APPLIED GENERAL TOPOLOGY

Best proximity point (pair) results via MNC in Busemann convex metric spaces

This paper is dedicated in the memory of Prof. Hans-Peter Künzi

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ABSTRACT

In this paper, we present a new class of cyclic (noncyclic) α-ψ and β-ψ condensing operators and survey the existence of best proximity points (pairs) as well as coupled best proximity points (pairs) in the setting of reflexive Busemann convex spaces. Then an application of the main existence result to study the existence of an optimal solution for a system of differential equations is demonstrated.

1. INTRODUCTION

In the present paper, we mainly focus on the study of best proximity points for certain classes of mappings $T : A \cup B \to A \cup B$ for which $T(A) \subseteq B$ and $T(B) \subseteq A$. Such mappings are called *cyclic mappings*. Likewise, if $T(A) \subseteq A$ and $T(B) \subseteq B$, then T is said to be a *noncyclic mapping*. For the noncyclic case, the point $(p, q) \in A \times B$ is said to be a *best proximity pair* for the mapping T provided that p and q are two fixed points of T which estimates the distance between two sets A and B, that is, $p = Tp$, $q = Tq$ and $d(p, q) = \text{dist}(A, B)$.

The first existence theorem of best proximity points (pairs) was established in [7] for cyclic (noncyclic) relatively nonexpansive mappings. We recall that the mapping $T : A\cup B \to A\cup B$ is called relatively nonexpansive, if $d(Tx,Ty) \leq d(x,y)$ for all $(x,y) \in A \times B$.

In [11] the main result of [7] was extended from Banach spaces to reflexive Busemann convex spaces.

Recently Gabeleh and Markin introduced a class of *cyclic (noncyclic) condensing operators* by using a notion of measure of noncompactness and then studied the existence of best proximity points (pairs) for such mappings in (strictly convex) Banach spaces (see Theorem 3.4 and Theorem 4.3 of [12]).

Subsequently Gabeleh and Künzi generalized the main conclusions of [12] from two points of view. One by considering a class of cyclic (noncyclic) condensing operators of integral type and another one by generalizing (strictly convex) Banach spaces to reflexive Busemann convex spaces (see Theorem 3.9 and Theorem 3.11 of [13]). We refer to articles [14, 18] in this direction. For more information about the existence of best proximity points for various classes of non-self mappings, one can see [17, 20, 21, 22].

In the current article, we introduce two new classes of cyclic (noncyclic) condensing operators, called $\alpha - \psi$ and $\beta - \psi$ condensing operators which are properly contain the class of cyclic (noncyclic) condensing operators introduced by Gabeleh and Markin in [12]. We establish the existence of best proximity points (pairs) for such mappings in the framework of reflexive Busemann convex spaces and then apply our conclusions to present a coupled best proximity point theorem by the notion of measure of noncompactness. Finally, the existence of an optimal solution for a system of second order differential equations by using the existence result of best proximity points for the considered extension of cyclic condensing operators is studied.

2. Preliminaries

In this section, we compile the main notions and notations we will work with along this paper.

2.1. **Geodesic spaces.** Let (X, d) be a metric space. By $\mathcal{B}(x_0; r)$ we denote the closed ball in the space X centered at $x_0 \in X$ with radius $r > 0$. Consider A and B two nonempty subsets of X. Define

$$\delta_x(A) = \sup\{d(x,y) : y \in A\} \text{ for all } x \in X,$$
$$\delta(A, B) = \sup\{\delta_x(B) : x \in A\},$$
$$\text{diam}(A) = \delta(A, A).$$

Throughout this article, we say that the pair (A, B) satisfies a property if both A and B satisfy that property. For example, (A, B) is closed if and only if both A and B are closed. Likewise, $(A, B) \subseteq (C, D)$ if and only if $A \subseteq C$ and $B \subseteq D$. The *proximal pair* of (A, B) is defined as

$$A_0 = \{x \in A : d(x, y') = \text{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B \colon d(x', y) = \text{dist}(A, B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty, bonded, closed and convex in a reflexive Banach space X, then (A_0, B_0) is a nonempty, bounded, closed and convex pair in X. We say that (A, B) is *proximinal* provided that $A = A_0$ and $B = B_0$.

In this paper we will mainly work with geodesic spaces.

Let $x, y \in X$. A geodesic path from x to y is a mapping $c \colon [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image of the mapping c forms a geodesic segment which joins x and y and will be denoted by $[x, y]$ whenever it is unique.

(X, d) is said to be a *(uniquely) geodesic space* if every two points in X can be joined by a (unique) geodesic path. A point $p \in X$ belongs to a geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, p) = td(x, y)$ and $d(y, p) = (1 - t)d(x, y)$ and for convenience we write $p = (1 - t)x \oplus ty$. In this situation $p = c(tl)$, where $c \colon [0, l] \to X$ is a geodesic path from x to y. A subset E of a geodesic space X is *convex* if it contains any geodesic segment joining each two points in E, that is, for any $x, y \in E$ we have $[x, y] \subseteq E$.

The (closed) convex hull of a subset E of a geodesic space X is the smallest (closed) convex set containing the set E which is denoted by $\text{con}(E)$ and $\overline{\text{con}}(E)$, respectively.

Lemma 2.1 ([5]). *Let E be a nonempty subset of a geodesic space X. Let $\mathcal{G}_1(E)$ denote the union of all geodesic segments with endpoints in E. Recursively, for $n \geq 2$ put $\mathcal{G}_n(E) = \mathcal{G}_1(\mathcal{G}_{n-1}(E))$. Then*

$$\text{con}(E) = \bigcup_{n=1}^{\infty} \mathcal{G}_n(E).$$

It is remarkable to note that in a Busemann convex space X the closure of $\text{con}(E)$ is convex and so, coincides with $\overline{\text{con}}(A)$ (see [10]).

A metric $d \colon X \times X \to \mathbb{R}$ of a uniquely geodesic space (X, d) is called convex if for any $x \in X$ and every geodesic path $c \colon [0, l] \to X$ we have

$$d(x, c(tl)) \leq (1 - t)d(x, c(0)) + td(x, c(l)), \quad \forall t \in [0, 1].$$

Furthermore, (X, d) is said to be *Busemann convex* ([6]) if for any two geodesics $c_1 \colon [0, l_1] \to X$ and $c_2 \colon [0, l_2] \to X$ one has

$$d\big(c_1(tl_1), c_2(tl_2)\big) \leq (1 - t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \quad \forall t \in [0, 1].$$

Equivalently, a geodesic metric space (X, d) is convex in the sense of Busemann if

$$d\big((1 - t)x \oplus ty, (1 - t)z \oplus tw\big) \leq (1 - t)d(x, z) + td(y, w),$$

for all $x, y, z, w \in X$ and $t \in [0, 1]$. It is well-known that Busemann convex spaces are uniquely geodesic and with convex metric. A very important class of Busemann convex spaces are CAT(0) spaces, that is, metric spaces of non-positive curvature in the sense of Gromov (see [5, 6] for a detailed discussion on CAT(0) spaces).

In the sequel we say that a geodesic metric space (X, d) is *strictly convex* ([3]) if for every $r > 0$, a, x and $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $x \neq y$, it is the case that $d(a, p) < r$, where $p \in [x, y] - \{x, y\}$. We mention that Busemann convex spaces are strictly convex with convex metric ([9]).

