.

(2) Straightforward. □

Let R be a G-graded R-module and consider $Spec_g(R)$, the spectrum of all graded prime ideals of R. The Zariski topology on $Spec_g(R)$ is defined in a similar way to that of $Spec(R)$. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then the set $\{V_R^g(I) | I$ is a graded ideal of $R\}$ satisfies the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$ (see [14]).

In [16], $Spec_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule N of M, let $V_*^g(N) = \{P \in Spec_g(M) | N \subseteq P\}$. In this case, the set $\zeta_*^g(M) = \{V_*^g(N) |$N is a graded submodule of M$\}$ contains the empty set and $Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-Top module if $\zeta_*^g(M)$ is closed under finite unions. In this case $\zeta_*^g(M)$ satisfies the axioms for the closed sets of a unique topology τ_*^g on $Spec_g(M)$. The topology $\tau_*^g(M)$ on $Spec_g(M)$ is called the quasi-Zariski topology. In the remainder of this section we use a similar method to define a topology on $Cl.Spec_g(M)$. To this end, For each graded submodule N of M, set

$$\mathbb{V}_*^g(N) = \{P \in Cl.Spec_g(M) | N \subseteq P\}.$$

Proposition 3.4. *Let M be a graded R-module. Then*

(1) *For each subset $E \subseteq h(M)$, $\mathbb{V}_*^g(E) = \mathbb{V}_*^g(N) = V_*^g(Gr_M^{cl}(N))$, where N is the graded submodule of M generated by E.*

(2) $\mathbb{V}_*^g(0) = Cl.Spec_g(M)$, *and* $\mathbb{V}_*^g(M) = \varnothing$.

(3) *If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of graded submodules of M, then $\bigcap_{\lambda\in\Lambda} \mathbb{V}_*^g(N_\lambda) = \mathbb{V}_*^g(\sum_{\lambda\in\Lambda} N_\lambda)$.*

(4) *For every pair N and K of graded submodules of M, we have $\mathbb{V}_*^g(N) \cup \mathbb{V}_*^g(K) \subseteq \mathbb{V}_*^g(N \cap K)$.*

Proof. The proof of $(2)-(4)$ is easy. So we just provide a proof for part (1). Assume that N is the graded submodule of M generated by $E \subseteq h(M)$. Then from $E \subseteq N \subseteq Gr_M^{cl}(N)$ we have

$$\mathbb{V}^g_*(Gr^{cl}_M(N)) \subseteq \mathbb{V}^g_*(N) \subseteq \mathbb{V}^g_*(E).$$

On the other hand, N is the smallest graded submodule of M containing E, so that if $P \in \mathbb{V}^g_*(E)$, then $P \in \mathbb{V}^g_*(N)$. Therefore $\mathbb{V}^g_*(E) = \mathbb{V}^g_*(N)$. Moreover $Gr^{cl}_M(N)$ is the intersection of all graded classical prime submodules of M containing N; so $\mathbb{V}^g_*(N) = \mathbb{V}^g_*(Gr^{cl}_M(N))$. Therefore $\mathbb{V}^g_*(E) = \mathbb{V}^g_*(N) = \mathbb{V}^g_*(Gr^{cl}_M(N))$. □

Now if we set

$$\eta^g_*(M) = \{\mathbb{V}^g_*(N) | \text{N is a graded submodule of M}\}$$

then $\eta^g_*(M)$ contains the empty set and $Cl.Spec_g(M)$. Moreover $\eta^g_*(M)$ is closed under arbitrary intersections, but it is not necessarily closed under finite unions.

Definition 3.5. *Let M be a graded R-module.*

(1) *We shall say that M is a g-Cl.Top module if $\eta^g_*(M)$ is closed under finite unions, i.e. for any graded submodules N and L of M there exists a graded submodule K of M such that $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(K)$.*
(2) *A graded classical prime submodule N of M will be called graded classical extraordinary, or g-Cl.extraordinary for short, if whenever K and L are graded classical semiprime submodules of M with $K \cap L \subseteq N$ then $K \subseteq N$ or $L \subseteq N$.*

Note that if M is a g-Cl.Top module, then $\eta^g_*(M)$ satisfies the axioms for the closed sets of a unique topology ϱ^g_* on $Cl.Spec_g(M)$. In this case, the topology $\varrho^g_*(M)$ on $Cl.Spec_g(M)$ is called the quasi-Zariski topology. Note that we are not excluding the trivial case where $Cl.Spec_g(M)$ is empty; that is every g-Cl.primeless modules is a g-Cl.Top module. The next result is a useful tool for characterizing g-Cl.Top modules.

