Appl. Gen. Topol. 23, no. 1 (2022), 79-90
doi:10.4995/agt.2022.15844

Applied General Topology

Closed ideals in the functionally countable subalgebra of $C(X)$

Amir Veisi

Faculty of Petroleum and Gas, Yasouj University, Gachsaran, Iran (aveisi@yu.ac.ir)

Communicated by A. Tamariz-Mascarúa

Abstract

In this paper, closed ideals in $C_c(X)$, the functionally countable subalgebra of $C(X)$, with the m_c-topology are studied. We show that if X is a CUC-space, then $C_c^*(X)$ with the uniform norm-topology is a Banach algebra. Closed ideals in $C_c(X)$ as a modified countable analogue of closed ideals in $C(X)$ with the m-topology, are characterized. For a zero-dimensional space X, we show that a proper ideal in $C_c(X)$ is closed if and only if it is an intersection of maximal ideals of $C_c(X)$. It is also shown that every ideal in $C_c(X)$ with the m_c-topology is closed if and only if X is a P-space if and only if every ideal in $C(X)$ with the m-topology is closed. Also, for a strongly zero-dimensional space X, it is proved that every properly closed ideal in $C_c^*(X)$ is an intersection of maximal ideals of $C_c^*(X)$ if and only if X is pseudocompact if and only if every properly closed ideal in $C^*(X)$ is an intersection of maximal ideals of $C^*(X)$. Finally, we show that if X is a P-space, then the family of e_c-ultrafilters and z_c-ultrafilter coincide.

2020 MSC: *54C30; 54C40; 13C11.*

Keywords: *zero-dimensional space; functionally countable subalgebra; m-topology; closed ideal; e_c-filter; e_c-ideal; P-space.*

1. Introduction

In what follows X stands for an infinite completely regular Hausdorff topological space (i.e., infinite Tychonoff space) and $C(X)$ as usual denotes the ring of all real-valued continuous functions on X. $C^*(X)$ designates the subring of $C(X)$ containing all those members which are bounded over X. For

Received 30 June 2021 – Accepted 09 October 2021

each $f \in C(X)$, the *zero-set* of f, denoted by $Z(f)$, is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f and the set of all zero-sets in X is denoted by $Z(X)$. An ideal I in $C(X)$ is called a *z-ideal* if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. The space βX is the *Stone-Čech compactification* of X and for any $p \in \beta X$, the maximal ideal M^p of $C(X)$ is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$. Moreover, M^p is fixed if and only if $p \in X$ (in which case, we put $M^p = M_p = \{f \in C(X) : p \in Z(f)\}$). Whenever $\frac{C(X)}{M^p} \cong \mathbb{R}$, then M^p is called *real*, else *hyper-real*, see [5, Chapter 8]. We recall that a *zero-dimensional space* is a Hausdorff space with a base consisting of clopen (closed-open) sets. A Tychonoff space X is called *strongly zero-dimensional* if for every finite cover $\{U_i\}_{i=1}^k$ of X by cozero-sets there exists a finite refinement $\{V_i\}_{i=1}^m$ of mutually disjoint open sets. A Tychonoff space X is strongly zero-dimensional if and only if βX is zero-dimensional, see [2].
The subring of $C(X)$ consisting of those functions with countable (resp. finite) image, which is denoted by $C_c(X)$ (resp. $C^F(X)$) is an $\mathbb{R}$-subalgebra of $C(X)$. The subring $C_c^*(X)$ of $C_c(X)$ consists of bounded elements of $C_c(X)$. So $C_c^*(X) = C^*(X) \cap C_c(X)$. The rings $C_c(X)$ and $C^F(X)$ are introduced and investigated in [3] and more studied in [1], [4], [9], [10] and [12]. A topological space X is called *countably pseudocompact*, briefly, *c-pseudocompact* if $C_c(X) = C_c^*(X)$. A nonempty subfamily $\mathcal{F}$ of $Z_c(X) := \{Z(f) : f \in C_c(X)\}$ is called a *z_c-filter* if it is a filter on X. For an ideal I in $C_c(X)$ and a z_c-filter $\mathcal{F}$, we define $Z_c[I] = \{Z(f) : f \in I\}$, $\cap Z_c[I] = \cap\{Z(f) : f \in I\}$ and $Z_c^{-1}[\mathcal{F}] = \{f \in C_c(X) : Z(f) \in \mathcal{F}\}$. It is observed that $\mathcal{F} = Z_c[Z_c^{-1}[\mathcal{F}]]$. Also, $Z_c[I]$ is a z_c-filter on X and $Z_c^{-1}[Z_c[I]] \supseteq I$. If the equality holds, then I is called a *z_c-ideal*. This means that if $f \in I$, $g \in C_c(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. So maximal ideals in $C_c(X)$ are z_c-ideals. In the same way, for an ideal I of $C_c^*(X)$ and a z_c-filter $\mathcal{F}$ on X, $E_c(I)$ is an e_c-filter and $E_c^{-1}(\mathcal{F})$ is an e_c-ideal. The counterpart notions are $E_c^{-1}(E_c(I)) \supseteq I$ and $E_c(E_c^{-1}(\mathcal{F})) = \mathcal{F}$, see [14]. By $\beta_0 X$, we mean the *Banaschewski compactification* of a zero-dimensional space X. If βX is zero-dimensional, then $\beta X = \beta_0 X$, see [13, Section 4.7] for more details. According to [1, Theorems 4.2, 4.8], for any $p \in \beta_0 X$, the maximal ideal M_c^p of $C_c(X)$ is the set of all $f \in C_c(X)$ for which $p \in \mathrm{cl}_{\beta_0 X} Z(f)$, or equivalently, it is the set of all $f \in C_c(X)$ for which $\pi_p \in \mathrm{cl}_{\beta X} Z(f)$. Moreover, M_c^p is fixed if and only if $p \in X$ (in which case, we put $M_c^p = M_{cp} = \{f \in C_c(X) : p \in Z(f)\}$). Let S be a subring of $C(X)$ and a topological space. An ideal I of S is called a *closed ideal* if $I = \mathrm{cl}_S I$, briefly, $I = \mathrm{cl} I$. The paper is organized as follows. In Section 2, we introduce the m_c-topology on $C_c(X)$ and derive some corollaries on the ideals of $C_c(X)$ and $C_c^*(X)$. We show that if X is a CUC-space, then $C_c^*(X)$ with the uniform-norm topology is a Banach algebra. It is shown that an ideal in $C_c(X)$ is a z-ideal if and only if it is a z_c-ideal. In [5], closed ideals in $C(X)$ with the m-topology are characterized. In Section 3, the countable analogue of this characterization is given. We show that a proper ideal in $C_c(X)$ is closed if and only if it is an intersection of maximal ideals in $C_c(X)$. It is also shown that every ideal