In the next section we will also work with reflexive geodesic spaces which is a generalization of the notion of reflexivity from Banach to geodesic spaces. A geodesic space X is said to be *reflexive* if for every decreasing chain $\{C_\alpha\} \subseteq X$ with $\alpha \in I$ such that C_α is nonempty, bounded, closed and convex for all $\alpha \in I$ we have that $\bigcap_{\alpha \in I} C_\alpha \neq \varnothing$. It was announced in [8] that a reflexive and Busemann convex space is complete. The following property of reflexive and Busemann convex spaces plays an important role in our coming discussions.

Proposition 2.2 ([11, Proposition 3.1]). *If (A, B) is a nonempty, closed and convex pair in a reflexive and Busemann convex space X such that B is bounded, then (A_0, B_0) is nonempty, bounded, closed and convex.*

We are now ready to state a main existence result of [11].

Theorem 2.3 ([11, Theorem 3.3, Theorem 3.4]). *Let (A, B) be a nonempty, compact and convex pair in a Busemann convex space X. Then every cyclic (noncyclic) relatively nonexpansive mapping defined on $A \cup B$ has a best proximity point (pair).*

We mention that the proof of Theorem 2.3 is based on the fact that any compact and convex pair in a geodesic space with convex metric has a geometric notion, called *proximal normal structure* (see Proposition 3.10 of [11]).
To state an extended version of Theorem 2.3 we recall the following concept.

Definition 2.4 ([12]). *Let (A, B) be a nonempty and bounded pair in a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) mapping. We say that T is compact whenever the pair $(\overline{T(A)}, \overline{T(B)})$ is compact.*

Theorem 2.5 ([13, Theorem 3.2 and Theorem 3.4]). *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that $T : A \cup B \to A \cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping. If T is compact, then T has a best proximity point (pair).*

We finish this section by stating the following auxiliary lemmas.

Lemma 2.6 ([13, Lemma 3.7]). *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d). Assume that $(E, F) \subseteq (A, B)$ is a nonempty and proximinal pair with $\mathrm{dist}(E, F) = \mathrm{dist}(A, B)$. Then the pair $(\overline{\mathrm{con}}(E), \overline{\mathrm{con}}(F))$ is proximinal with*

$$\mathrm{dist}(\overline{\mathrm{con}}(E), \overline{\mathrm{con}}(F)) = \mathrm{dist}(A, B).$$

Lemma 2.7 ([13, Lemma 3.8]). *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded.*

Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Suppose $\mathcal{U}_0 := A_0$ and $\mathcal{V}_0 := B_0$ and for all $n \in \mathbb{N}$ define

$$\mathcal{U}_n = \overline{\text{con}}(T(\mathcal{U}_{n-1})), \quad \mathcal{V}_n = \overline{\text{con}}(T(\mathcal{V}_{n-1})).$$

Then $\{(\mathcal{U}_{2n}, \mathcal{V}_{2n})\}$ is a descending sequence of nonempty, bounded, closed, convex and proximinal pairs which are T-invariant, that is, T is cyclic on $\mathcal{U}_{2n} \cup \mathcal{V}_{2n}$ for all $n \in \mathbb{N}$. Moreover,

$$\mathcal{U}_{n+2} \subseteq \mathcal{V}_{n+1} \subseteq \mathcal{U}_n \subseteq \mathcal{V}_{n-1}, \quad \text{dist}(\mathcal{U}_{2n}, \mathcal{V}_{2n}) = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

2.2. **Measure of noncompactness on metric spaces.** In this section we recall some basic notions of measure of noncompactness which will be used in introducing new classes of condensing operators. Let (X, d) be a metric space. By Σ we denote the collection of all bounded subsets of X.

Definition 2.8. A function $\mu : \Sigma \to [0, \infty)$ is called a measure of noncompactness (MNC for brief) if it satisfies the following conditions:
(i) $\mu(A) = 0$ iff A is relatively compact,
(ii) $\mu(A) = \mu(\overline{A})$ for all $A \in \Sigma$,
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$.

We can survey the following useful properties of an MNC, easily;
(1) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$,
(2) $\mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$,
(3) If A is a finite set, then $\mu(A) = 0$,
(4) If $\{A_n\}$ is a decreasing sequence of nonempty, bounded and closed subsets of X such that $\lim_{n \to \infty} \mu(A_n) = 0$, then $A_\infty := \cap_{n \geq 1} A_n$ is nonempty and compact.
In the case that (X, d) is a geodesic metric space, we say μ is *invariant w.r.t. the convex hull*, whenever

$$\mu(\text{con}(A)) = \mu(A), \quad \forall A \in \Sigma.$$

From now on, we assume that the considered MNC is invariant w.r.t. convex hulls.
Two well-known MNSs are due to Kuratowski and Hausdorff which are denoted by α and χ, respectively and define as below:

$\alpha(A) = \inf\{\varepsilon > 0 \mid A \subseteq \cup_{j=1}^n E_j : \ E_j \in \Sigma, \ \text{diam}(E_j) \leq \varepsilon, \ \forall \ 1 \leq j \leq n < \infty\},$

$\chi(A) = \inf\{\varepsilon > 0 \mid A \subseteq \cup_{j=1}^n \mathcal{B}(x_j; r_j) : \ x_j \in X, \ r_j \leq \varepsilon, \ \forall \ 1 \leq j \leq n < \infty\},$

for all $A \in \Sigma$ (see [2] for more details).

3. Best proximity points (pairs)

We begin our investigations by recalling the following class of functions which will be considered as control functions to introduce a new class of cyclic (non-cyclic) condensing operators.

Let Ψ denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_{n \to \infty} \psi^n(t) = 0$, for every $t > 0$, where ψ^n denotes n^{th} iteration of ψ. It is evident that for every nondecreasing $\psi \in \Psi$, for each $t > 0$, $\psi(t) < t$.

Let $\alpha : X \times X \to [0, +\infty)$ be a mapping. $T : X \to X$ is called α-*admissible* ([23]) if for every $x, y \in X$, $\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

3.1. **Results for α-ψ-condensing operator.** Let us introduce the following new class of cyclic (noncyclic) mappings.

Definition 3.1. Let (A, B) be a nonempty and convex pair in a Banach space X and μ an MNC on X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic (noncyclic) $\alpha - \psi$-condensing operator if for any nonempty, bounded, closed, convex, proximinal and T-invariant pair $(K_1, K_2) \subseteq (A, B)$ with $\text{dist}(K_1, K_2) = \text{dist}(A, B)$ and $x \in X$, we have

$$\alpha(x, Tx)\mu(T(K_1) \cup T(K_2)) \leq \psi(\mu(K_1 \cup K_2))$$

where $\psi \in \Psi$ is a nondecreasing function and $\alpha : X \times X \to [0, +\infty)$.