Theorem 3.6. *Let M be a graded R-module. Then, the following statements are equivalent:*

(i) *M is a g-Cl.Top module.*
(ii) *Every graded classical prime submodule of M is g-Cl.extraordinary.*
(iii) *$\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(N \cap L)$ for any graded classical semiprime submodules N and L of M.*

Proof. The result is clear when $Cl.Spec_g(M) = \varnothing$. So assume that $Cl.Spec_g(M) \neq \varnothing$.

$(i) \Rightarrow (ii)$ Let M be a g-Cl.Top module. Assume that P is a graded classical prime submodule of M and that N, L are graded classical semiprime submodules of M with $N \cap L \subseteq P$. By assumption, there exists a graded submodule K of M with $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(K)$. Since N is a graded classical semiprime submodule, $N = \bigcap_{i\in I} P_i$ in which $\{P_i\}_{i\in I}$ is a collection of graded classical prime submodules of M. For every $i \in I$, we have

$$P_i \in \mathbb{V}^g_*(N) \subseteq \mathbb{V}^g_*(K) \Rightarrow K \subseteq P_i \Rightarrow K \subseteq \bigcap_{i\in I} P_i = N$$

Similarly, $K \subseteq L$. So $K \subseteq N \cap L$. Now we have

$$\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) \subseteq \mathbb{V}^g_*(N \cap L) \subseteq \mathbb{V}^g_*(K) = \mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L).$$

Consequently, $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(N \cap L)$. Now from $N \cap L \subseteq P$ we have $P \in \mathbb{V}^g_*(N \cap L) = \mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L)$. Hence either $P \in \mathbb{V}^g_*(N)$ or $P \in \mathbb{V}^g_*(L)$, that is either $N \subseteq P$ or $L \subseteq P$. So P is g-Cl.extraordinary.

$(ii) \Rightarrow (iii)$ Suppose that every graded classical prime submodule of M is g-Cl.extraordinary. Assume that N and L are two graded classical semiprime submodules of M. Clearly $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) \subseteq \mathbb{V}^g_*(N \cap L)$. Now assume that $P \in \mathbb{V}^g_*(N \cap L)$. Then $N \cap L \subseteq P$. Since P is g-Cl.extraordinary, we have $N \subseteq P$ or $L \subseteq P$, that is either $P \in \mathbb{V}^g_*(N)$ or $P \in \mathbb{V}^g_*(L)$. Therefore $\mathbb{V}^g_*(N \cap L) \subseteq \mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L)$, and so $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(N \cap L)$.

$(iii) \Rightarrow (i)$ Let N, L be two graded submodules of M. We can assume that $\mathbb{V}^g_*(N)$ and $\mathbb{V}^g_*(L)$ are both nonempty, for otherwise $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(N)$ or $\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(L)$. We know that $Gr^{cl}_M(N)$ and $Gr^{cl}_M(L)$ are both graded classical semiprime submodules of M. Setting $K = Gr^{cl}_M(N) \cap Gr^{cl}_M(L)$ we have:

$$\mathbb{V}^g_*(N) \cup \mathbb{V}^g_*(L) = \mathbb{V}^g_*(Gr^{cl}_M(N)) \cup \mathbb{V}^g_*(Gr^{cl}_M(L)) = \mathbb{V}^g_*(Gr^{cl}_M(N) \cap Gr^{cl}_M(L)) = \mathbb{V}^g_*(K)$$

by (iii). Hence M is a g-Cl.Top module. □

Corollary 3.7. *Every g-Cl.Top module is a g-Top module.*

Proof. Assume that M is a g-Cl.Top module. Let P be a graded prime submodule of M. Since every graded prime L-submodule is a graded classical prime submodule, P is g-Cl.extraordinary by Proposition 3.6. Hence it is g-extraordinary. Now the result follows from [16, Theorem 2.3]. □

Theorem 3.8. *Let M be a g-Cl.Top R-module. Then,*

1. *For every graded submodule K of M, the R-module M/K is a g-Cl.Top module.*
2. *The graded R_P-module M_P is a g-Cl.Top module for every graded prime ideal P of R.*
3. *If $Gr^{cl}_M(N) = N$ for every graded submodule N of M, then M is a graded distributive module.*

Proof. There will be nothing to prove if M has no graded classical prime submodules. So assume that $Cl.Spec_g(M) \neq \varnothing$.