in $C_c(X)$ is closed if and only if X is a P-space if and only if every ideal in $C(X)$ is closed. For a strongly zero-dimensional space X, we prove that every properly closed ideal in $C_c^*(X)$ is an intersection of maximal ideals of $C_c^*(X)$ if and only if X is pseudocompact if and only if every properly closed ideal in $C^*(X)$ is an intersection of maximal ideals of $C^*(X)$. Finally, we show that if X is a P-space, then the family of e_c-ultrafilters and z_c-ultrafilter coincide.

2. Some properties of ideals in $C_c(X)$

The m-topology on $C(X)$ was first introduced and studied by Hewitt [8], the generalizing work of E. H. Moore. In his article, he demonstrated that certain classes of topological spaces X can be characterized by topological properties of $C(X)$ with the m-topology. For example, he showed that X is pseudocompact if and only if $C(X)$ with the m-topology is first countable. Several authors have investigated the topological properties of X via properties of $C(X)$, for more information, one can refer to [6] and [11]. The *m-topology* on $C(X)$ is defined by taking the sets of the form

$$B(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\},$$

as a base for the neighborhood system at f, for each $f \in C(X)$ and each positive unit u of $C(X)$. The *m_c-topology* (in brief, m_c) on $C_c(X)$ is determined by considering the sets of the form

$$B(f,u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\},$$

as a base for the neighborhood system at f, for each $f \in C_c(X)$ and each positive unit u of $C_c(X)$. The *uniform topology*, or the *u_c-topology* (in brief, u_c) on $C_c(X)$ is defined by taking the sets of the form

$$B(f,\varepsilon) = \{g \in C_c(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in X\},$$

as a base for the neighborhood system at f, for each $f \in C_c(X)$ and each $\varepsilon > 0$. Equivalently, a base at f is given by all sets

$$B(f,u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\},$$

where u is a positive unit of $C_c^*(X)$. We observe that $u_c \subseteq m_c$. It is shown in [15] that $u_c = m_c$ if and only if X is countably pseudocompact. The u_c-topology turns $C_c(X)$ into a metric space with $d(f,g) = \|f-g\| = \sup\{|f(x) - g(x)| : x \in X\}$. Also, the m_c-topology is contained in the relative m-topology. We remind a well-known result that due to Rudin, Pelczynski and Semadeni which asserts that a compact Hausdorff space X is functionally countable (i.e., $C(X) = C_c(X)$) if and only if X is scattered. So if X is a compact scattered space or a countable space, then $C(X) = C_c(X)$, and thus the m_c-topology and the m-topology coincide.

Proposition 2.1. *Let I be an ideal in $C_c(X)$ (resp. $C_c^*(X)$) and the topology on $C_c(X)$ be the m_c-topology. Then:*

- (i) *$cl\, I$ is an ideal in $C_c(X)$ (resp. $C_c^*(X)$) and hence I is contained in a closed ideal.*
- (ii) *If I is a proper ideal, then $cl\, I$ is also a proper ideal and hence there is no proper dense ideal in $C_c(X)$ (resp. $C_c^*(X)$).*

Proof. We provide the proof for which case I is an ideal in $C_c(X)$. In the same way, the proof holds for the ideal I in $C_c^*(X)$. (i). Clearly, the result holds if $I = C_c(X)$. Suppose that $I \subsetneqq C_c(X)$. Let $f, g \in \mathrm{cl}I$, $h \in C_c(X)$ and u be a positive unit of $C_c(X)$. Then for some $f' \in B(f, \frac{u}{2}) \cap I$, and $g' \in B(g, \frac{u}{2}) \cap I$, we have $f' + g' \in B(f + g, u) \cap I$. To show that $fh \in \mathrm{cl}I$, we consider the positive unit

$$u_1 = \frac{u}{(|h| + 1)(u + 1)} \in C_c(X).$$

Therefore, for some $f_1 \in B(f, u_1) \cap I$ we have that $|fh - f_1h| < u_1|h| < u$. So $f_1h \in B(fh, u) \cap I$. Moreover, if $f \in \mathrm{cl}I$, then also $-f \in \mathrm{cl}I$. Thus, $\mathrm{cl}I$ contains both $f + g$ and fh. So $\mathrm{cl}I$ is ideal. (ii). Suppose that I is a proper ideal in $C_c(X)$ and $\mathrm{cl}I = C_c(X)$. Consider the constant function $1 \in \mathrm{cl}I$ and $0 < \varepsilon < 1$. Hence, the nonempty set $B(1, \varepsilon) \cap I$ contains a nonzero element of $C_c(X)$, f say. Since $1 - \varepsilon < f(x) < 1 + \varepsilon$ for each $x \in X$, we have $Z(f) = \varnothing$, i.e., f is a unit of $C_c(X)$, which is impossible (because $f \in I$). Thus, $\mathrm{cl}I \subsetneqq C_c(X)$, and we are done. □

The next result is now immediate.

Corollary 2.2. *Any maximal ideal in $C_c(X)$ (resp. $C_c^*(X)$) and hence any intersection of maximal ideals in $C_c(X)$ (resp. $C_c^*(X)$) is closed.*

Definition 2.3. An ideal I in a commutative ring with unity R is called a *z-ideal* in R if for each $a \in I$, we have $M_a \subseteq I$, here M_a is the intersection of all maximal ideals in R containing a.

Evidently, each maximal ideal in R is a z-ideal. This notion of z-ideal is consistent with the notion of z-ideals in $C(X)$, see [5, 4A(5)].