Example 3.2. Let $X = BC(\mathbb{R}_+)$ be a Banach space and conisder $A = \{x \in BC(\mathbb{R}_+) : \|x\| \leq 1\}$ and $B = \{x \in BC(\mathbb{R}_+) : \|x\| > 1\}$. Let us define $T : A \cup B \to A \cup B$ as

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in A; \\[2mm] 2x - 2, & \text{if } x \in B. \end{cases}$$

Clearly T is noncyclic mapping. Let $K_1 = A$ and $K_2 = B$. Then for $\mu(E) = diam(E)$, we have

$$\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\}$$

$$= \max\left\{ \sup\left\{\|Tx(t) - Ty(t)\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{\|Tx(t) - Ty(t)\| : x(t), y(t) \in K_2\right\} \right\}$$

$$= \max\left\{ \sup\left\{\|\dfrac{x(t)}{2} - \dfrac{y(t)}{2}\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{\|2x(t) - 2 - 2y(t) + 2\| : x(t), y(t) \in K_2\right\} \right\}$$

$$= \max\left\{ \sup\left\{\dfrac{1}{2}\|x(t) - y(t)\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{2\|x(t) - y(t)\| : x(t), y(t) \in K_2\right\} \right\}$$

$$= 2\mu(T(K_2)) = 2\mu(K_1 \cup K_2)$$

which shows that T is not noncyclic condensing mapping. But, T is noncyclic α-ψ-condensing, indeed if we define $\alpha : X \times X \to X$ as

$$\alpha(x, Tx) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\alpha(x, Tx)\mu(T(K_1) \cup T(K_2)) \leq \psi(\mu(K_1 \cup K_2))$, where $\psi(t) = \frac{t}{2}, t \geq 0$.

Now we are ready to state and prove our first existence result for a best proximity point.

Theorem 3.3. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an MNC on X and $T : A \cup B \to$*

$A \cup B$ is a cyclic relatively nonexpansive α-admissible α-ψ-condensing operator. Then T has a best proximity point if there exists a point x_0 in X such that $\alpha(x_0, Tx_0) \geq 1$.

Proof. Let us define two sequences:

(i) $\{(\mathcal{U}_{2n}, \mathcal{V}_{2n})\}$ consisting of nonempty, bounded, closed, convex, proximinal and T-invariant pairs as defined in Lemma 2.7.

(ii) Let $x_0 \in X$ and $\{x_n\}$ such that $x_n = Tx_{n-1}$ for every $n \geq 1$.

Since $\alpha(x_0, Tx_0) \geq 1$, α-admissibility of T implies that $\alpha(x_1, x_2) \geq 1$. Applying recursion, we get $\alpha(x_n, x_{n+1}) \geq 1$, for every $n \geq 0$.

We note that if for some $k \in \mathbb{N}$ we have $\max\{\mu(\mathcal{U}_{2k}), \mu(\mathcal{V}_{2k})\} = 0$, then $T : \mathcal{U}_{2k} \cup \mathcal{V}_{2k} \to \mathcal{U}_{2k} \cup \mathcal{V}_{2k}$ is a compact and cyclic relatively nonexpansive mapping and so the result follows from Theorem 2.5. Assume that $\max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} > 0$ for all $n \in \mathbb{N}$. Since T is a $\alpha - \psi$-condensing operator,

$$
\begin{aligned}
\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) &\leq \alpha(x_n, x_{n+1})\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \\
&= \alpha(x_n, Tx_n)\mu(\overline{\mathrm{con}}(T(\mathcal{U}_{2n})) \cup \overline{\mathrm{con}}(T(\mathcal{V}_{2n}))) \\
&\leq \alpha(x_n, Tx_n)\mu(T(\mathcal{U}_{2n}) \cup T(\mathcal{V}_{2n})) \\
&\leq \psi(\mu(\mathcal{U}_{2n}) \cup \mu(\mathcal{V}_{2n})) \\
&< \mu(\mathcal{U}_{2n}) \cup \mu(\mathcal{V}_{2n}).
\end{aligned}
$$

Repeating this pattern we get the following inequality

$$
\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \leq \psi^n(\mu(\mathcal{U}_0 \cup \mathcal{V}_0)),
$$

which yields us

$$
\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \to 0, \text{ as } n \to \infty.
$$

That is,

$$
\lim_{n \to \infty} \max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} = 0.
$$

Let

$$
\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_{2n}, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_{2n}.
$$

By the properties of the measure of noncompactness, the pair $(\mathcal{U}_\infty, \mathcal{V}_\infty) \subseteq (A, B)$ is nonempty, compact and convex with $\mathrm{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \mathrm{dist}(A, B)$. On the other hand, $T : \mathcal{U}_\infty \cup \mathcal{V}_\infty \to \mathcal{U}_\infty \cup \mathcal{V}_\infty$ is a cyclic relatively nonexpansive mapping. Now Theorem 2.3 ensures that T has a best proximity point. $\qquad\square$

We now state the noncyclic version of Theorem 3.3 in order to study the existence of best proximity pairs.

Theorem 3.4. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an MNC on X and $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive α-admissible and α-ψ-condensing operator. Then T admits a best proximity pair if there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Proof. Consider the sequence of pairs $(\mathcal{U}_n, \mathcal{V}_n) \subseteq (A, B)$ for $n \in \mathbb{N} \cup \{0\}$ as in Lemma 2.7. It follows from the proof of Theorem 3.11 of [13] that $\{(\mathcal{U}_n, \mathcal{V}_n)\}$ is a descending sequence of nonempty, bounded, closed, convex and proximinal pairs with $\operatorname{dist}(\mathcal{U}_n, \mathcal{V}_n) = \operatorname{dist}(A, B)$ which are T-invariant. If there exists some $k \in \mathbb{N}$ for which $\max\{\mu(\mathcal{U}_k), \mu(\mathcal{V}_k)\} = 0$, then from Theorem 2.5 the result will be concluded.

Now let $x_0 \in X$. By α-admissibility of T and $\alpha(x_0, Tx_0) \geq 1$, we get $\alpha(x_n, x_{n+1}) \geq 1$. Suppose that $\max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By a discussion as in the proof of Theorem 3.3, we can see that $\lim_{n \to \infty} \max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} = 0$. Now, if we set

$$\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_n, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_n,$$

then $(\mathcal{U}_\infty, \mathcal{V}_\infty)$ is a nonempty, compact and convex pair with $\operatorname{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \operatorname{dist}(A, B)$. Hence, the existence of a best proximity pair for the mapping T is deduced from Theorem 2.3. $\qquad\square$

3.2. **Results for β-ψ-condensing operator.** Now we recall the concept of β-admissibility which is introduced by Rehman et al. in [19], with a slight modification as follows.

Definition 3.5. Let $\beta : 2^X \to [0, +\infty)$. A mapping $T : X \to X$ is called β-admissible if for every $M_1, M_2 \in 2^X$, we have $\beta(M_1 \cup M_2) \geq 1 \implies \beta(\overline{con}(T(M_1) \cup T(M_2))) \geq 1$.

We now introduce a new class of cyclic (noncyclic) operators in the following definition.

Definition 3.6. Let (A, B) be a nonempty and convex pair in a Banach space X and μ an MNC on X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic (noncyclic) β-ψ-condensing operator if for any nonempty, bounded, closed, convex, proximinal and T-invariant pair $(K_1, K_2) \subseteq (A, B)$ with $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$, we have

$$\beta(K_1 \cup K_2)\mu(T(K_1) \cup T(K_2)) \leq \psi(\mu(K_1 \cup K_2))$$

where $\psi \in \Psi$ is a nondecreasing function and $\beta : 2^X \to [0, +\infty)$.

