Applied General Topology

Appl. Gen. Topol. 22, no. 1 (2021), 79-89
doi:10.4995/agt.2021.13608

Ideal spaces

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Communicated by A. Tamariz-Mascarúa

Abstract

Let $C_\infty(X)$ denote the family of real-valued continuous functions which vanish at infinity in the sense that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact in X for all $n \in \mathbb{N}$. It is not in general true that $C_\infty(X)$ is an ideal of $C(X)$. We define those spaces X to be ideal space where $C_\infty(X)$ is an ideal of $C(X)$. We have proved that nearly pseudocompact spaces are ideal spaces. For the converse, we introduced a property called "RCC" property and showed that an ideal space X is nearly pseudocompact if and only if X satisfies "RCC" property. We further discussed some topological properties of ideal spaces.

2010 MSC: *54F65; 54G20; 54D45; 54D60; 54D99.*

Keywords: *rings of continuous functions; $C_K(X)$ and $C_\infty(X)$; nearly pseudocompact spaces;* ***RCC*** *properties.*

1. Introduction

In this paper, by a space we shall mean completely regular Hausdorff space, unless otherwise mentioned. As usual $C(X)$ and $C^*(X)$ are real-valued continuous and bounded continuous functions respectively. They are commutative rings with 1 under usual pointwise addition and multiplication. Rigorous and systematic developments of these two rings are made in the classic monograph of L. Gillman and M. Jerrison entitled *"Rings of Continuous Functions"*[7]. In fact most of the symbols, definitions and results are hired from the above book. There are two important classes of subrings of $C(X)$, namely, $C_\infty(X)$ and $C_K(X)$, where $C_\infty(X)$ is the family of all those continuous functions such

Received 28 April 2020 – Accepted 29 October 2020

that $\{x : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$ and $C_K(X)$ is the family of all functions $f \in C(X)$ whose support, that is $cl_X(X \backslash Z(f))$, is compact, where $Z(f) = \{x \in X : f(x) = 0\}$. Both the subrings are in fact subrings of $C^*(X)$ also. Even more they are ideal in $C^*(X)$. But though $C_K(X)$ is an ideal of $C(X)$, $C_\infty(X)$ need not be an ideal of $C(X)$. Immediate example can be cited if we count $X = \mathbb{N}$. In this paper we have worked on those spaces X for which $C_\infty(X)$ is an ideal of $C(X)$. For the sake of convenience, we define these spaces as ideal space. In this context it ought to be relevant to mention that Azarpanah et. al in [14], [2], already did some works in this area. They have shown that every pseudocompact space is an ideal space and every ideal space is pseudocompact if the space is locally compact. They also introduced ∞-compact space, the space where $C_K(X) = C_\infty(X)$, which is trivially an ideal space.

In this paper we have shown that every nearly pseudocompact space, introduced by Henriksen and Rayburn in [10], is also ideal space. We have further introduced a criteria, so called ***RCC*** property, a generalization of locally compact property, to go for converse. In fact we have shown that a nearly pseudocompact space is ideal space and an ideal space is nearly pseudocompact if and only if the space satisfies ***RCC*** property. Throughout the paper we have given many examples and counter examples. Henriksen and Rayburn in their paper [10] have shown that every anti-locally realcompact space, i.e., having no point with realcompact neighbourhood, is a nearly pseudocompact space. All the examples of nearly pseudocompact spaces that they cited, are anti-locally realcompact but they did not produce any example of a nearly pseudocompact space which is not anti-locally realcompact. Here we have cited such example [Example 4.14]. At the end we tried to explore few topological properties of ideal spaces and finally have shown that if $X$and Y are nearly pseudocompact, then $X \times Y$ is nearly pseudocompact if and only if $X \times Y$ is ideal space.

2. Preliminaries

As we already mentioned that most of the basic symbols and terminologies followed the book, The Rings of Continuous Functions, by L. Gillman and M. Jerrison [7], yet for ready references, we include few basic notations, definitions and related results that will be repeatedly used here. For each $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero set of f. The complement of zero set is called cozero set, denoted as coz f. For each space X, βX is the largest compactification of X where every compact-valued continuous function can be continuously extended, referred as Stone-Čech compactification of X. It is also the largest compactification of X in which X is C^*-embedded. Similarly υX is the largest realcompact subspace of βX in which X is C-embedded. A space is realcompact if it can be embedded as a closed subspace in the product of reals. The υX is referred as Hewitt-Nachbin completion of X or simply Hewitt realcompactification of X. Compactness and realcompactness can be easily characterized respectively by showing $X = \beta X$ and $X = \upsilon X$. A space X is pseudocompact (i.e, every realvalued continuous function on X is bounded) if and only if $\beta X = \upsilon X$ or equivalently $\beta X \backslash X = \upsilon X \backslash X$. Both realcompactness and pseudocompactness are significant generalizations of compactness. However the realcompact and pseudocompact jointly enforce compact. A subset S of X is relatively pseudocompact if every continuous function on X is bounded over S. R.L. Blair and M.A. Swardson proved that a subset S of a space X is relatively pseudocompact if $cl_{\beta X} S \subseteq \upsilon X$ [3, Proposition 2.6]. Henriksen and Rayburn in their paper [10] defined a space X to be nearly pseudocompact if $\upsilon X \backslash X$ is dense in $\beta X \backslash X$, a generalization of pseudocompact space. They have proved the following characterization of nearly pseudocompact spaces.