Proposition 2.4. *Let X be zero-dimensional and I be an ideal in $C_c^*(X)$. Then I is a z-ideal if and only if $g \in I$ whenever $Z(f^\beta) \subseteq Z(g^\beta)$ with $f \in I$ and $g \in C_c^*(X)$, where f^β is the extension of f to βX.*

Proof. $(\Rightarrow)$: Let $f \in I$, $g \in C_c^*(X)$ and $Z(f^\beta) \subseteq Z(g^\beta)$ and let M_f be the intersection of all the maximal ideals in $C_c^*(X)$ containing f. By the assumption, $M_f \subseteq I$. Let M be a maximal ideal in $C_c^*(X)$ containing f. According to [9, Corollary 2.11], M has a form of $M_c^{*p} = \{h \in C_c^*(X) : h^\beta(p) = 0\}$, for some $p \in \beta X$. Now, $Z(f^\beta) \subseteq Z(g^\beta)$ implies that $g \in M$. Hence, $g \in I$.
$(\Leftarrow)$: Let $f \in I$ and $g \in M_f$. Then $f \in M_c^{*p}$ implies that $g \in M_c^{*p}$, i.e., $Z(f^\beta) \subseteq Z(g^\beta)$. Therefore, by the hypothesis, $g \in I$. □

Lemma 2.5. *Let X be zero-dimensional and I be an ideal in $C_c(X)$. Then I is a z-ideal if and only if it is a z_c-ideal.*

Proof. $(\Rightarrow)$: Let I be a z-ideal in $C_c(X)$, $f \in I$ and $Z(f) \subseteq Z(g)$ with $g \in C_c(X)$. We have to show that $g \in I$. Since I is a z-ideal, we have $M_f \subseteq I$, where M_f is the intersection of all the maximal ideals in $C_c(X)$ containing f. It suffices to show that $g \in M_f$. So let M_c^p ($p \in \beta_0 X$) be any maximal ideal in $C_c(X)$ which contains f, we have to show that $g \in M_c^p$ (see [1, Theorem

4.2]). Indeed $f \in M_c^p$ implies that $p \in \mathrm{cl}_{\beta_0 X} Z(f)$ which further implies that $p \in \mathrm{cl}_{\beta_0 X} Z(g)$, by the assumption, $Z(f) \subseteq Z(g)$. Hence, $g \in M_c^p$. Thus, I becomes a z_c-ideal in $C_c(X)$.
$(\Leftarrow)$: Let I be a z_c-ideal in $C_c(X)$ and $f \in I$. We must show $M_f \subseteq I$. Let $g \in M_f$. Then $f \in M_c^p$ gives $g \in M_c^p$, where $p \in \beta_0 X$. Equivalently, $\mathrm{cl}_{\beta_0 X} Z(f) \subseteq \mathrm{cl}_{\beta_0 X} Z(g)$. So $Z(f) = \mathrm{cl}_{\beta_0 X} Z(f) \cap X \subseteq \mathrm{cl}_{\beta_0 X} Z(g) \cap X = Z(g)$. Now, the assumption yields that $g \in I$. $\square$

Proposition 2.6. *If I is a closed ideal in $C_c(X)$, then I is a z_c-ideal.*

Proof. Suppose that $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C_c(X)$. To show that $g \in I$, we show that $g \in \mathrm{cl}I$ because $I = \mathrm{cl}I$. Let $u \in C_c(X)$ be a positive unit and let us define a function $h : X \to \mathbb{R}$ as follows:

$$h(x) = \begin{cases} \frac{g(x) - \frac{u(x)}{2}}{f(x)} & \text{where } \; g(x) \geq \frac{u(x)}{2}, \\ 0 & \text{where } \; |g(x)| \leq \frac{u(x)}{2}, \\ \frac{g(x) + \frac{u(x)}{2}}{f(x)} & \text{where } \; g(x) \leq -\frac{u(x)}{2}. \end{cases}$$

From the continuity of h on the three closed sets $(g - \frac{u}{2})^{-1}([0, \infty))$, $(g + \frac{u}{2})^{-1}([0, \infty)) \cap (g - \frac{u}{2})^{-1}((-\infty, 0])$, and $(g + \frac{u}{2})^{-1}((-\infty, 0])$, which their union is X, we infer that $h \in C(X)$. Moreover, since the ranges of g, u and f are countable, the range of h is also countable, i.e., $h \in C_c(X)$. Thus, $fh \in I$. Furthermore, it is easy to see that $|g(x) - f(x)h(x)| < u(x)$ for every $x \in X$, i.e., $fh \in B(g, u) \cap I$ and thus $g \in \mathrm{cl}I$, which completes the proof. $\square$

The next example shows that the converse of the above proposition is not true in general.

Example 2.7. Consider the zero-dimensional space $X = \mathbb{Q} \times \mathbb{Q}$, $p = (0, 0) \in X$, and put $O_p = \{f \in C(X) : p \in \mathrm{int}_X Z(f)\}$ (note, $C_c(X) = C(X)$ because X is countable). Recall that O_p is a z_c-ideal. We now claim that O_p is not a closed ideal in $C(X)$. To see this, consider $f(x, y) = \frac{|x| + |y|}{1 + |x| + |y|} \in C(X)$ and let u be a fixed positive unit of $C(X)$. Define a function g by

$$g(x, y) = \begin{cases} 0 & \text{where } \; f(x, y) \leq \frac{u(x,y)}{2}, \\ f(x, y) - \frac{u(x,y)}{2} & \text{where } \; f(x, y) \geq \frac{u(x,y)}{2}. \end{cases}$$

Obviously, $g \in C(X)$. Let $G = \{(x, y) \in X : f(x, y) < \frac{u(x,y)}{2}\}$. Then $p \in G \subseteq Z(g)$ and therefore $g \in O_p$, in fact, $g \in B(f, u) \cap O_p$. It follows that $f \in \mathrm{cl}_{C(X)} O_p$. On the other hand, the set $Z(f) = \{p\}$ is not open in X. Hence, $f \in \mathrm{cl}_{C(X)} O_p \setminus O_p$. i.e., O_p is not a closed ideal in $C(X)$.