Example 3.7. Let $X = BC(\mathbb{R}_+)$ be a Banach space and conisder $A = \{x \in BC(\mathbb{R}_+) : ||x|| \leq 1\}$ and $B = \{x \in BC(\mathbb{R}_+) : ||x|| > 1\}$. Let us define $T : A \cup B \to A \cup B$ as

$$Tx = \begin{cases} \dfrac{x}{3}, & \text{if } x \in A; \\[2ex] 2x - \frac{3}{2}, & \text{if } x \in B. \end{cases}$$

Clearly T is noncyclic mapping. Let $K_1 = A$ and $K_2 = B$. Then for $\mu(E) = diam(E)$, we have

$$\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\}$$

$$= \max\left\{ \sup\left\{\|Tx(t) - Ty(t)\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{\|Tx(t) - Ty(t)\| : x(t), y(t) \in K_2\right\}\right\}$$

$$= \max\left\{ \sup\left\{\|\frac{x(t)}{3} - \frac{y(t)}{3}\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{\|2x(t) - \frac{3}{2} - 2y(t) + \frac{3}{2}\| : x(t), y(t) \in K_2\right\}\right\}$$

$$= \max\left\{ \sup\left\{\frac{1}{3}\|x(t) - y(t)\| : x(t), y(t) \in K_1\right\}, \ \sup\left\{2\|x(t) - y(t)\| : x(t), y(t) \in K_2\right\}\right\}$$

$$= 2\mu(T(K_2)) = 2\mu(K_1 \cup K_2),$$

which shows that T is not noncyclic condensing mapping. But, T is noncyclic β-ψ-condensing. Indeed if we define $\beta : 2^X \to [0, +\infty)$ as $\beta(K_1 \cup K_2) = 1$, then $\beta(K_1 \cup K_2)\mu(T(K_1) \cup T(K_2)) \leq \psi(\mu(K_1 \cup K_2))$, where $\psi(t) = \frac{t}{2}, t \geq 0$.

Theorem 3.8. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an MNC on X and $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive β-admissible, β-ψ-condensing operator. Then T has a best proximity point if there exist nonempty, bounded, closed and convex $X_0, Y_0 \subseteq X$ such that $\beta(X_0 \cup Y_0) \geq 1$.*

Proof. Let us define sequences:
(i) $\{(\mathcal{U}_{2n}, \mathcal{V}_{2n})\}$ consisting of nonempty, bounded, closed, convex, proximinal and T-invariant pairs as defined in the Lemma 2.7.

We note that if for some $k \in \mathbb{N}$ we have $\max\{\mu(\mathcal{U}_{2k}), \mu(\mathcal{V}_{2k})\} = 0$, then $T : \mathcal{U}_{2k} \cup \mathcal{V}_{2k} \to \mathcal{U}_{2k} \cup \mathcal{V}_{2k}$ is a compact and cyclic relatively nonexpansive mapping and so the result follows from Theorem 2.5. Assume that $\max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} > 0$ for all $n \in \mathbb{N}$. Since $\beta(\mathcal{U}_0 \cup \mathcal{V}_0) \geq 1$, β-admissibility of T implies that $\beta(\mathcal{U}_1 \cup \mathcal{V}_1) \geq 1$. Applying recursion, we get $\beta(\mathcal{U}_{2n} \cup \mathcal{U}_{2n}) \geq 1$, for every $n \geq 0$. Since T is a $\beta - \psi$-condensing operator,

$$\begin{aligned}
\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) &\leq \beta(\mathcal{U}_{2n} \cup \mathcal{U}_{2n})\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \\
&= \beta(\mathcal{U}_{2n} \cup \mathcal{U}_{2n})\mu(\overline{\mathrm{con}}(T(\mathcal{U}_{2n})) \cup \overline{\mathrm{con}}(T(\mathcal{V}_{2n}))) \\
&\leq \beta(\mathcal{U}_{2n} \cup \mathcal{U}_{2n})\mu(T(\mathcal{U}_{2n}) \cup T(\mathcal{V}_{2n})) \\
&\leq \psi(\mu(\mathcal{U}_{2n}) \cup \mu(\mathcal{V}_{2n})).
\end{aligned}$$

Repeating this pattern we get the following inequality

$$\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \leq \psi^n(\mu(\mathcal{U}_0 \cup \mathcal{V}_0)),$$

which yields us

$$\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \to 0, \ \text{as } n \to \infty.$$

That is,

$$\lim_{n \to \infty} \max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} = 0.$$

Let

$$\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_{2n}, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_{2n}.$$

By the properties of the measure of noncompactness, the pair $(\mathcal{U}_\infty, \mathcal{V}_\infty) \subseteq (A, B)$ is nonempty, compact and convex with $\mathrm{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \mathrm{dist}(A, B)$. On the other hand, $T : \mathcal{U}_\infty \cup \mathcal{V}_\infty \to \mathcal{U}_\infty \cup \mathcal{V}_\infty$ is a cyclic relatively nonexpansive mapping. Now Theorem 2.3 ensures that T has a best proximity point. □

We now state the noncyclic version of Theorem 3.8 in order to study the existence of best proximity pairs.

Theorem 3.9. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an* MNC *on X and $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive β-admissible and $\beta - \psi$-condensing operator. Then T admits a best proximity pair if there exist nonempty, bounded, closed and convex $X_0, Y_0 \subseteq X$ such that $\beta(X_0 \cup Y_0) \geq 1$.*

Proof. Consider the sequence of pairs $(\mathcal{U}_n, \mathcal{V}_n) \subseteq (A, B)$ for $n \in \mathbb{N} \cup \{0\}$ as in Lemma 2.7. It follows from the proof of Theorem 3.11 of [13] that $\{(\mathcal{U}_n, \mathcal{V}_n)\}$ is a descending sequence of nonempty, bounded, closed, convex and proximinal pairs with $\mathrm{dist}(\mathcal{U}_n, \mathcal{V}_n) = \mathrm{dist}(A, B)$ which are T-invariant. If there exists some $k \in \mathbb{N}$ for which $\max\{\mu(\mathcal{U}_k), \mu(\mathcal{V}_k)\} = 0$, then from Theorem 2.5 the result will be concluded.
Now by β-admissibility of T and $\beta(\mathcal{U}_0, \mathcal{V}_0) \geq 1$, we get $\beta(\mathcal{U}_n, \mathcal{V}_n) \geq 1$. Suppose that

$$\max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} > 0, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

By a discussion as in the proof of Theorem 3.8, we can see that $\lim_{n\to\infty} \max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} = 0$. Now, if we set

$$\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_n, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_n,$$

then $(\mathcal{U}_\infty, \mathcal{V}_\infty)$ is a nonempty, compact and convex pair with $\mathrm{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \mathrm{dist}(A, B)$. Hence, the existence of a best proximity pair for the mapping T is deduced from Theorem 2.3. □

4. Coupled best proximity point results

In this section, we study the existence of *coupled best proximity points* in the setting of Busemann convex spaces. To this end, we recall the following concepts which first appeared in the Ph.D Thesis of the first author ([15]).

Let (A, B) be a nonempty pair in a metric space (X, d) and $S : (A \times A) \cup (B \times B) \to A \cup B$ be a cyclic mapping, that is, $S(A \times A) \subseteq B$ and $S(B \times B) \subseteq A$. A point $(u, v) \in (A \times A) \cup (B \times B)$ is called a coupled best proximity point for the mapping S provided that

$$d(u, S(u, v)) = d(v, S(v, u)) = \mathrm{dist}(A, B).$$

We also need the following lemma.

Lemma 4.1 ([1])**.** *Suppose that $\mu_1, \mu_2, \cdots, \mu_n$ are measures of noncompactness on the metric spaces $X_1, X_2, \cdots, X_n$, respectively. Moreover, assume that the function $\Theta : [0, \infty)^n \to [0, \infty)$ is convex and $\Theta(x_1, x_2, \cdots, x_n) = 0$ if and only if $x_j = 0$ for all $j = 1, 2, \cdots, n$. Then*

$$\mu(E) = \Theta\big(\mu_1(E_1), \mu_2(E_2), \cdots, \mu_n(E_n)\big),$$

defines a measure of noncompactness on $X_1 \times X_2 \times \cdots \times X_n$, where E_j denotes the natural projection of E into E_j for $j = 1, 2, \cdots, n$.