(1) By Proposition 3.3, the graded classical prime submodules of M/K are just the submodules N/K where N is a graded classical prime submodule of M with $K \subseteq N$. Consequently, any graded classical semiprime submodule of M/K is of the form S/K in which S is a graded classical semiprime submodule of M with $K \subseteq S$. Assume that S_1/K and S_2/K are two graded classical semiprime submodules of M/K. Then, by Theorem 3.6, $\mathbb{V}^g_*(S_1) \cup \mathbb{V}^g_*(S_2) = \mathbb{V}^g_*(S_1 \cap S_2)$ since M is a $g - Cl.Top$ module. Thus $\mathbb{V}^g_*(S_1/K) \cup \mathbb{V}^g_*(S_2/K) = \mathbb{V}^g_*(S_1/K \cap S_2/K)$. It follows from Theorem 3.6 that M/K is a $g - Cl.Top$ module.

(2) By Theorem 3.6, it is enough to show that every graded classical prime submodule of M_P is g-Cl.extraordinary. Let N be a graded classical prime submodule of M_P, and let $S_1 \cap S_2 \subseteq N$ for some graded classical semiprime submodules S_1, S_2 of M_P. Clearly, $N \cap M$ is a proper graded submodule of M. Assume that $a, b \in h(R)$ and $m \in h(M)$ are such that $abm \in N \cap M$. Then, $a/1, b/1 \in h(R_P)$ and $m/1 \in h(M_P)$ with $(a/1)(b/1)(m/1) = (abm)/1 \in N$. It follows that either $(a/1)(m/1) \in N$ or $(b/1)(m/1) \in N$ since N is graded classical prime. Therefore, either $am \in N \cap M$ or $bm \in N \cap M$. This implies that $N \cap M$ is a graded classical prime submodule of M. Hence N is g-Cl.extraordinary by Theorem 3.6. As another consequence, $S_1 \cap M$ and $S_2 \cap M$ are graded classical semiprime submodules of M with $(S_1 \cap M) \cap (S_2 \cap M) \subseteq N \cap M$. Therefore, $S_1 \cap M \subseteq N \cap M$ or $S_2 \cap M \subseteq N \cap M$. It follows that either $S_1 = (S_1 \cap M)R_P \subseteq (N \cap M)R_P = N$ or $S_2 = (S_2 \cap M)R_P \subseteq (N \cap M)R_P = N$. Hence N is a g-Cl.extraordinary submodule of M_P.

(3) For every graded submodules N, K and L of M we have:

$$\begin{aligned}
(K+L)\cap N &= Gr^{cl}_M((K+L)\cap N)\\
&= \bigcap\{P \mid P \in \mathbb{V}^g_*((K+L)\cap N)\}\\
&= \bigcap\{P \mid P \in \mathbb{V}^g_*(K+L) \cup \mathbb{V}^g_*(N)\}\\
&= \bigcap\{P \mid P \in (\mathbb{V}^g_*(K) \cap \mathbb{V}^g_*(L)) \cup \mathbb{V}^g_*(N)\}\\
&= \bigcap\{P \mid P \in (\mathbb{V}^g_*(K) \cup \mathbb{V}^g_*(N)) \cap (\mathbb{V}^g_*(L) \cup \mathbb{V}^g_*(N))\}\\
&= \bigcap\{P \mid P \in (\mathbb{V}^g_*(K\cap N)) \cap (\mathbb{V}^g_*(L\cap N))\}\\
&= \bigcap\{P \mid P \in \mathbb{V}^g_*((K\cap N)+(L\cap N))\}\\
&= Gr^{cl}_M((K\cap N)+(L\cap N)) = (K\cap N)+(L\cap N).
\end{aligned}$$

Thus M is graded distributive. □

Let M be a g-Cl.Top module and let $X = Cl.Spec_g(M)$. We know that any closed subset of X is of the form $\mathbb{V}^g_*(N)$ for some graded submodule N of M. But now the question arises as to what open subsets of X look like. To say that any open subset of X is of the form $X - \mathbb{V}^g_*(N)$ for some graded prime submodule N of M, though true, is not very helpful. For every subset S of $h(M)$, define

$$X_S = X - \mathbb{V}^g_*(S)$$

In particular, if $S = \{f\}$, we denote X_S be X_f.