Theorem 2.1. *A space X is nearly pseudocompact if and only if $X = X_1 \cup X_2$, where X_1 is a regular closed* ***almost locally compact*** *pseudocompact subset, and X_2 is a regular closed* ***anti-locally realcompact*** *subset and $int(X_1 \cap X_2) = \varnothing$*

In the year 1976, Rayburn in his paper [13] defined hard set in X. A subset H of X is hard in X if H is closed in $X \cup K$ where $K = cl_{\beta X}(\upsilon X \backslash X)$. It is clear that every hard set in X is closed in X. However he has also provided a characterization of hardness of a closed subset in this paper as follows.

Theorem 2.2. *A closed subset F of X is hard in X if and only if there exists a compact set K such that for any open neighbourhood V of K, there exists a realcompact subset P of X so that $F \backslash V$ and $X \backslash P$ can be completely separated in X.*

In particular H is hard if it is completely separated from the complement of a realcompact subset of X.

Henriksen and Rayburn in [10] described few more characterizations as follows.

Theorem 2.3. *A space X is nearly pseudocompact if and only if every hard set is compact if and only if every regular hard set (i.e, a regular closed set which is hard in X) is compact.*

They have further proved the following theorem which is relevant in this paper.

Theorem 2.4. *Every regular closed subset of a nearly pseudocompact space is a nearly pseudocompact space.*

In the year 2005, Mitra and Acharyya in their paper [12] introduced two subrings of $C(X)$, $C_H(X)$ and $H_\infty(X)$, analogical to the rings $C_K(X)$ and $C_\infty(X)$. As per their definition, $C_H(X) = \{f \in C(X) : cl_X(X \backslash Z(f))$ is hard in X$\}$ and $H_\infty(X) := \{f \in C(X) : \{x \in X : |f(x)| \geqq \frac{1}{n}\}$is hard in X$\}$. They have shown that $C_H(X)$ is an ideal of $C(X)$. But no conclusion was made regarding $H_\infty(X)$. However they have given the following characterization of nearly pseudocompact spaces using $C_H(X)$ and $H_\infty(X)$.

Theorem 2.5. *The following statements are equivalent for a space X:*

1. (1) X is nearly pseudocompact.
2. (2) $C_K(X) = C_H(X)$.
3. (3) $C_\infty(X) = H_\infty(X)$
4. (4) $C_H(X) \subseteq C^*(X)$
5. (5) $H_\infty(X) \subseteq C^*(X)$
6. (6) $H_\infty(X) \bigcap C^*(X) = C_\infty(X)$
7. (7) $C_H(X) \bigcap C^*(X) = C_K(X)$

3. Ideal spaces-a generalization of nearly pseudocompact space

In this section we shall formally study ideal spaces. We begin with the definition of ideal spaces that we already introduced above.

Definition 3.1. A space X is called an ideal space if $C_\infty(X)$ is an ideal of $C(X)$.

Azarpanah and Soundarajan in [2], gave a nice characterization of ideal space.

Theorem 3.2. *A space X is an ideal space if and only if every locally compact σ-compact subset is relatively pseudocompact if and only if every open locally compact subset is relatively pseudocompact.*

They further proved that within the class of local compact spaces, the notions of ideal space and pseudocompact space are identical.

However here use of locally compact condition does not require to prove that every pseudocompact space is ideal. On the other hand, it will be shown here that, ideal space generalizes nearly pseudocompact and ∞-compact space. That the ∞-compact space is ideal trivially follows from its definition, hence we shall mainly concentrate on nearly pseudocompact space. Before showing that, we shall first show that $H_\infty(X)$ is indeed an ideal of $C(X)$ which was unanswered in the paper of Mitra and Acharyya [12]. In fact we shall show that $H_\infty(X) = C_{RC}(X)$, where $C_{RC}(X) = \{f \in C(X) : \text{ coz f is realcompact}\}$ and that $C_{RC}(X)$ is an ideal of $C(X)$, follows from the fact that realcompact co-zero sets are closed under finite, in fact countable union and realcompactness is a co-zero hereditary property. The notion of $C_{RC}(X)$ was introduced by T. Isiwata in [11]. Further, Azarpanah et.al in [[1], Theorem 3.8] produced another characterization of this ideal. Here we established the following result.

