Applied General Topology

Appl. Gen. Topol. 23, no. 1 (2022), 157-167
doi:10.4995/agt.2022.15356

Boyd-Wong contractions in $\mathcal{F}$-metric spaces and applications

Ashis Bera [a], Lakshmi Kanta Dey [a], Sumit Som [b], Hiranmoy Garai [a] and Wutiphol Sintunavarat [c]

[a] Department of Mathematics, National Institute of Technology Durgapur 713209, India (beraashis.math@gmail.com,lakshmikdey@yahoo.co.in,hiran.garai24@gmail.com)
[b] Department of Mathematics, School of Basic and Applied Sciences, Adamas University, Barasat-700126, West Bengal, India (somkakdwip@gmail.com)
[c] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand. (wutiphol@mathstat.sci.tu.ac.th)

Communicated by I. Altun

Abstract

The main aim of this paper is to study the Boyd-Wong type fixed point result in the $\mathcal{F}$-metric context and apply it to obtain some existence and uniqueness criteria of solution(s) to a second order initial value problem and a Caputo fractional differential equation. We substantiate our obtained result by finding a suitable non-trivial example.

2020 MSC: *47H10; 54A20; 54E50.*

Keywords: *$\mathcal{F}$-metric space; fractional differential equation; Boyd-Wong fixed point theorem.*

1. Introduction and Preliminaries

After the invention of metric spaces by Frèchet, many mathematicians have generalized the metric structure by making some changes in the original definition of a metric given by Frèchet. Most of the generalizations are made by making some changes in the triangle inequality of the original definition. Some well-known metrics of such generalizations are b-metric due to Czerwik [10], rectangular metric due to Branciari [7], $b_v(s)$- metric due to Mitrović and

Received 06 April 2021 – Accepted 15 December 2021

Radenović [20], JS-metric due to Jleli and Samet [16] etc. After all such types of generalizations, recently in 2018, Jleli and Samet [15] introduced another such abstraction, which they denominate as $\mathcal{F}$-metric. They defined this metric structure by means of a certain class $\mathcal{F}$, which contains the set of functions $f:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0<s<t\Rightarrow f(s)\leq f(t)$;

($\mathcal{F}_2$) for every sequence $(t_n)\subseteq(0,\infty)$,

$$\lim_{n\to\infty} t_n=0 \Longleftrightarrow \lim_{n\to+\infty} f(t_n)=-\infty.$$

The definition of an $\mathcal{F}$-metric based on the idea of a class $\mathcal{F}$ is as follows:

Definition 1.1 ([15])**.** A function $D:X\times X\to[0,\infty)$, X being a nonempty set, is called an $\mathcal{F}$-metric on X if there exists $(f,\alpha)\in\mathcal{F}\times[0,\infty)$ such that the following conditions hold:

(D1) for $(x,y)\in X\times X$, $\ D(x,y)=0\Longleftrightarrow x=y$;

(D2) $D(x,y)=D(y,x)$ for all $(x,y)\in X\times X$;

(D3) for every $(x,y)\in X\times X$, for each $N\in\mathbb{N}, N\geq 2$, and for every $(u_i)_{i=1}^N\subseteq X$ with $(u_1,u_N)=(x,y)$, we have

$$D(x,y)>0\Longrightarrow f(D(x,y))\leq f\left(\sum_{i=1}^{N-1} D(u_i,u_{i+1})\right)+\alpha.$$

Furthermore, Jleli and Samet [15] also introduced the concepts of $\mathcal{F}$-openness, $\mathcal{F}$-convergence, $\mathcal{F}$-Cauchyness and $\mathcal{F}$-completeness as follows:

Definition 1.2 ([15])**.** A subset $\mathcal{O}$ of an $\mathcal{F}$-metric space (X,D) is said to be $\mathcal{F}$-open if for every $x\in\mathcal{O}$, there is some $r>0$ such that $B_D(x,r)\subseteq\mathcal{O}$, where

$$B_D(x,r)=\{y\in X: D(x,y)<r\}.$$

From the above definition, it is easy to see that the family of all $\mathcal{F}$-open subsets of an $\mathcal{F}$-metric space (X,D) is a topology on X.