Proposition 3.9. *The set $\{X_f \mid f \in h(M)\}$ is a basis for the quasi-Zariski topology on X.*

Proof. Let U be a non-void open subset in X. Then $U = X - \mathbb{V}^g_*(N)$ for some graded submodule N of M. Assume that N is generated by some subset $E \subseteq h(M)$. Then we have

$$U = X - \mathbb{V}^g_*(N) = X - \mathbb{V}^g_*(E) = X - \mathbb{V}^g_*(\textstyle\bigcup_{f\in E}\{f\}) = X - \bigcap_{f\in E}\mathbb{V}^g_*(f) = \bigcup_{f\in E}(X - \mathbb{V}^g_*(f)) = \bigcup_{f\in E} X_f$$

Therefore the set $\{X_f \mid f \in h(M)\}$ is a basis for X. □

A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-void open sets in X intersect. Let X be a topological space. A subset $A \subseteq X$ is said to be dense in X if and only if $A \cap G \neq \varnothing$ for every non-void open subset $G \subseteq X$. Therefore X is irreducible if and only if every non-void open subset of X is dense.

Lemma 3.10. *Let M be a graded R-module. Then, $N := Gr^{cl}_M(0)$ is a graded classical prime submodule of M if and only if $Cl.Spec_g(M)$ is irreducible.*

Proof. Set $X = Cl.Spec_g(M)$. Assume first that N is a graded classical prime submodule of M. Let $U, V \subseteq X$ be non-void open subsets. Pick $P \in U$. Now, $U = X \setminus \mathbb{V}^g_*(E)$ for some $E \subseteq h(M)$. Then $P \in U$ implies that $E \nsubseteq P$. Moreover, from $N \subseteq P$ we have $E \nsubseteq N$, so that $N \in U$. Similarly, $N \in V$. Hence $N \in U \cap V$, and thus $U \cap V \neq \varnothing$. Therefore, X is irreducible. Conversely, assume that N is not a graded classical prime submodule of M. So there exist $a, b \in h(R)$ and $m \in h(M)$ such that $am, bm \notin N$, but $abm \in N$. Both X_{am} and X_{bm} are open in X. Also $am \notin N \Rightarrow \mathbb{V}^g_*(am) \neq X$ so $X_{am} \neq \varnothing$. Similarly, $X_{bm} \neq \varnothing$. Now we have,

$$X_{am} \cap X_{bm} = X_{abm} = X - \mathbb{V}^g_*(abm) \subseteq X - \mathbb{V}^g_*(N) = \varnothing$$

Therefore, X is not irreducible. $\square$

4. Homomorphisms and graded classical prime spectrum of modules

In our discussion so far we have concerned ourselves with the graded classical prime spectrum of but one graded module at any given time. A natural question to ask is what relationships on their respective graded classical prime spectra are induced by a homomorphism between two rings. In this section, we address this question.

Let M and M' be two graded R-modules and let $\phi : M \to M'$ be a graded R-homomorphism. The inverse image of a graded classical prime submodule of M' is a graded classical prime submodule of M. For every graded submodule of M', we write $\phi^{-1}(N') = {N'}^c$, the contraction of N' to M. Also, if N is a graded submodule of M, then we write $R\phi(N) = N^e$, the graded submodule of M' generated by $\phi(N)$, the extension of N to M'. Let $X = Cl.Spec_g(M)$ and $Y = Cl.Spec_g(M')$. Thus if $P \in Y$, then $P^c \in X$. So we see that ϕ induces a map $\phi^* : Y \to X$ defined by $\phi^*(P) = P^c$, for all $P \in Y$. Before continuing, we introduce a more explicit notation:

$$P \in \mathbb{V}^g_{*M}(E) \text{ means that } E \subseteq h(M) \text{ and } E \subseteq P \in X$$

and

$$Q \in \mathbb{V}^g_{*M'}(F) \text{ means that } F \subseteq h(M') \text{ and } F \subseteq P \in Y$$

Proposition 4.1. *Let M and M' be two graded R-modules and let $\phi : M \to M'$ be a graded R-homomorphism. Let $X = Cl.Spec_g(M)$, $Y = Cl.Spec_g(M')$, and let $\phi^* : Y \to X$ be the induced map.*