A *Banach algebra* B is an algebra that is a Banach space with a norm that satisfies $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in B$, and there exists a unit element $e \in B$ such that $ex = xe = x$, $\|e\| = 1$.

In [7, Definition 2.2], a topological space X is called a *countably uniform closed-space*, briefly, a *CUC-space*, if whenever $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of functions of $C_c(X)$ and $f_n \to f$ uniformly, then f belongs to $C_c(X)$.

Theorem 2.8. *If X is a CUC-space, then $C_c^*(X)$ with the supremum-norm topology is a Banach algebra.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of functions in $C_c^*(X)$. Given $\varepsilon > 0$, we can find a natural number N such that $\|f_n - f_m\| \leq \varepsilon$ for every $m, n > N$. Thus, $|f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in X$ and all $m, n > N$. Let $x \in X$ be fixed and a_x be the limit of the numerical sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ (note, $\mathbb{R}$ is a Banach space). Now, define $f : X \to \mathbb{R}$ by $f(x) = a_x$. Let n be fixed, then $|f_n(x) - \lim_{m\to\infty} f_m(x)| \leq \varepsilon$ for each $x \in X$ and each $m > N$. So $\|f_n - f\| \leq \varepsilon$. Since n is arbitrary, we get $f_n \to f$ in the norm, uniformly. Consequently, $f \in C(X)$. Furthermore, our assumption implies that $f \in C_c(X)$. Moreover, $\|f\| \leq \|f - f_n\| + \|f_n\|$ gives f is bounded. Hence, $C_c^*(X)$ is a Banach space. The proof is completed by the fact that $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in C_c^*(X)$. $\square$

3. Closed ideals in $C_c(X)$ and $C_c^*(X)$ (with the m_c-topology)

We need the next statement which is the counterpart of [5, 1D(1)] for $C_c(X)$.

Proposition 3.1. *If $f, g \in C_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in C_c(X)$.*

Proposition 3.2. *Let X be a zero-dimensional space, $f \in C_c(\beta_0 X)$ and let f_0 be the restriction of f on X. Then $\mathrm{int}_{\beta_0 X} Z(f) \subseteq \mathrm{cl}_{\beta_0 X} Z(f_0) \subseteq Z(f)$.*

Proof. Let $p \in \mathrm{int}_{\beta_0 X} Z(f)$ and V be an open set in $\beta_0 X$ containing p. Since X is dense in $\beta_0 X$, we have $\varnothing \neq V \cap \mathrm{int}_{\beta_0 X} Z(f) \cap X \subseteq V \cap Z(f_0)$. So $p \in \mathrm{cl}_{\beta_0 X} Z(f_0)$. For the second inclusion, since $Z(f_0) \subseteq Z(f)$, we have that $\mathrm{cl}_{\beta_0 X} Z(f_0) \subseteq \mathrm{cl}_{\beta_0 X} Z(f) = Z(f)$. $\square$

Corollary 3.3. *Let X be zero-dimensional and $p \in \beta_0 X$. Then*

(i) $\bigcap_{f\in M_c^p} \mathrm{cl}_{\beta_0 X} Z(f) = \{p\}$.
(ii) *If $p \in X$, then $\bigcap_{f\in M_{cp}} Z(f) = \{p\}$, i.e., M_{cp} is fixed.*

Proof. (i). Recall that $f \in M_c^p$ if and only if $p \in \mathrm{cl}_{\beta_0 X} Z(f)$ (see [1, Theorem 4.2]). Therefore, $p \in \bigcap_{f\in M_c^p} \mathrm{cl}_{\beta_0 X} Z(f)$. Now, we claim that the latter intersection is the singleton set $\{p\}$. On the contrary, suppose that this set contains an element $q \in \beta_0 X$ distinct from p. Since $\beta_0 X$ is zero-dimensional, by [3, Proposition 4.4], there exists $g \in C_c(\beta_0 X)$ such that $p \in \mathrm{int}_{\beta_0 X} Z(g)$ and $g(q) = 1$. Let g_0 be the restriction of g on X. Then by Proposition 3.2, $\mathrm{cl}_{\beta_0 X} Z(g_0)$ contains p but not q. This means that $g_0 \in M_c^p \setminus M_c^q$ which is a contradiction, so (i) holds. (ii). Clearly, $\bigcap_{f\in M_{cp}} Z(f) = \bigcap_{f\in M_{cp}} \mathrm{cl}_{\beta_0 X} Z(f) \cap X = \{p\}$. $\square$

In a similar way to Proposition 3.2 and Corollary 3.3, we get:

Proposition 3.4. *For a Tychonoff space X and $f \in C^*(X)$, we have that $\mathrm{int}_{\beta X} Z(f^\beta) \subseteq \mathrm{cl}_{\beta X} Z(f) \subseteq Z(f^\beta)$, where f^β is the extension of f to βX. Moreover, if $p \in \beta X$, then $\bigcap_{f\in M^p} \mathrm{cl}_{\beta X} Z(f) = \{p\}$. In particular, if $p \in X$, then $\bigcap_{f\in M_p} Z(f) = \{p\}$, i.e., M_p is fixed.*

Proposition 3.5. *Let X be zero-dimensional, $p \in \beta_0 X$ and π_p be its corresponding point of βX in characterizing of maximal ideals in $C_c(X)$. Then $M_c^p \cap C_c^*(X) \subseteq M^{*\pi_p} \cap C_c^*(X)$. Particularly, if X is strongly zero-dimensional, then $M_c^p \cap C_c^*(X) \subseteq M^{*p} \cap C_c^*(X)$.*

Proof. In view of [1, Theorems 4.2, 4.8], we have

$$M_c^p = \{f \in C_c(X) : p \in \mathrm{cl}_{\beta_0 X} Z(f)\} = \{f \in C_c(X) : \pi_p \in \mathrm{cl}_{\beta X} Z(f)\}.$$

Let $f \in M_c^p \cap C_c^*(X)$. Then $\pi_p \in \mathrm{cl}_{\beta X} Z(f)$ and hence $f^\beta(\pi_p) = 0$, by Proposition 3.4. Therefore, $f \in M^{*\pi_p} \cap C_c^*(X)$. The second part follows from the assumption, i.e., $\beta_0 X = \beta X$ and so $\pi_p = p$. $\square$

Remark 3.6. Replacing T with $\beta_0 X$ in [1, Proposition 3.2] implies that for any two zero-sets Z_1 and Z_2 in $Z_c(X)$, we get $cl_{\beta_0 X}(Z_1 \cap Z_2) = cl_{\beta_0 X} Z_1 \cap cl_{\beta_0 X} Z_2$.