Definition 1.3 ([15])**.** Let (X,D) be an $\mathcal{F}$-metric space and (x_n) be a sequence in X.

(1) We say that (x_n) is $\mathcal{F}$-convergent to $x\in X$ if for every $\mathcal{F}$-open subset $\mathcal{O}_x$ of X containing x, there exists some $N\in\mathbb{N}$ such that $x_n\in\mathcal{O}_x$ for all $n\geq N$.

(2) We say that (x_n) is an $\mathcal{F}$-Cauchy sequence if $\lim\limits_{n,m\to\infty} D(x_n,x_m)=0$.

(3) We say that X is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some point in X.

Remark 1.4. If (x_n) is a sequence in an $\mathcal{F}$-metric space (X,D), then (x_n) is $\mathcal{F}$-convergent to $x\in X$ if and only if $\lim\limits_{n\to\infty} D(x_n,x)=0$. In addition, the limit of an $\mathcal{F}$-convergent sequence is unique.

Among all the mathematical theories, which make all such generalized structures interesting and important, fixed point theory is one of these. Throughout the last few decades, many renowned mathematicians have achieved a lot of well-known metric fixed point theorems in these structures (see in [1, 3, 23, 25, 26] and references therein). However, if a distance space is metrizable, then sometimes it may happen that some metric fixed point theorems directly follow from the metrizability result of the spaces. Still, it is essential to note that some well-known fixed point results can't be obtained from the fact that the space is metrizable.

In this paper, we deal with one such structure, $\mathcal{F}$-metric space, which is metrizable, and at the same time, some fixed point theorems like the Banach contraction principle follows from its metric counterpart using metrizability result. However, the general non-linear contractions like Boyd-Wong contraction [8] can't be obtained from the metrizability result. Som et al. proved the above facts in [24]. Thus the study of Boyd-Wong fixed point theorem in the context of $\mathcal{F}$-metric spaces seems to be interesting. We could not establish an analogous result in this setting to that of usual metric spaces, i.e., it is not known whether a mapping satisfying the Boyd-Wong contraction condition in an $\mathcal{F}$-complete $\mathcal{F}$-metric space possesses a fixed point or not. We leave it as an open question in Section 2. So it is challenging work to have Boyd-Wong type result in $\mathcal{F}$-metric spaces assuming some mild additional hypotheses. We succeed in this direction and propose Boyd-Wong type result in $\mathcal{F}$-metric spaces. Moreover, we apply our result in the context of special types of ordinary and Caputo fractional differential equations. We present the above-mentioned results in Section 3.

2. Boyd-Wong type results in the $\mathcal{F}$-metric structure

In the previous section, we have already mentioned that we need some additional hypotheses either on an $\mathcal{F}$-metric space X or on a mapping $T : X \to X$ to get a fixed point of T satisfying the Boyd-Wong contractive condition. In this section, we present such additional hypotheses via the following theorem.

Theorem 2.1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that f is continuous and $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing upper semi-continuous mapping from right such that $\psi(t) < t$ for all $t > 0$. Suppose that $T : X \to X$ is a mapping such that*

$$D(Tx, Ty) \leq \psi(D(x, y)) \tag{2.1}$$

for all $x, y \in X$. Further, assume that

$$f(t) > f(\psi(t)) + \alpha \tag{2.2}$$

for all $t \in (0, \infty)$. Then T has a unique fixed point and $(T^n x)$ converges to that fixed point for all $x \in X$.

Proof. Let x_0 be an arbitrary point in X. Define a sequence (x_n) by setting $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If $x_{n^*-1} = x_{n^*}$ for some $n^* \in \mathbb{N}$, then the proof is done. So, we now assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then from the given condition, we have

$$\begin{aligned} D(x_n, x_{n+1}) &= D(Tx_{n-1}, Tx_n) \\ &\leq \psi\left(D(x_{n-1}, x_n)\right) \\ &< D(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, the sequence $(D(x_n, x_{n+1}))$ is a strictly decreasing sequence and also this sequence is bounded below. Therefore, $\lim_{n\to\infty} D(x_n, x_{n+1})$ exists. Let $\lim_{n\to\infty} D(x_n, x_{n+1}) =: l \geq 0$. If $l > 0$, then by the property of ψ, we get