The next lemma was given in [16]. For the convenience of the reader, we include a proof of this fact.

Lemma 4.2 (see [16, Lemma 4.2])**.** *Let (A, B) be a nonempty pair in a metric space (X, d). Consider the product space $X \times X$ with the metric*

$$d_\infty\big((x_1, y_1), (x_2, y_2)\big) = \max\{d(x_1, x_2), d(y_1, y_2)\}, \quad \forall(x_1, y_1), (x_2, y_2) \in X \times X.$$

Then the pair (A, B) is proximinal in X if and only if $(A \times A, B \times B)$ is proximinal in $X \times X$.

Proof. Note that $\operatorname{dist}(A \times A, B \times B) = \operatorname{dist}(A, B)$. In fact, for any $(a, a') \in A \times A, (b, b') \in B \times B$ we have

$$
\begin{aligned}
\operatorname{dist}(A \times A, B \times B) &= \inf_{\big((a,a'),(b,b')\big) \in (A \times A) \times (B \times B)} d_\infty\big((a, a'), (b, b')\big) \\
&= \inf_{\big((a,a'),(b,b')\big) \in (A \times A) \times (B \times B)} \max\{d(a, b), d(a', b')\} \\
&= \operatorname{dist}(A, B).
\end{aligned}
$$

Suppose (A, B) is proximinal and $(a, a') \in A \times A$, then we can find $b, b' \in B$ such that $d(a, b) = d(a', b') = \operatorname{dist}(A, B)$. Thus for an element $(b, b') \in B \times B$ we have $d_\infty\big((a, a'), (b, b')\big) = \operatorname{dist}(A, B)\big(= \operatorname{dist}(A \times A, B \times B)\big)$, that is, $(A \times A)_0 = A \times A$. By a similar manner, $(B \times B)_0 = B \times B$ and so the pair $\big((A \times A), (B \times B)\big)$ is proximinal in $X \times X$. Now assume that $\big((A \times A), (B \times B)\big)$ is proximinal and $a \in A$, then $(a, a) \in A \times A$ and so there exists a point $(b, b') \in B \times B$ for which $d_\infty\big((a, a), (b, b')\big) = \operatorname{dist}(A, B)$ which deduces that $d(a, b) = d(a, b') = \operatorname{dist}(A, B)$, that is, $A_0 = A$. Equivalently, $B_0 = B$ which implies that (A, B) is proximinal and this completes the proof of lemma. $\square$

We now state our first coupled best proximity point result.

Theorem 4.3. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an MNC on X and $S : (A \times A) \cup (B \times B) \to A \cup B$ is a cyclic mapping satisfying following conditions.*

(i) *Let $(K_1, K_2) \subseteq (A, B)$ and $(K_1', K_2') \subseteq (A, B)$ be nonempty, bounded, closed, convex, proximinal and S-invariant pairs with $\operatorname{dist}(K_1, K_2) =$*

$\text{dist}(A, B) = \text{dist}(K_1', K_2')$, $\gamma : X^2 \times X^2 \to [0, +\infty)$ *such that for any* $(x_1, y_1) \in X \times X$ *we have*

$$\gamma((x_1, y_1), (x_2, y_2))\mu((S(K_1 \times K_1') \cup S(K_2 \times K_2'))) \leq \frac{1}{2}\psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big),$$

where $\psi \in \Psi$ *is nondecreasing.*

(ii) *For all* $(x, y), (u, v) \in X \times X$ *and* $\gamma\big((x, y), (u, v)\big) \geq 1$ *we have*

$$\gamma\big((S(x, y), S(y, x)), (S(u, v), S(v, u)\big) \geq 1.$$

(iii) *Moreover,* $d(S(x_1, x_2), S(y_1, y_2)) \leq d_\infty\big((x_1, y_1), (x_2, y_2)\big), \quad \forall (x_1, x_2) \in A \times A, \ \forall(y_1, y_2) \in B \times B.$

(iv) *Furthermore, there exists* $(x_0, y_0) \in X \times X$ *such that*

$$\gamma\big((x_0, y_0), (S(x_0, y_0), S(y_0, x_0))\big) \geq 1,$$

$$\gamma\big((y_0, x_0), (S(y_0, x_0), S(x_0, y_0))\big) \geq 1.$$

Then S admits a coupled best proximity point.

Proof. Set $\widetilde{\mu}(E) := \max\{\mu(E_1), \mu(E_2)\}$, where E_j denotes the natural projection of E into E_j for $j = 1, 2$. Thus by Lemma 4.1 $\widetilde{\mu}$ is an MNC on $X \times X$. Let $T : (A \times A) \cup (B \times B) \to (A \times A) \cup (B \times B)$ defined by

$$T(u, v) = (S(u, v), S(v, u)), \quad \forall (u, v) \in (A \times A) \cup (B \times B).$$

Note that if $(u, v) \in A \times A$, then by the fact that S is cyclic, $(S(u, v), S(v, u)) \in B \times B$, that is, $T(A \times A) \subseteq B \times B$. Equivalently, $T(B \times B) \subseteq A \times A$. Therefore, T is cyclic on $(A \times A) \cup (B \times B)$.

Besides, for any $((x_1, x_2), (y_1, y_2)) \in (A \times A) \times (B \times B)$ we have

$$\begin{aligned}
d_\infty\Big(T(x_1, x_2), T(y_1, y_2)\Big) &= d_\infty\Big((S(x_1, x_2), S(x_2, x_1)), (S(y_1, y_2), S(y_2, y_1))\Big) \\
&= \max\{d(S(x_1, x_2), S(y_1, y_2)), d(S(x_2, x_1), S(y_2, y_1))\} \\
&\leq \max\{d_\infty((x_1, y_1), (x_2, y_2)), d_\infty((x_2, y_2), d(x_1, y_1))\} \\
&= d_\infty((x_1, x_2), (y_1, y_2)).
\end{aligned}$$

Hence, T is relatively nonexpansive.

Let us now define $\alpha : X^2 \times X^2 \to [0, +\infty)$ as follows:

$$\alpha((x_1, y_1), (x_2, y_2)) = \min\{\gamma((x_1, y_1), (x_2, y_2)), \gamma((y_1, x_1), (y_2, x_2))\}.$$

By our hypothesis (ii), it is clear that whenever $\alpha((x_1, y_1), (x_2, y_2)) \geq 1$, we have $\alpha(T(x_1, y_1), T(x_2, y_2)) \geq 1$, which shows that T is α-admissible. Also by hypothesis (iv) it is clear that there exists $(x_0, y_0) \in X \times X$ such that $\alpha((x_0, y_0), (y_0, x_0)) \geq 1$.