Remark 3.7. ([1, Remark 4.12]) If X is zero-dimensional and $f, g \in C_c(X)$, then $\mathrm{cl}_{\beta_0 X} Z(f)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$ if and only if there exists $h \in C_c(X)$ such that $Z(g) \subseteq \mathrm{coz}(h) \subseteq Z(f)$.

Proposition 3.8. *Let X be zero-dimensional and I a proper ideal in $C_c(X)$ and let $V_c(I) = \{p \in \beta_0 X : M_c^p \supseteq I\}$. Then:*

(i) $V_c(I) = \bigcap_{g \in I} cl_{\beta_0 X} Z(g)$.

(ii) *If $f \in C_c(X)$ and $cl_{\beta_0 X} Z(f)$ is a neighborhood of $V_c(I)$, then $f \in I$.*

Proof. (i). This is easily obtained from the fact that $g \in M_c^p$ if and only if $p \in \mathrm{cl}_{\beta_0 X} Z(g)$. (ii). Suppose that

$$V_c(I) = \bigcap_{g \in I} \mathrm{cl}_{\beta_0 X} Z(g) \subseteq \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f).$$

Then we have $\bigcup_{g \in I} \big(\beta_0 X \setminus \mathrm{cl}_{\beta_0 X} Z(g)\big) \supseteq \beta_0 X \setminus \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)$. Hence, the collection

$$\mathcal{C} = \{\mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f),\ \beta_0 X \setminus \mathrm{cl}_{\beta_0 X} Z(g) : g \in I\}$$

is an open cover for the compact set $\beta_0 X$. Therefore, there is a finite number of elements of I; $g_1, g_2, \ldots, g_n$ say, such that

$$\begin{aligned}\beta_0 X &= \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f) \cup \big(\beta_0 X \setminus \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)\big) \\ &= \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f) \cup \Big(\bigcup_{i=1}^{n} (\beta_0 X \setminus \mathrm{cl}_{\beta_0 X} Z(g_i))\Big).\end{aligned}$$

Now, we have that

$$\Big(\bigcap_{i=1}^n \mathrm{cl}_{\beta_0 X} Z(g_i)\Big) \cap \Big(\beta_0 X \setminus \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)\Big) = \varnothing.$$

Thus, $\bigcap_{i=1}^n \mathrm{cl}_{\beta_0 X} Z(g_i) \subseteq \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)$. Since I is a proper ideal, the element $g = \sum_{i=1}^n g_i^2$ of I is not a unit of $C_c(X)$ and hence $Z(g) = \bigcap_{i=1}^n Z(g_i) \neq \varnothing$. From Remark 3.6 we conclude that

$$\mathrm{cl}_{\beta_0 X} Z(g) = \mathrm{cl}_{\beta_0 X}\big(\bigcap_{i=1}^n Z(g_i)\big) = \bigcap_{i=1}^n \mathrm{cl}_{\beta_0 X} Z(g_i) \subseteq \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f).$$

This leads us $\mathrm{cl}_{\beta_0 X} Z(f)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$. In view of Remark 3.7, there exists $h \in C_c(X)$ such that $Z(g) \subseteq \mathrm{coz}(h) \subseteq Z(f)$. So $Z(f)$ is a neighborhood of $Z(g)$. By Proposition 3.1, we get $f \in I$. □

Lemma 3.9. *Let X be zero-dimensional and $g \in C_c(X)$. Then for any neighborhood $B(g,u)$ of g in the m_c-topology, there exists some $f_u \in B(g,u)$ such that $cl_{\beta_0 X} Z(f_u)$ is a neighborhood of $cl_{\beta_0 X} Z(g)$.*

Proof. If $\mathrm{cl}_{\beta_0 X} Z(g)$ is an open set in $\beta_0 X$, then we set $f_u = g$. In general, we define a function $f_u : X \to \mathbb{R}$ by

$$f_u(x) = \begin{cases} g(x) - \frac{u(x)}{2} & \text{where } \; g(x) \geq \frac{u(x)}{2}, \\ 0 & \text{where } \; |g(x)| \leq \frac{u(x)}{2}, \\ g(x) + \frac{u(x)}{2} & \text{where } \; g(x) \leq -\frac{u(x)}{2}. \end{cases}$$

It is clear that $f_u \in C(X)$ and further since the range of g and u is countable, we get $f_u \in C_c(X)$. Moreover, $f_u \in B(g,u)$. To establish the conclusion, consider the function h below

$$h(x) = \begin{cases} \big(g(x) + \frac{u(x)}{2}\big)\big(g(x) - \frac{u(x)}{2}\big) & \text{where } \; |g(x)| \leq \frac{u(x)}{2}, \\ 0 & \text{where } \; |g(x)| \geq \frac{u(x)}{2}. \end{cases}$$

We observe that $h \in C_c(X)$. Furthermore, $Z(g) \subseteq \mathrm{coz}\,(h) \subseteq Z(f_u)$. Now, Remark 3.7 implies that $\mathrm{cl}_{\beta_0 X} Z(f_u)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$, and we are through. □

Theorem 3.10. *Let X be zero-dimensional and I a proper ideal in $C_c(X)$ and let $V_c(I)$ be the same as the set in Proposition 3.8 $\big(V_c(I) = \bigcap_{g \in I} cl_{\beta_0 X} Z(g)\big)$. Let*

$$J = \{f \in C_c(X) : cl_{\beta_0 X} Z(f) \supseteq V_c(I)\}, \text{ and } \bar{I} = \cap\{M_c^p : M_c^p \supseteq I\}.$$

Then:

(i) *$\bar{I}$ is a closed ideal in $C_c(X)$ containing I.*
(ii) *$J = \bar{I}$, in other words, J is the kernel of the hull of I in the structure space of $C_c(X)$.*
(iii) *$V_c(I) = V_c(\bar{I})$.*
(iv) *$cl\, I = \bar{I}$.*

Proof. (i). It follows from Corollary 2.2. (ii). Let $f \in J$ and M_c^p $(p \in \beta_0 X)$ be a maximal ideal in $C_c(X)$ containing I. Then

$$V_c(I) \supseteq V_c(M_c^p) \text{ and so } \mathrm{cl}_{\beta_0 X} Z(f) \supseteq V_c(I) \supseteq V_c(M_c^p) = \{p\} \tag{3.1}$$

(note, the last equality follows from Corollary 3.3). Therefore, $f \in M_c^p$ and thus $f \in \bar{I}$, i.e., $J \subseteq \bar{I}$. For the reverse inclusion, we show that if $f \notin J$, then $f \notin \bar{I}$. Since $f \notin J$, there exists $q \in \beta_0 X$ such that $q \in V_c(I) \setminus \mathrm{cl}_{\beta_0 X} Z(f)$. Therefore, $g \in M_c^q$ for every $g \in I$ and hence $I \subseteq M_c^q$. But $f \notin M_c^q$. Thus, M_c^q is a maximal ideal containing I but not f. This yields that $f \notin \bar{I}$. (iii). Using (ii) and the definition of J, we have $V_c(\bar{I}) = V_c(J) \supseteq V_c(I)$. On the other hand, the inclusion $I \subseteq \bar{I}$ implies that $V_c(\bar{I}) \subseteq V_c(I)$. So (iii) holds.

(iv). By (i), $\mathrm{cl}I \subseteq \bar{I}$. Now, suppose that $g \in \bar{I}$ and u is a positive unit of $C_c(X)$. We claim that $B(g,u) \cap I \neq \varnothing$. According to Lemma 3.9, there exists $f_u \in C_c(X)$ such that $f_u \in B(g,u)$, and $\mathrm{cl}_{\beta_0 X} Z(f_u)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$. Now, it remains to show that $f_u \in I$. From (iii), we infer that $V_c(I) = V_c(\bar{I}) \subseteq \mathrm{cl}_{\beta_0 X} Z(g) \subseteq \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f_u)$. Proposition 3.8(ii) now yields that $f_u \in I$. Therefore, $f_u \in B(g,u) \cap I$ and so $g \in \mathrm{cl}I$, i.e., $\bar{I} \subseteq \mathrm{cl}I$. ☐

It is known that a proper ideal in $C(X)$ with the m-topology is closed if and only if it is an intersection of maximal ideals in $C(X)$ (see [5, 7Q(2)]). The next theorem involves the countable analogue characterization of closed ideals in $C_c(X)$. Using Theorem 3.10(iv) and Corollary 2.2, we obtain:

Theorem 3.11. *Let X be zero-dimensional and the topology on $C_c(X)$ be the m_c-topology. Then a proper ideal in $C_c(X)$ is closed if and only if it is an intersection of maximal ideals of $C_c(X)$.*

Theorem 3.12. *Let X be zero-dimensional and the topology on $C_c(X)$ (resp. $C(X)$) be the m_c-topology (resp. the m-topology). Then the following statements are equivalent.*

(i) *Every ideal in $C(X)$ is closed.*
(ii) *X is a P-space.*
(iii) *Every ideal in $C_c(X)$ is closed.*
(iv) *Every prime ideal in $C_c(X)$ is closed.*

Proof. (i) $\Leftrightarrow$ (ii). It follows from [5, 4J(9), 7Q(2)].

(ii) $\Rightarrow$ (iii). By [3, Proposition 5.3], X is a CP-space. Now, the result is obtained by [3, Theorem 5.8(7)] and Corollary 2.2.

(iii) $\Rightarrow$ (iv). It is evident.

(iv) $\Rightarrow$ (ii). According to [3, Corollary 5.7], it is enough to show that X is a CP-space. Let P be a prime ideal in $C_c(X)$, then by [1, Lemma 4.11(4)], P is contained in a unique maximal ideal M_c^p of $C_c(X)$, where $p \in \beta_0 X$. Now, by the assumption and Theorem 3.11, we get $P = M_c^p$, i.e., X is a CP-space. ☐

Theorem 3.13. *Let X be strongly zero-dimensional and the topology on $C_c^*(X)$ (resp. $C^*(X)$) be the m_c-topology (resp. the m-topology). Then the following statements are equivalent.*

(i) *Every properly closed ideal in $C_c^*(X)$ is an intersection of maximal ideals of $C_c^*(X)$.*
(ii) *X is pseudocompact.*
(iii) *Every properly closed ideal in $C^*(X)$ is an intersection of maximal ideals of $C^*(X)$.*

Proof. A maximal ideal in $C_c^*(X)$ is of the form $M_c^{*p} = \{f \in C_c^*(X) : f^\beta(p) = 0\}$, where $p \in \beta X$. Also, $M_c^{*p} = M^{*p} \cap C_c^*(X)$, see [9, Corollaries 2.10, 2.11].

(i) $\Rightarrow$ (ii). Suppose that X is not pseudocompact, so $C_c^*(X) \subsetneqq C_c(X)$, by [9, Theorem 6.3]. Hence, $C_c(X)$ contains an unbounded element, f say. So for some $p \in \beta X$ and the maximal ideal M_c^p of $C_c(X)$, we have $|M_c^p(f)|$ is infinitely large ([9, Proposition 2.4]). In other words, M_c^p is hyper-real, i.e., $\mathbb{R} \subsetneqq \frac{C_c(X)}{M_c^p}$. Hence, by [9, Corollary 2.13], $M_c^p \cap C_c^*(X)$ is not a maximal ideal in $C_c^*(X)$. Using Proposition 3.5, we infer that

$$M_c^p \cap C_c^*(X) \subsetneqq M^{*p} \cap C_c^*(X). \tag{3.2}$$

Furthermore, since the maximal ideal M_c^p is closed in $C_c(X)$ (Corollary 2.2), the ideal $M_c^p \cap C_c^*(X)$ is also closed in $C_c^*(X)$. We now claim that the latter closed ideal cannot be an intersection of maximal ideals of $C_c^*(X)$. Otherwise,