$$l = \lim_{n\to\infty} D(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \psi(D(x_{n-1}, x_n)) \leq \psi(l) < l,$$

which is a contradiction. Therefore, we obtain

$$l = \lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{2.3}$$

Next, we will show that $(T^n x)$ is an $\mathcal{F}$-Cauchy sequence. We will prove it by contradiction. If possible let $(T^n x)$ be not an $\mathcal{F}$-Cauchy sequence. Then there exist $\varepsilon > 0$ and subsequences (x_{m_k}) and (x_{n_k}) of (x_n) with $m_k > n_k \geq k$ such that

$$D(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{2.4}$$

for each $k \in \mathbb{N}$. We Choose n_k as the smallest number not exceeding m_k for which the equation (2.4) holds. Then we have

$$D(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.5}$$

Now, from (D_3) and (2.5), we have

$$\begin{aligned} f(\varepsilon) &\leq f(D(x_{m_k}, x_{n_k})) \\ &\leq f\left(D(x_{m_k}, x_{m_k+1}) + D(x_{m_k+1}, x_{n_k})\right) + \alpha \\ &\leq f\left(D(x_{m_k}, x_{m_k+1}) + \psi(D(x_{m_k}, x_{n_k-1})\right) + \alpha \\ &\leq f\left(D(x_{m_k}, x_{m_k+1}) + \psi(\varepsilon)\right) + \alpha. \end{aligned}$$

Now, letting $k \to \infty$ in both sides of the above equation, we get

$$f(\varepsilon) \leq f(\psi(\varepsilon)) + \alpha,$$

which is a contradiction. This shows that the sequence (x_n) is an $\mathcal{F}$-Cauchy sequence. Since X is $\mathcal{F}$-complete, there is a point $x^* \in X$ such that

$$\lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.6}$$

Next, we prove that x^* is a fixed point of T. We prove it by contradiction. Let $Tx^* \neq x^*$. Then by (D_3), we have

$$f(D(Tx^*, x^*)) \leq f\Big(D(Tx^*, Tx_n) + D(Tx_n, x^*)\Big) + \alpha \tag{2.7}$$

for all $n \in \mathbb{N}$. If there are two natural numbers n_1 and n_2 such that $D(x_{n_1}, x^*) = 0$ and $D(x_{n_2}, x^*) = 0$, we obtain $x_{n_1} = x^* = x_{n_2}$, which is a contradiction. So we may choose a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $D(x_{n_l}, x^*) \neq 0$ for all $l \in \mathbb{N}$. Using the given condition, (2.7) and $(\mathcal{F}_1)$, we get

$$\begin{aligned} f(D(Tx^*, x^*)) &\leq f\Big(D(Tx^*, Tx_{n_l}) + D(Tx_{n_l}, x^*)\Big) + \alpha \\ &\leq f\Big(\psi(D(x^*, x_{n_l})) + D(x_{n_l+1}, x^*)\Big) + \alpha \\ &\leq f\Big(D(x^*, x_{n_l}) + D(x_{n_l+1}, x^*)\Big) + \alpha. \end{aligned} \tag{2.8}$$

Therefore, by the condition $(\mathcal{F}_2)$ and the equation (2.6), we have

$$\lim_{l\to\infty} f\Big(D(x^*, x_{n_l}) + D(x_{n_l+1}, x^*)\Big) + \alpha = -\infty,$$

which is a contradiction. Hence, $Tx^* = x^*$. For the uniqueness, let T has two fixed points, say x^* and y^* such that $x^* \neq y^*$. Then

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \psi(D(x^*, y^*)) < D(x^*, y^*),$$

which is impossible. Hence, this theorem is proved. □

Next, we provide an example to validate our obtained result.