(1) *ϕ^* is continuous.*
(2) *If N is a graded submodule of M, then ${\phi^*}^{-1}(\mathbb{V}^g_{*M}(N)) = \mathbb{V}^g_{*M'}(N^e)$.*
(3) *If ϕ is an epimorphism, then ϕ^* is a homeomorphism from Y onto the closed subset $\mathbb{V}^g_{*M}(Ker(\phi))$ of X.*

Proof. (1) It is enough to show that if U is open in X, then ${\phi^*}^{-1}(U)$ is open if Y. For every subset $E \subseteq h(M)$ and $Q \in Y$, we have

$$\begin{array}{rcl} \phi(E) \subseteq Q & \Leftrightarrow & E \subseteq \phi^{-1}(Q) \\ & \Leftrightarrow & \phi^*(Q) \in \mathbb{V}^g_{*M}(E) \\ & \Leftrightarrow & Q \in {\phi^*}^{-1}(\mathbb{V}^g_{*M}(E)) \end{array}$$

Hence, if $f \in h(M)$ and $Q \in Y$, then

$$\begin{array}{rcl} Q \in Y_{\phi(f)} & \Leftrightarrow & \phi(f) \notin Q \\ & \Leftrightarrow & Q \notin {\phi^*}^{-1}(\mathbb{V}^g_{*M}(f)) \\ & \Leftrightarrow & Q \in {\phi^*}^{-1}(X) - {\phi^*}^{-1}(\mathbb{V}^g_{*M}(f)) \\ & \Leftrightarrow & Q \in {\phi^*}^{-1}(X - \mathbb{V}^g_{*M}(f)) = {\phi^*}^{-1}(X_f) \end{array}$$

Therefore, ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$. In particular, if U is open in X, then ${\phi^*}^{-1}(U)$ is open if Y. Hence ϕ^* is continuous.

(2) Assume that $Q \in Y$. Then,

$$\begin{array}{rcl} Q \in {\phi^*}^{-1}(\mathbb{V}^g_{*M}(N)) & \Leftrightarrow & \phi(N) \subseteq Q \\ & \Leftrightarrow & Q \in \mathbb{V}^g_{*M'}(\phi(N)) \\ & \Leftrightarrow & Q \in \mathbb{V}^g_{*M'}(N^e) \end{array}$$

Therefore, ${\phi^*}^{-1}(\mathbb{V}^g_{*M}(N)) = \mathbb{V}^g_{*M'}(N^e)$.

(3) Suppose that ϕ is an epimorphism. Then, there exists a one-to-one correspondence between graded submodules of M' and graded submodules of M containing $Ker(\phi)$. Under this correspondence, graded classical prime submodules of M' correspond to the graded classical prime submodules of M containing $Ker(\phi)$. Therefore, $\phi^* : Y \to \mathbb{V}^g_{*M}(Ker(\phi))$ is bijective. As ϕ^* is continuous by (1), it suffices to prove that ϕ^* is an open map. Assume that U is an open subset of Y. Then, without loss of generality, we may assume that $U = Y_f$ for some $f \in h(M')$. In this case,

$$\begin{array}{rcl} \phi^*(U) = \phi^*(Y_f) & = & \{\phi^*(Q) | Q \in Y \text{ and } f \notin Q\} \\ & = & \{P \in \mathbb{V}^g_{*M}(Ker(\phi)) | \phi^{-1}(f) \notin P\} \\ & = & X_{\phi^{-1}(f)} \cap \mathbb{V}^g_{*M}(Ker(\phi)) \end{array}$$

This implies that $\phi^*(U)$ is an open subset of $\mathbb{V}^g_{*M}(Ker(\phi))$, that is ϕ^* is an open map. Consequently, $\phi^* : Y \to \mathbb{V}^g_{*M}(Ker(\phi))$ is a homeomorphism. $\square$

Corollary 4.2. *Let M be a graded R-module, and let N be the graded classical nil-radical of M. Then $Cl.Spec_g(M)$ and $Cl.Spec_g(M/N)$ are naturally homeomorphic.*

Proof. By Proposition 4.1, the canonical graded R-epimorphism $f : M \to M/N$ induces the homeomorphism $f^* : Cl.Spec_g(M/N) \to \mathbb{V}^g_{*M}(Ker(f))$. Now the result follows from $\mathbb{V}^g_{*M}(Ker(f)) = \mathbb{V}^g_{*M}(N) = Cl.Spec_g(M)$. □

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