$$M_c^p \cap C_c^*(X) = \bigcap_{q \in A \subseteq \beta X} \Big(M^{*q} \cap C_c^*(X)\Big), \tag{3.3}$$

for a subset A of βX. Notice that by (3.2), $A \neq \varnothing$ since $p \in A$. Now, we claim that $A = \{p\}$. On the contrary, suppose that A contains an element q distinct from p. We can take $f \in C_c(\beta X)$ such that $Z(f)$ is a neighborhood of p and $f(q) = 1$ (note, by the assumption, βX is zero-dimensional). Let f_0 be the restriction of f on X. Then the compactness of βX gives f and hence f_0 are bounded, i.e., $f_0 \in C_c^*(X)$. By density of X in βX, we get $f = f_0^\beta$, where f_0^β is the extension of f_0 to βX. Due to Proposition 3.2, we infer that $p \in \mathrm{cl}_{\beta X} Z(f_0)$, since $p \in \mathrm{int}_{\beta X} Z(f)$. Hence, $f_0 \in M_c^p \cap C_c^*(X)$. On the other hand, since $q \notin Z(f)$, we have that $f_0 \notin M^{*q}$. Therefore, $f_0 \in M_c^p \cap C_c^*(X) \setminus (M^{*q} \cap C_c^*(X))$, which contradicts the equation in (3.3). So $A = \{p\}$ and hence $M_c^p \cap C_c^*(X) = M^{*p} \cap C_c^*(X)$. But this also contradicts (3.2). Thus, if X is not pseudocompact, then there exists a closed ideal in $C_c^*(X)$ which is not an intersection of maximal ideals of $C_c^*(X)$, and we are done.

(ii) $\Rightarrow$ (i). Since X is pseudocompact, $C(X) = C^*(X)$ gives $C_c(X) = C_c^*(X)$. Now, it follows from Theorem 3.11.

(ii) $\Leftrightarrow$ (iii). It follows from [5, 7Q(3)]. □

We end the article with some results on e_c-filters on X and e_c-ideals in $C_c^*(X)$, for more details, see [14, Section 2]. Let $p \in \beta X$ and f^β be the extension of $f \in C^*(X)$ to βX. Let us recall that

$$M_c^{*p} = \{f \in C_c^*(X) : f^\beta(p) = 0\} = M^{*p} \cap C_c^*(X), \text{ and } O_c^{*p} = O_c^p \cap C_c^*(X),$$

where

$$M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}, \text{ and } O_c^p = \{f \in C_c(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}.$$

Lemma 3.14. *Let X be strongly zero-dimensional and $p \in \beta X$. Then*
$$E_c(M_c^{*p}) = Z_c[O_c^p] = Z_c[O_c^{*p}] = E_c(O_c^{*p}).$$

Proof. By the hypothesis, $\beta X = \beta_0 X$. To get the result, we show the following chain of inclusions holds.

$$E_c(M_c^{*p}) \subseteq Z_c[O_c^p] \subseteq Z_c[O_c^{*p}] \subseteq E_c(O_c^{*p}) \subseteq E_c(M_c^{*p}). \tag{3.4}$$

To establish the first inclusion, let $E_\varepsilon^c(f) := \{x \in X : |f(x)| \leq \varepsilon\} \in E_c(M_c^{*p})$, where $f \in M_c^{*p}$ and $\varepsilon > 0$. Then $f^\beta(p) = 0$. Notice that $E_\varepsilon^c(f) = Z((|f| - \varepsilon) \vee 0)$

and

$$\mathrm{cl}_{\beta X} Z((|f|-\varepsilon)\vee 0) = \mathrm{cl}_{\beta X} E^c_\varepsilon(f) = \{q\in\beta X : |f^\beta(q)|\le\varepsilon\}. \tag{3.5}$$

Hence, $p\in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z((|f|-\varepsilon)\vee 0)$, in other words, $(|f|-\varepsilon)\vee 0\in O^p_c$.
Here, we are going to show the last equality in (3.5). Let $q\in\beta X$ such that $|f^\beta(q)|\le\varepsilon$. Since X is dense in βX, there exists a net $(x_\lambda)_{\lambda\in\Lambda}\subseteq X$ converging to q and so $f(x_\lambda)=f^\beta(x_\lambda)\to f^\beta(q)$. Moreover, $|f(x_\lambda)|\to|f^\beta(q)|$. Now, let V be an open set in βX containing q. Then for some $\lambda_0\in\Lambda$ and each $\lambda\ge\lambda_0$, we have $x_\lambda\in V$. Furthermore, $|f^\beta(q)|\le\varepsilon$ yields that $|f(x_\lambda)|\le\varepsilon$. Hence, $V\cap E^c_\varepsilon(f)\ne\varnothing$, i.e., $q\in \mathrm{cl}_{\beta X} E^c_\epsilon(f)$.

The second inclusion in (3.4) follows from the fact that $Z(f)=Z(\frac{f}{1+|f|})$, where $f\in O^p_c$ (and thus $\frac{f}{1+|f|}\in O^{*p}_c$). To verify the third inclusion, we let $f\in O^{*p}_c$ and show that $Z(f)\in E_c(O^{*p}_c)$. Since p does not belong to the closed set $F:=\beta X\setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$ and βX is zero-dimensional, by [3, Proposition 4.4], there is some $g\in C_c(\beta X)=C^*_c(\beta X)$ such that $p\in\mathrm{int}_{\beta X} Z(g)$ and $g(F)=\{1\}$. Let g_0 be the restriction of g on X. Then by Proposition 3.2, $p\in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g_0)$. So $g_0\in O^{*p}_c$ and hence $E^c_\varepsilon(g_0)\in E_c(O^{*p}_c)$ for all $\varepsilon>0$. Let $0<\varepsilon<1$ be fixed. Since X is dense in βX, the open set $\{q\in\beta X : |g(q)|<\varepsilon\}$ intersects X nontrivially (since it contains p). Therefore,

$$\begin{aligned}\varnothing\ne\{q\in\beta X : |g(q)|\le\varepsilon\}\cap X &= \{x\in X : |g_0(x)|\le\varepsilon\}\\ &= E^c_\varepsilon(g_0)\subseteq(\beta X\setminus F)\cap X\subseteq Z(f).\end{aligned}$$