Example 2.2. Let $X = \mathbb{N}$ and consider the mapping $D : X \times X \to [0, \infty)$, defined by

$$D(x, y) = \begin{cases} |x - y|, & \text{if } x \text{ and } y \text{ both are even or both are odd;} \\ 3|x - y| + 5, & \text{if any one of } x \text{ and } y \text{ is even and the other is odd.} \end{cases}$$

Then (X, D) is an $\mathcal{F}$-metric space with $f(t) = \ln t$ and $\alpha = \ln 3$. Also, X is $\mathcal{F}$-complete. Let us define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 2, & \text{if } x \text{ is even;} \\ 4, & \text{if } x \text{ is odd.} \end{cases}$$

We define a mapping $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{1}{4}t \text{ for all } t \in [0, \infty).$$

Then ψ is a nondecreasing upper semicontinuous from right on $[0, \infty)$ and $\psi(t) < t$ for all $t > 0$. In this case it is clear that f is continuous and satisfies $f(t) > f(\psi(t)) + \alpha$ for all $t \in (0, \infty)$.

Let $x, y \in X$ be arbitrary. If x, y both are even or both are odd, then it is obvious that $D(Tx, Ty) \leq \psi(D(x, y))$. If x is even and y is odd, then $D(Tx, Ty) = 2$ and $D(x, y) = 3|x-y|+5 \geq 2$, which implies that $\psi(D(x, y)) \geq 2$. Hence,

$$D(Tx, Ty) \leq \psi(D(x, y))$$

for all $x, y \in X$. Thus, all conditions of Theorem 2.1 hold. So by this theorem, T has a unique fixed point in X. Note that $z = 2$ is the unique fixed point of T.

Now we give the open question, which we mentioned in the previous section, as follows:

Question 2.3. *Does there exist a fixed point free self mapping T which satisfies the Boyd-Wong contractive condition in an $\mathcal{F}$-complete $\mathcal{F}$-metric space?*

3. Applications of Boyd-Wong contractions

The aims of this section is to give applications of the fixed result for Boyd-Wong contractions in the previous section.

3.1. **Application to a second-order IVP.** In this part, we apply our result to the following initial value problem of the second-order differential equation:

$$\begin{cases} \dfrac{d^2x}{d^2t} + \omega^2 x = g(t, x(t)) \\ \\ x(0) = a, x'(0) = b, \end{cases} \tag{3.1}$$

where $x \in C([0,T], \mathbb{R})$ is an unknown function, $\omega(\neq 0), a, b \in \mathbb{R}$ and g is a continuous function from $[0,T] \times \mathbb{R}^+$ into $\mathbb{R}$.

The above differential equation plays a crucial role in different engineering problems of activation of a spring governed by an exterior force. It can be easily shown that the given differential equation (3.1) is equivalent to the integral equation

$$x(t) = \int_0^t G(t,u) g(u, x(u)) du + a\cos(wt) + b\sin(wt),\ t \in [0,T],$$

where $G(t,u)$ is the Green's function defined by

$$G(x,t) = \frac{1}{\omega}\sin(\omega(t-u))H(t-u),$$

where H is the Heaviside unit function.

We like to study the existence of solution(s) of the differential equation (3.1) (studying the above equivalent integral equation) using our obtained result (Theorem 2.1). For this, we need to consider an underlying $\mathcal{F}$-metric space as (X, D), where X is the set of all real-valued continuous functions defined on $[0,T]$, and D is defined by

$$D(x,y) = \|x-y\|_\infty = \max_{t\in[0,T]} |x(t) - y(t)| \tag{3.2}$$

for all $x, y \in X$. Then clearly (X, D) is an $\mathcal{F}$-metric space with $f(t) = \ln t$ and $\alpha = 0$.

Now we have the following theorem.

Theorem 3.1. *Consider the following differential equation (3.1) under the assumptions:*

1. *g is a continuous function;*
2. *there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that ψ is upper semi-continuous from right and $\psi(t) < t$ for all $t > 0$, and*
$$|g(t,r) - g(t,s)| \leq \omega^2 \psi(|r-s|)$$
for all $t \in [0,T]$ and $r, s \in \mathbb{R}$.

Then the differential equation (3.1) *has a unique solution in* X*.*

Proof. Consider the $\mathcal{F}$-metric space (X, D) as in (3.2). Then by assumption (2), it is clear that $f(t) > f(\psi(t)) + \alpha$ for all $t \in (0, \infty)$. Let us now define a mapping $T : X \to X$ for each $x \in X$ by

$$(Tx)(t) = a\cos(wt) + b\sin(wt) + \int_0^t G(t,u)g(u,x(u))du$$

for all $t \in [0, T]$. Then the existence of fixed point(s) of the mapping T is equivalent to the existence of the solution(s) of the IVP (3.1).