Moreover, we have

$$
\begin{aligned}
&\alpha((x_1, y_1), (x_2, y_2)) \widetilde{\mu}(T(K_1 \times K_1') \cup T(K_2 \times K_2')) \\
&= \alpha((x_1, y_1), (x_2, y_2)) \max\{\widetilde{\mu}\big(T(K_1 \times K_1')\big), \widetilde{\mu}\big(T(K_2 \times K_2')\big)\} \\
&= \alpha((x_1, y_1), (x_2, y_2)) \max\{\widetilde{\mu}\big(S(K_1 \times K_1') \times S(K_1' \times K_1)\big), \widetilde{\mu}\big(S(K_2 \times K_2') \times S(K_2' \times K_2)\big)\} \\
&= \alpha((x_1, y_1), (x_2, y_2)) \max \big\{ \max\{\mu\big(S(K_1 \times K_1')\big), \mu\big(S(K_1' \times K_1)\big)\}, \\
&\hspace{8cm} \max\{\mu\big(S(K_2 \times K_2')\big), \mu\big(S(K_2' \times K_2)\big)\}\big\} \\
&= \alpha((x_1, y_1), (x_2, y_2)) \max \big\{ \max\{\mu\big(S(K_1 \times K_1')\big), \mu\big(S(K_2 \times K_2')\big)\}, \\
&\hspace{8cm} \max\{\mu\big(S(K_1' \times K_1)\big), \mu\big(S(K_2' \times K_2)\big)\}\big\} \\
&= \alpha((x_1, y_1), (x_2, y_2)) \max \big\{\mu\big(S(K_1 \times K_1') \cup S(K_2 \times K_2')\big), \mu\big(S(K_1' \times K_1) \cup S(K_2' \times K_2)\big)\big\} \\
&\leq \alpha((x_1, y_1), (x_2, y_2)) \big[\mu\big(S(K_1 \times K_1') \cup S(K_2 \times K_2')\big) + \mu\big(S(K_1' \times K_1) \cup S(K_2' \times K_2)\big)\big] \\
&\leq \frac{1}{2}\psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big) + \frac{1}{2}\psi\big(\max\{\mu(K_1' \cup K_1), \mu(K_2' \cup K_2)\}\big) \quad \text{by hypothesis}(i) \\
&= \psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big) \\
&= \psi\big(\max\big\{\max\{\mu(K_1), \mu(K_1')\}, \max\{\mu(K_2), \mu(K_2')\}\big\}\big) \\
&= \psi\big(\max\big\{\widetilde{\mu}(K_1 \times K_1'), \widetilde{\mu}(K_2 \times K_2')\big\}\big) = \psi\big(\widetilde{\mu}((K_1 \times K_1') \cup (K_2 \times K_2'))\big).
\end{aligned}
$$

This implies that T is a α-ψ-condensing operator. Now, Theorem 3.3 ensures that T has a best proximity point, called $(p, q) \in (A \times A) \cup (B \times B)$. That is,

$$
\begin{aligned}
\text{dist}(A, B) = d_\infty\big((p, q), T(p, q)\big) &= d_\infty\big((p, q), (S(p, q), S(q, p))\big) \\
&= \max\{d(p, S(p, q)), d(q, S(q, p))\}.
\end{aligned}
$$

Thus (p, q) is a coupled best proximity point of S. $\qquad\square$

Theorem 4.4. *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d) such that B is bounded and let closed convex hulls of finite sets be compact. Assume that μ is an MNC on X and $S : (A \times A) \cup (B \times B) \to A \cup B$ is a cyclic mapping satisfying following conditions.*

(i) *For all nonempty, bounded, closed, convex, proximinal and S-invariant pairs $(K_1, K_2) \subseteq (A, B)$ and $(K_1', K_2') \subseteq (A, B)$ with $\text{dist}(K_1, K_2) = \text{dist}(A, B) = \text{dist}(K_1', K_2')$ and $\gamma : 2^{X \times X} \to [0, +\infty)$ such that*

$$
\gamma(K_1 \times K_2)\mu((S(K_1 \times K_1') \cup S(K_2 \times K_2'))) \leq \frac{1}{2}\psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big),
$$

where $\psi \in \Psi$ is nondecreasing.

(ii) *For any $(U, V) \in X \times X$ and $\gamma(U, V) \geq 1$ we have*

$$
\gamma\big(\overline{con}(S(U \times V) \times S(V \times U))\big) \geq 1.
$$

(iii) *Moreover, $d(S(x_1, x_2), S(y_1, y_2)) \leq d_\infty\big((x_1, y_1), (x_2, y_2)\big), \quad \forall(x_1, x_2) \in A \times A, \ \forall(y_1, y_2) \in B \times B,$*

(iv) *Furthermore, there exists closed and convex $X_0, Y_0 \subseteq X$ such that $\gamma\big((X_0 \times Y_0))\big) \geq 1$ and $\gamma\big(Y_0 \times X_0\big) \geq 1$.*

Then S admits a coupled best proximity point.

Proof. Let us set $\widetilde{\mu}(E) := \max\{\mu(E_1), \mu(E_2)\}$, where E_j denotes the natural projection of E into E_j for $j = 1, 2$. Thus by Lemma 4.1 $\widetilde{\mu}$ is an MNC on $X \times X$. Then following proof of Theorem 4.3 it is easy to show T is cyclic on $(A \times A) \cup (B \times B)$ and T is relatively nonexpansive.

Let us now define $\beta : 2^{X \times X} \to [0, +\infty)$ as follows:

$$\beta(E_1 \times E_2) = \min\{\gamma(E_1 \times E_2), \gamma(E_2 \times E_1)\}.$$

By our hypothesis (ii), it is clear that whenever $\beta(E_1 \times E_2) \geq 1$, we have $\beta(\overline{\mathrm{con}}(T(E_1 \times E_2))) \geq 1$, which shows that T is β-admissible. Also by hypothesis (iv) it is clear that there exist $X_0, Y_0 \subseteq X$ such that $\beta(X_0 \times Y_0) \geq 1$.

Moreover, we have

$$\beta(K_1 \times K_2)\widetilde{\mu}(T(K_1 \times K_1') \cup T(K_2 \times K_2'))$$
$$= \beta(K_1 \times K_2) \max\{\widetilde{\mu}\big(T(K_1 \times K_1')\big), \widetilde{\mu}\big(T(K_2 \times K_2')\big)\}$$
$$= \beta(K_1 \times K_2) \max\{\widetilde{\mu}\big(S(K_1 \times K_1') \times S(K_1' \times K_1)\big), \widetilde{\mu}\big(S(K_2 \times K_2') \times S(K_2' \times K_2)\big)\}$$
$$= \beta(K_1 \times K_2) \max\big\{\max\{\mu\big(S(K_1 \times K_1')\big), \mu\big(S(K_1' \times K_1)\big)\},$$
$$\max\{\mu\big(S(K_2 \times K_2')\big), \mu\big(S(K_2' \times K_2)\big)\}\big\}$$

$$= \beta(K_1 \times K_2) \max\big\{\max\{\mu\big(S(K_1 \times K_1')\big), \mu\big(S(K_2 \times K_2')\big)\},$$
$$\max\{\mu\big(S(K_1' \times K_1)\big), \mu\big(S(K_2' \times K_2)\big)\}\big\}$$
$$= \beta(K_1 \times K_2) \max\big\{\mu\big(S(K_1 \times K_1') \cup S(K_2 \times K_2')\big), \mu\big(S(K_1' \times K_1) \cup S(K_2' \times K_2)\big)\big\}$$
$$\leq \beta(K_1 \times K_2)\big[\mu\big(S(K_1 \times K_1') \cup S(K_2 \times K_2')\big) + \mu\big(S(K_1' \times K_1) \cup S(K_2' \times K_2)\big)\big]$$
$$\leq \frac{1}{2}\psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big) + \frac{1}{2}\psi\big(\max\{\mu(K_1' \cup K_1), \mu(K_2' \cup K_2)\}\big) \quad \text{by hypothesis}(i)$$
$$= \psi\big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\big)$$
$$= \psi\big(\max\big\{\max\{\mu(K_1), \mu(K_1')\}, \max\{\mu(K_2), \mu(K_2')\}\big\}\big)$$
$$= \psi\big(\max\big\{\widetilde{\mu}(K_1 \times K_1'), \widetilde{\mu}(K_2 \times K_2')\big\}\big) = \psi\big(\widetilde{\mu}\big((K_1 \times K_1') \cup (K_2 \times K_2')\big)\big).$$

This implies that T is a β-ψ-condensing operator. Now, Theorem 3.8 ensures that T has a best proximity point, called $(p, q) \in (A \times A) \cup (B \times B)$. Thus (p, q) is a coupled best proximity point of S.