Now, since the z_c-filter (in fact, the e_c-filter) $E_c(O^{*p}_c)$ contains $E^c_\varepsilon(g_0)$ and $E^c_\varepsilon(g_0)\subseteq Z(f)$, we infer that $Z(f)\in E_c(O^{*p}_c)$, and we are done.
Finally, the last inclusion in (3.4) follows from the inclusion $O^{*p}_c\subseteq M^{*p}_c$ and the fact that E_c preserves the order, see [14, Corollary 2.1]. □

Theorem 3.15. *Let X be a P-space and $\mathcal{F}$, an e_c-filter on X. Then $\mathcal{F}$ is an e_c-ultrafilter if and only if it is a z_c-ultrafilter.*

Proof. $(\Rightarrow)$: By [5, 4K(7), 6M(1), 16O], every P-space is strongly zero- dimensional (see also [15, Proposition 2.12]). By [5, 7L], we have $O^p=M^p$ for every $p\in\beta X$. Therefore, $O^p_c=O^p\cap C_c(X)=M^p\cap C_c(X)=M^p_c$ (note, $\beta X=\beta_0 X$). Let $\mathcal{F}$ be an e_c-ultrafilter on X. Then $E_c^{-1}(\mathcal{F})$ is a maximal ideal in $C^*_c(X)$, see [14, Proposition 2.14]. Therefore, $E_c^{-1}(\mathcal{F})=M^{*p}_c$ for some $p\in\beta X$. By Lemma 3.14, we have

$$\mathcal{F}=E_c(E_c^{-1}(\mathcal{F}))=E_c(M^{*p}_c)=Z_c[O^p_c]=Z_c[M^p_c].$$

Since M^p_c is a maximal ideal in $C_c(X)$, $\mathcal{F}$ is a z_c-ultrafilter.
$(\Leftarrow)$: Suppose that $\mathcal{F}$ is a z_c-ultrafilter. Then $Z_c^{-1}[\mathcal{F}]$ is a maximal ideal in $C_c(X)$. So $Z_c^{-1}[\mathcal{F}]=M^p_c$ for some $p\in\beta X$. Therefore,

$$\mathcal{F}=Z_c[Z_c^{-1}[\mathcal{F}]]=Z_c[M^p_c]=E_c(M^{*p}_c).$$

Since M^{*p}_c is a maximal ideal in $C^*_c(X)$, $\mathcal{F}$ is an e_c-ultrafilter. □

Corollary 3.16. *For a strongly zero-dimensional space X and $p\in\beta X$, M^{*p}_c is the only e_c-ideal in $C^*_c(X)$ containing O^{*p}_c.*

Proof. Let J be an e_c-ideal in $C_c^*(X)$ which contains O_c^{*p}. Then $E_c^{-1}(E_c(O_c^{*p})) \subseteq E_c^{-1}(E_c(J)) = J$. By Lemma 3.14, $E_c(M_c^{*p}) = E_c(O_c^{*p})$ and therefore

$$M_c^{*p} = E_c^{-1}(E_c(M_c^{*p})) = E_c^{-1}(E_c(O_c^{*p})) \subseteq J.$$

So $M_c^{*p} = J$, and we are through. □

Acknowledgements. *The author is grateful to the referee for useful comments and recommendations towards the improvement of the paper.*

References

[1] F. Azarpanah, O. A. S. Karamzadeh, Z. Keshtkar and A. R. Olfati, On maximal ideals of $C_c(X)$ and uniformity its localizations, Rocky Mountain Journal of Mathematics 48, no. 2 (2018), 345–382.

[2] R. Engelking, General Topology, Sigma Ser. Pure Math., Vol. 6, Heldermann Verlag, Berlin, 1989.

[3] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, On the functionally countable subalgebra of $C(X)$, Rend. Sem. Mat. Univ. Padova 129 (2013), 47–69.

[4] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, $C(X)$ versus its functionally countable subalgebra, The Bulletin of the Iranian Mathematical Society 45, no. 1 (2019), 173–187.

[5] L. Gillman and M. Jerison, Rings of Continuous Functions, Springer-Verlag, Berlin/Heidelberg/New York, 1976.

[6] J. Gómez-Pérez and W. W. McGovern, The m-topology on $C_m(X)$ revisited, Topology Appl. 153 (2006), 1838–1848.

[7] A. Hayati, M. Namdari and M. Paimann, On countably uniform closed spaces, Quaestiones Mathematicae 42, no. 5 (2019), 593–604.

[8] E. Hewitt, Rings of real-valued continuous functions, I, Trans. Amer. Math. Soc. 64 (1948), 45–99.

[9] O. A. S. Karamzadeh and Z. Keshtkar, On c-realcompact spaces, Quaestiones Mathematicae 41, no. 8 (2018), 1135–1167.

[10] O. A. S. Karamzadeh, M. Namdari and S. Soltanpour, On the locally functionally countable subalgebra of $C(X)$, Appl. Gen. Topol. 16, no. 2 (2015), 183–207.

[11] G. D. Maio, L. Hola, D. Holy and R. A. McCoy, Topologies on the space of continuous functions, Topology Appl. 86 (1998), 105–122.

[12] M. Namdari and A. Veisi, Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image, Inter. Math. Forum 7 (2012), 561–571.

[13] J. R. Porter and R. G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer-Verlag, 1988.

[14] A. Veisi, e_c-Filters and e_c-ideals in the functionally countable subalgebra of $C^*(X)$, Appl. Gen. Topol. 20, no. 2 (2019), 395–405.

[15] A. Veisi, On the m_c-topology on the functionally countable subalgebra of $C(X)$, Journal of Algebraic Systems 9, no. 2 (2022), 335–345.

[16] A. Veisi and A. Delbaznasab, Metric spaces related to Abelian groups, Appl. Gen. Topol. 22, no. 1 (2021), 169–181.