Now, for each $x, y \in X$ and each $t \in [0, T]$, by applying the conditions (1) and (2), we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= |a\cos(wt) + b\sin(wt) + \int_0^t G(t,u)g(u,x(u))du \\
&\quad -a\cos(wt) - b\sin(wt) - \int_0^t G(t,u)g(u,y(u))du| \\
&= \left|\int_0^t G(t,u)g(u,x(u))du - \int_0^t G(t,u)g(u,y(u))du\right| \\
&\le \int_0^t G(t,u)|g(u,x(u)) - g(u,y(u))|du \\
&\le \int_0^t G(t,u)\omega^2\psi(|x(u) - y(u)|)du \\
&\le \omega^2\psi(\|x - y\|_\infty) \sup_{t\in[0,T]} \int_0^t G(t,u)du \\
&= \omega^2\psi(\|x(u) - y(u)\|_\infty) \sup_{t\in[0,T]} \int_0^t \frac{1}{\omega}\sin(\omega(t-u))du \\
&\le \psi(\|x - y\|_\infty).
\end{aligned}$$

Therefore, we get

$$\|(Tx)(t) - (Ty)(t)\|_\infty \le \psi(\|x - y\|_\infty).$$

This yields that

$$D(Tx, Ty) \le \psi(D(x, y)).$$

By Theorem 2.1, T has a unique fixed point, say x. Therefore, x is the unique solution of the second-order differential equation (3.1) in X. □

3.2. **Application to Caputo fractional differential equations.** In this part, we discuss on Caputo fractional differential equation and apply our result to this equation. Before going further, we first recall some basic definitions of fractional derivatives. We first start with the definition of fractional integral operators as follows.

Definition 3.2.

- The fractional integral operator of order $q \in (0, \infty)$ (denoted by I_0^q) is defined as follows:

$$I_0^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(u)}{(t-u)^{1-q}} du. \tag{3.3}$$

- The Riemann-Liouville fractional derivative of order $q > 0$ is defined as follows:

$$D_0^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t \frac{f(u)}{(t-u)^{q-m+1}} du, & \text{if } m-1 < q < m; \\ \frac{d^m f(t)}{dt^m}, & \text{if } q = m, \end{cases}$$

  where m is a positive integer and $m - 1 < q < m$.
- The Caputo fractional derivative of order $q > 0$ with $m - 1 < q < m$ is defined as follows:

$$D_0^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t \frac{f^(m)(u)}{(t-u)^{q-m+1}} du, & \text{if } m-1 < q < m; \\ \frac{d^m f(t)}{dt^m}, & \text{if } q = m. \end{cases}$$

Now, we recall the following lemma due to [18], which will be needed for the application.

Lemma 3.3 ([18]). *For $q > 0$, the homogeneous fractional differential equation $D_0^q g(t) = 0$ has a solution*

$$g(t) = c_1 + c_2 t + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3, \ldots, n$ and $n = [q] + 1$.

For more informations concerning the fractional calculus, one can see [12, 13, 14, 22] and the references therein.

Now, we consider the boundary value problem (BVP) as follows:

$$\begin{cases} {}^cD_0^q x(t) - g(t, x(t)) = 0, 0 \le t \le 1, 1 < q \le 2 \\ x'(0) = 0, x(0) - \beta x(1) = \int_0^r h(u, x(u)) du, r \in (0,1), \beta \neq 1, \end{cases} \tag{3.4}$$

where $x \in C([0,1], \mathbb{R})$ is an unknown function and $g, h : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ are continuous functions.