$\square$

Remark 4.5. It is remarkable to note that by a classical theorem due to S. Mazur, the convex hull of a compact set in a Banach space is again relatively compact. But this fact may not be true in geodesic metric spaces in general (see [4] for more information about the geodesic spaces which satisfies the Mazur's theorem). Therefore, we should consider the compactness assumption of closed convex hulls of finite sets in our main existence theorems.

Remark 4.6. It is worth noticing that the class of reflexive and Busemann convex spaces contains the class of reflexive and strictly convex Banach spaces as a subclass. For example consider $X = \mathbb{R}^2$ with *radial metric* defined with

$$
d\Big((x_1, y_1), (x_2, y_2)\Big) = \begin{cases} \rho\Big((x_1, y_1), (x_2, y_2)\Big); & \text{if} \quad (0,0), (x_1, y_1), (x_2, y_2) \text{ are colinear,} \\ \rho\Big((x_1, y_1), (0,0)\Big) + \rho\Big((x_2, y_2), (0,0)\Big); & \text{otherwise,} \end{cases}
$$

where ρ denotes the usual Euclidean metric on $\mathbb{R}^2$. Then (X, d) is a complete $\mathbb{R}$-tree and so is a reflexive and Busemann convex space (see [8] for more details). Note that the radial metric does not induced with any norm.

5. APPLICATION

This section is dedicated to prove a result which shows the existence of optimum solutions of a system of second order differential equation with two initial conditions.

Let $\tau, \gamma \in \mathbb{R}^+$, $\mathcal{I} = [0, \tau]$ and $(E, \|.\|)$ be a Banach space. Let $B_1 = B(\alpha_0, \gamma)$, $B_2 = B(\beta_0, \gamma)$ where $\alpha_0, \beta_0 \in E$. We consider the following system of second order differential equation with two initial conditions

$$
x''(s) = f(s, x(s)), \ x(0) = \alpha_0, \ x'(0) = \alpha_1,
$$

(5.1)

$$
y''(s) = g(s, y(s)), \ y(0) = \beta_0, \ y'(0) = \beta_1,
$$

where, $f : \mathcal{I} \times B_1 \to \mathbb{R}$, $g : \mathcal{I} \times B_2 \to \mathbb{R}$ are continuous functions such that $\|f(s, x)\| \leq A_1$, $\|g(s, y)\| \leq A_2$, $s \in \mathcal{I}$ and $\alpha_1, \beta_1 \in E$. Twice integrating (5.1) and usage of initial conditions yields us

$$
x(s) = \alpha_0 + \int_0^s (\alpha_1 + (s - r)f(r, x(r))dr,
$$

(5.2)

$$
y(s) = \beta_0 + \int_0^s (\beta_1 + (s - r)g(r, x(r))dr.
$$

It is clear that the systems (5.1) and (5.2) are equivalent to each other. Let $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{S} = C(\mathcal{J}, E)$ be a Banach space of continuous mappings from $\mathcal{J}$ into E endowed with supremum norm and consider

$$
\mathcal{S}_1 = C(\mathcal{J}, B_1) = \{x : \mathcal{J} \to B_1 : x \in \mathcal{S}, \ x(0) = \alpha_0, \ x'(0) = \alpha_1\},
$$

$$
\mathcal{S}_2 = C(\mathcal{J}, B_2) = \{y : \mathcal{J} \to B_2 : y \in \mathcal{S}, \ y(0) = \beta_0, \ y'(0) = \beta_1\}.
$$

So, $(\mathcal{S}_1, \mathcal{S}_2)$ is $\mathcal{NBCC}$ pair in $\mathcal{S}$. Now, for every $x \in \mathcal{S}_1$ and every $y \in \mathcal{S}_2$, we have

$$
\|x - y\| = \sup_{s \in \mathcal{J}} \|x(s) - y(s)\| \geq \|\alpha_0 - \beta_0\|.
$$

So, $\text{dist}(\mathcal{S}_1, \mathcal{S}_2) = \|\alpha_0 - \beta_0\|$. Let us define operator $T : \mathcal{S}_1 \cup \mathcal{S}_2 \to \mathcal{S}$ as follows:

$$
Tx(s) = \begin{cases} \beta_0 + \int_0^s \left(\beta_1 + (s - r)g(r, x(r))\right)dr, \ x \in \mathcal{S}_1, \\ \\ \alpha_0 + \int_0^s \left(\alpha_1 + (s - r)f(r, x(r))\right)dr, \ x \in \mathcal{S}_2. \end{cases}
$$

It is clear that T is cyclic operator. It is known that $w \in \mathcal{S}_1 \cup \mathcal{S}_2$ is an optimum solution of the system (5.2) if $\|w - Tw\| = \text{dist}(\mathcal{S}_1 \cup \mathcal{S}_2)$ is satisfied. Equivalently, w is the best proximity point of the operator T. Before proving the existence of an optimum solution of system (5.2) we recall an extension of the mean values theorem for integrals, which is presented according to our notations.

Theorem 5.1 ([12]). *For $\mathcal{I}, \mathcal{J}, B_1, B_2, f$ and g as given in the discussion above with $s \in J$ we have*

$$\alpha_0 + \int_0^s (\alpha_1 + (s-r)f(r,x(r)))dr \in \alpha_0 + s\,\overline{con}\big(\{\alpha_1 + (s-r)f(r,x(r)) : r \in [0,s]\}\big)$$

and

$$\beta_0 + \int_0^s (\beta_1 + (s-r)g(r,x(r)))dr \in \beta_0 + s\,\overline{con}\big(\{\beta_1 + (s-r)g(r,x(r)) : r \in [0,s]\}\big).$$

The following theorem shows the existence of optimum solutions for the system (5.1).

Theorem 5.2. Let μ be an arbitrary MNC on $\mathcal{S}$, $\tau(\tau A_2 + \|\beta_1\|) \leq \gamma$, $\tau(\tau A_1 + \|\alpha_1\|) \leq \gamma$ and $\tau \leq 1$. The system (5.1) has an optimal solution if the following condition holds true:

(1) For any bounded pair $(K_1, K_2) \subseteq (B_1, B_2)$, there is a nondecreasing function $\psi \in \Psi$ such that

$$\mu(f(\mathcal{J} \times K_1) \cup g(\mathcal{J} \times K_2)) < \frac{1}{s}\psi(\mu(N_1 \cup N_2)).$$

(2) For each $x \in \mathcal{S}_1$ and for all $y \in \mathcal{S}_2$,

$$\|g(r,x(r)) - f(r,y(r))\| \leq \frac{1}{s^2}(\|x(r) - y(r)\| - \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\|s).$$