Using Lemma 3.3 and the above BVP, we get

$$x(t) = I_0^q g(t, x(t)) + c_1 + c_2 t.$$

Using the boundary conditions, we have

$$\begin{aligned} x(t) &= I_0^q g(t,x(t)) + \frac{\beta}{1-\beta} I_0^q g(1,x(1)) + \frac{1}{1-\beta}\int_0^r h(u,x(u))du \\ &= \frac{1}{\Gamma(q)}\int_0^t \frac{g(u,x(u))}{(t-u)^{1-q}}du + \frac{\beta}{1-\beta}\frac{1}{\Gamma(q)}\int_0^1 \frac{g(u,x(u))}{(1-u)^{1-q}}du + \frac{1}{1-\beta}\int_0^r h(u,x(u))du \\ &= \frac{1}{\Gamma(q)}\left[\int_0^t \frac{g(u,x(u))}{(t-u)^{1-q}}du + \frac{\beta}{1-\beta}\int_0^1 \frac{g(u,x(u))}{(1-u)^{1-q}}du\right] + \frac{1}{1-\beta}\int_0^r h(u,x(u))du \\ &= \frac{1}{\Gamma(q)}\int_0^1 G(t,u)g(u,x(u))du + \frac{1}{1-\beta}\int_0^r h(u,x(u))du, \end{aligned}$$

where

$$G(t,u) = \begin{cases} \frac{1}{(t-u)^{1-q}} + \frac{\beta}{1-\beta}\frac{1}{(1-u)^{1-q}}, & 0 \le u \le t \le 1; \\ \frac{\beta}{1-\beta}\frac{1}{(1-u)^{1-q}}, & 0 \le t \le u \le 1. \end{cases}$$

Thus, we see that the BVP (3.4) is equivalent to the integral equation

$$x(t) = \frac{1}{\Gamma(q)}\int_0^1 G(t,u)g(u,x(u))du + \frac{1}{1-\beta}\int_0^r h(u,x(u))du. \tag{3.5}$$

Now we are in a position to present a result concerning the existence of a solution of the above BVP.

Theorem 3.4. *Consider the BVP (3.4) under the following assumptions:*

(1) *g, h are continuous functions;*
(2) *there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that ψ is upper semi-continuous mapping from right and $\psi(t) < t$ for all $t > 0$*

$$\max\{|g(t,a) - g(t,b)|, |h(t,a) - h(t,b)|\} \le K\psi(|a-b|)$$

for all $t \in [0,1]$ and $a, b \in \mathbb{R}$, where $K := \frac{\Gamma(q)-\beta\Gamma(q)}{1+\Gamma(q)-\beta}$.

Then the equation (3.4) has a unique solution in X.

Proof. Consider the $\mathcal{F}$-metric space (X, D) considered in (3.2). Then by assumption (2), it is clear that $f(t) > f(\psi(t)) + \alpha$ for all $t \in (0,\infty)$. Note that α is zero in this case. Let us now define a mapping $T : X \to X$ for each $x \in X$ by

$$(Tx)(t) = \frac{1}{\Gamma(q)}\int_0^1 G(t,u)g(u,x(u))du + \frac{1}{1-\beta}\int_0^r h(u,x(u))du$$

for all $t \in [0,1]$. Then a point $x \in X$ is a solution of the BVP (3.4) if and only if x is a fixed point of T.

Let $x, y \in C[0,1]$ and $t \in [0,1]$. Then we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \Big| \frac{1}{\Gamma(q)} \int_0^1 G(t,u) g(u, y(u)) du + \frac{1}{1-\beta} \int_0^r h(u, y(u)) du \\
&\quad - \frac{1}{\Gamma(q)} \int_0^1 G(t,u) g(u, y(u)) du + \frac{1}{1-\beta} \int_0^r h(u, y(u)) du \Big| \\
&= \frac{1}{\Gamma(q)} \int_0^1 G(t,u) |g(u, x(u)) - g(u, y(u))| du \\
&\quad + \frac{1}{1-\beta} \int_0^r |h(u, x(u)) - h(u, y(u))| du \\
&\leq \frac{1}{\Gamma(q)} \int_0^1 G(t,u) K\psi(|x-y|) du \\
&\quad + \frac{1}{1-\beta} \int_0^r K\psi(|x-y|) du \\
&\leq K\psi(\|x-y\|_\infty) \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(q)} \int_0^1 G(t,u) du + \frac{r}{1-\beta} \right\} \\
&\leq K\psi(\|x-y\|_\infty) \left(\frac{1}{\Gamma(q)} + \frac{1}{1-\beta} \right) \\
&\leq \psi(\|x-y\|_\infty)
\end{aligned}$$

and so

$$\|Tx(t) - Ty(t)\|_\infty \leq \psi(\|x-y\|_\infty).$$

Then

$$D(Tx, Ty) \leq \psi(D(x,y))$$

for all $x, y \in X$. Hence, from Theorem 2.1, T has a unique fixed point, and hence the Caputo fractional differential equation (3.4) has a unique solution. □