Proof. As the system (5.1) and (5.2) are equivalent to each other, in order to show (5.1) has an optimal solution it is sufficient to show (5.2) has an optimal solution. From the above discussion it is clear that the operator T is cyclic. Our first task is to show that $T(\mathcal{S}_1)$ is a bounded and equicontinuous subset of $\mathcal{S}_2$. For each $x \in \mathcal{S}_1$,

$$\begin{aligned}
\|Tx(t)\| &= \|\beta_0 + \int_0^s (\beta_1 + (s-r)g(r,x(r))dr\| \\
&\leq \|\beta_0\| + \int_0^s \|\beta_1 + (s-r)g(r,x(r)\|dr \\
&\leq \|\beta_0\| + \tau(\|\beta_1\| + \tau A_2) \\
&\leq \|\beta_0\| + \gamma.
\end{aligned}$$

Thus $T(\mathcal{S}_1)$ is bounded. Now for $s, s' \in J$ and $x \in \mathcal{S}_1$,

$$
\begin{aligned}
\|Tx(s) - Tx(s')\| &= \left\| \int_0^s (\beta_1 + (s-r)g(r, x(r))dr - \int_0^{s'} (\beta_1 + (s-r)g(r, x(r))dr \right\| \\
&\leq \left| \int_s^{s'} \left\| \beta_1 + (s-r)g(r, x(r) \right\| dr \right| \\
&\leq (\tau A_2 + \|\beta_1\|) \left\| s - s' \right\| \\
&\leq M|s - s'|, \text{ where } M = \tau A_2 + \|\beta_1\|,
\end{aligned}
$$

which means that $T(\mathcal{S}_1)$ is equicontinuous. By a similar argument $T(\mathcal{S}_2)$ is a bounded and equicontinuous subset of $\mathcal{S}_1$. Thus an application of the Arzela-Ascoli theorem concludes that $(\mathcal{S}_1, \mathcal{S}_2)$ is relatively compact.

Now our aim is to show that T is a relatively nonexpansive cyclic β-ψ-condensing operator. For each $(x, y) \in \mathcal{S}_1 \times \mathcal{S}_2$ with the help of assumption (2), we have

$$
\begin{aligned}
&\|Tx(s) - Ty(s)\| \\
&= \left\| \beta_0 + \int_0^s (\beta_1 + (s-r)g(r, x(r))dr - \left[\alpha_0 + \int_0^s (\alpha_1 + (s-r)f(r, x(r))dr\right] \right\| \\
&\leq \|\beta_0 - \alpha_0\| + \left\| \int_0^s \left[(\beta_1 - \alpha_1) + (s-r)(g(r, x(r)) - f(r, x(r)))\right] dr \right\| \\
&\leq \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\| \ s + \left\| s \int_0^s \left(g(r, x(r)) - f(r, x(r))dr \right\| \\
&\leq \|\beta_0 - \alpha_0\| + \|\beta_1 - \alpha_1\| \ s + (\|x(s) - y(s)\| - \|\beta_0 - \alpha_0\| - \|\beta_1 - \alpha_1\| \ s) \\
&= \|x(s) - y(s)\|.
\end{aligned}
$$

This means that T is relatively nonexpansive. In order to show that T is cyclic α-ψ-condensing, suppose that the pair $(K_1, K_2) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ is $\mathcal{NBCC}$, proximinal, T-invariant and $\text{dist}(K_1, K_2) = \text{dist}(\mathcal{S}_1, \mathcal{S}_2)(= \|\alpha_0 - \beta_0\|)$. Now

using Theorem 5.1 and assumption (1) we have

$$\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\}$$

$$= \max\left\{\sup_{s\in\mathcal{J}}\left\{\mu(\{Tx(s) : x \in K_1\})\right\}, \sup_{s\in\mathcal{J}}\left\{\mu(\{Ty(s) : y \in K_2\})\right\}\right\}$$

$$= \max\left\{\sup_{s\in\mathcal{J}}\left\{\mu\left(\left\{\beta_0 + \int_0^s (\beta_1 + (s-r)g(r,x(r)))dr : x \in K_1\right\}\right)\right\},\right.$$

$$\left.\sup_{s\in\mathcal{J}}\left\{\mu\left(\left\{\alpha_0 + \int_0^s (\alpha_1 + (s-r)f(r,x(r)))dr : x \in K_1\right\}\right)\right\}\right\}$$

$$= \max\left\{\sup_{s\in\mathcal{J}}\left\{\mu\left(\left\{\beta_0 + s\ \overline{con}(\{\beta_1 + (s-r)g(r,x(r)) : r \in [0,s]\})\right\}\right)\right\},\right.$$

$$\left.\sup_{s\in\mathcal{J}}\left\{\mu\left(\left\{\alpha_0 + s\ \overline{con}(\{\alpha_1 + (s-r)f(r,x(r)) : r \in [0,s]\})\right\}\right)\right\}\right\}$$

$$= \max\left\{\mu\left(\left\{\beta_0 + s\ (\beta_1 + \overline{con}(\{g(r,x(r)) : r \in [0,s]\}))\right\}\right),\right.$$

$$\left.\mu\left(\left\{\alpha_0 + s\ (\alpha_1 + \overline{con}(\{f(r,x(r)) : r \in [0,s]\}))\right\}\right)\right\}$$

$$\leq \max\left\{s\mu(\{g(\mathcal{J} \times K_1)\}), s\mu(\{f(\mathcal{J} \times K_2)\})\right\}$$

$$= s\mu\left(\{f(\mathcal{J} \times K_1) \cup g(\mathcal{J} \times K_2)\}\right) \leq s\frac{\psi(\mu(K_1 \cup K_2))}{s}.$$

Thus we get

$$\mu(T(K_1 \cup T(K_2))) \leq \psi(\mu(K_1 \cup K_2).$$

Furthermore, we define the function $\beta : 2^E \to [0, +\infty)$ as follows: $\beta(K_1 \cup K_2) = 1$. The latter implies that

$$\beta(K_1 \cup K_2)\mu(T(K_1 \cup T(K_2))) \leq \psi(\mu(K_1 \cup K_2)).$$

Thus necessary requirements of Theorem 3.4 are satisfied. So the operator T has a best proximity point and hence the system (5.1) has an optimal solution. $\square$

Remark 5.3. We know that, the Banach space $\mathcal{C}(J, X)$ in Theorem 5.2 is not reflexive and the reflexivity condition in Theorem 3.4 is essential. In fact, the reflexivity condition in Theorem 3.4 was used to prove that the proximal pair (A_0, B_0) is nonempty. But in Theorem 5.2 the proximal pair $(\mathcal{S}_1, \mathcal{S}_2)$ is nonempty, automatically because of the fact that $(\alpha_0, \beta_0) \in \mathcal{S}_1 \times \mathcal{S}_2$.

As an application of Theorem 5.2, we conclude the following existence result of a solution for a system of second order differential equations satisfying the same two initial conditions.

Corollary 5.4. *Under the notations defined above in this section and the assumptions of Theorem 5.2 if $\alpha_0 = \beta_0 = x_0$ and $\alpha_1 = \beta_1 = x_1$, then the system*

$$x^{''}(s) = f(s, x(s)), \ x(0) = x_0, \ x^{'}(0) = x_1,$$

$$y^{''}(s) = g(s, y(s)), \ y(0) = x_0, \ y^{'}(0) = x_1,$$

has a solution.

6. Conclusions

This work elicited some best proximity point (pair) theorems for cyclic (non-cyclic) $(\alpha - \psi)$ and $(\beta - \psi)$ condensing operators in the framework of reflexive Busemann convex spaces and by considering an appropriate measure of non-compactness. As an application of the main existence result of best proximity points, we survey the existence of an optimal solution for a system of differential equations.

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