Acknowledgements. *The Research is funded by the Ministry of Human Resource and Development, Government of India and by the Council of Scientific and Industrial Research (CSIR), Government of India under the Grant Number:* 25(0285)/18/*EMR-II. This project is funded by National Research Council of Thailand (NRCT) N41A640092.*

References

[1] I. Altun, M. Aslantas and H. Sahin, KW-type nonlinear contractions and their best proximity points, Numerical Functional Analysis and Optimization 42, no. 8 (2021), 935–954.

[2] T. V. An, N. V. Dung and V. T. L. Hang, A new approach to fixed point theorems on G-metric spaces, Topology Appl. 160, no. 12 (2013), 1486–1493.

[3] M. Aslantas, Best proximity point theorems for proximal b-cyclic contractions on b-metric spaces, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 70, no. 1 (2021), 483–496.

[4] H. Aydi, M. Jleli and B. Samet, On the absence of global solution for some q-difference inequalities, Adv. Difference Equ. 2019, no. 1 (2019), 1–9.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181.

[6] A. Bera, H. Garai, B. Damjanovic and A. Chanda, Some interesting results on $\mathcal{F}$-metric spaces, Filomat 33, no. 10 (2019), 3257–3268.

[7] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57 (2000), 31–37.

[8] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20, no. 2 (1969), 458–464.

[9] A. Chanda, S. Mondal, L. K. Dey and S. Karmakar, C^*-algebra-valued contractive mappings with its application to integral equations, Indian J. Math. 59, no. 1 (2017), 107–124.

[10] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Univ. Osstrav. 1, no. 1 (1993), 5–11.

[11] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena 46 (1988), 263–276.

[12] K. Diethelm, N. J. Ford and A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynam. 29 (2002), 3–22.

[13] M. F. Elettreby, A. A. Al-Raezah and T. Nabil, Fractional order model of two-prey one-predator system, Math. Probl. Eng. 12 (2017), 1–13.

[14] H. Garai, L. K. Dey and A. Chanda, Positive solutions to a fractional thermostat model in Banach spaces via fixed point results, J. Fixed Point Theory Appl. 20, no. 3 (2018), 1–24.

[15] M. Jleli and B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl. 20, no. 3 (2018), 1–20.

[16] M. Jleli and B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl. 2015, no. 1 (2015), 1–14.

[17] M. A. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal. 73, no. 9 (2010), 3123-3129.

[18] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier (2006).

[19] Z. D. Mitrović, A note on the results of Suzuki, Miculescu and Mihail, J. Fixed Point Theory Appl. 21, no. 1 (2019), 1–4.

[20] Z. D. Mitrović and S. Radenović, The Banach and Reich contractions in $b_v(s)$-metric spaces. J. Fixed Point Theory Appl. 19, no. 4 (2017), 3087–3095.

[21] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7, no. 2 (2006), 289–297.

[22] H. Sahin, Existence and uniqueness results for nonlinear fractional differential equations via new Q-function, Adv. Oper. Theory 7, no. 1 (2022), 1–21.

[23] K. Sawangsup and W. Sintunavarat, On solving nonlinear matrix equations in terms of b-simulation functions in b-metric spaces with numerical solutions, Comput. Appl. Math. 37 (2018), 5829–5843.

[24] S. Som, A. Bera and L. K. Dey, Some remarks on the metrizability of $\mathcal{F}$-metric spaces, J. Fixed Point Theory Appl. 22, no. 1 (2020), 1–7.

[25] O. Yamaod, W. Sintunavarat, On new homotopy results via a generalize extended simulation function in b-metric spaces, J. Nonlinear Convex Anal. 20, no. 10 (2019), 2225–2238.

[26] O. Yamaod and W. Sintunavarat, Discussion of hybrid JS-contractions in b-metric spaces with applications to the existence of solution for nonlinear integral equations, Fixed Point Theory 22, no. 2 (2021), 899–912.