Theorem 3.3. $H_\infty(X) = C_{RC}(X)$.

Proof. We know that $X \setminus Z(f) = \cup_n [x \in X : |f(x)| \gneqq \frac{1}{n}]$, for $f \in C(X)$. If $f \in H_\infty(X)$, $[x \in X : |f(x)| \geq \frac{1}{n}]$ is hard in X and hence realcompact. Since $[x \in X : |f(x)| \gneqq \frac{1}{n}]$ is a co-zero subset of $[x \in X : |f(x)| \geq \frac{1}{n}]$, $[x \in X : |f(x)| \gneqq \frac{1}{n}]$ is also realcompact. As countable union of realcompact co-zero set is realcompact, so $X \setminus Z(f)$ is also realcompact. So $f \in C_{RC}(X)$.

Conversely, if $f \in C_{RC}(X)$, then the zero set $[x \in X : |f(x)| \geq \frac{1}{n}]$ is completely separated from the complement of the realcompact co-zero set $X \backslash Z(f)$, for all n. Hence the $[x \in X : |f(x)| \geq \frac{1}{n}]$ is hard in X, for all n. Thus $f \in H_\infty(X)$. □

Corollary 3.4. *Every nearly pseudocompact space is an ideal space.*

Proof. Since a nearly pseudocompact space, $H_\infty(X) = C_\infty(X)$ [Theorem 2.5] and $H_\infty(X)$ is always an ideal of $C(X)$, $C_\infty(X)$ is also an ideal of $C(X)$. □

However the following example shows that the converse is not true.

Example 3.5. Let X_1 be a non-compact realcompact space where the closure of the set D of points having compact neighbourhood is compact, e.g. $[(-\infty, 0) \cap \mathbb{Q}] \cup [0,1] \cup [(1,\infty) \cap \mathbb{Q}]$. Then $X_1 \times [0, \omega_1]$, where $[0, \omega_1]$ is the space of all ordinals less than or equal to the first uncountable ordinal ω_1, is an ideal space which is not nearly pseudocompact. The reason is quite simple. If we take U, a locally compact, σ-compact space, then $U \subseteq D \times \omega_1$. Now $clD \times \omega_1$, being pseudocompact space, U is relatively pseudocompact; hence an ideal space. But it is not nearly pseudocompact, as $X_1 \times \{0\}$ being a regular closed subset of $X_1 \times [0, \omega_1]$, not nearly pseudocompact as it is non-compact real-compact.

In the above example, U is actually contained in a compact space as the right projection of U into $[0, \omega_1]$ is also locally compact and σ-compact and hence is bounded by a compact subset K. Thus U is then contained in $clD \times K$.

This type of space is called ∞-compact space.

Definition 3.6. A space is called ∞-compact if $C_K(X) = C_\infty(X)$.

From [2, Proposition 2.1], we know the following result.

Theorem 3.7. *A space is ∞-compact if and only if every open locally compact σ-compact subset is bounded by a compact set.*

This tells us that every ∞-compact space is an ideal space. The following example is an ideal space which is not ∞-compact.

Example 3.8. As in Example 3.5, we take X_1 making product with Tychonoff plank T. That it is an ideal space but not nearly pseudocompact follows along the same lines of argument as in example 3.5. We show here that it is not even ∞-compact. $U = \cup_n (1/4, 1/2) \times ([0, \omega_1] \times \{n\})$ is open locally compact σ-compact subset. But U is not contained in any compact set as then, it's projection into Tychonoff plank would be covered by a compact set. But the right edge, the copy of $\mathbb{N}$, being a closed subset of U would be compact, which is not true.

We therefore have the following parallel strictly forward implications.

compact $\hookrightarrow$ *pseudocompact* $\hookrightarrow$ *nearly pseudocompact* $\hookrightarrow$ *ideal space.*
compact $\hookrightarrow \infty -$ *compact* $\hookrightarrow$ *ideal space.*

Next we give an example of a pseudocompact space which is not ∞-compact.

Example 3.9. We take Tychonoff plank $[0,\omega_1]\times[0,\omega_0]\backslash\{(\omega_1,\omega_0)\}$. The union of the horizontal line $[0,\omega_1]\times\{n\}$, $n\in\mathbb{N}$ is open, locally compact, σ-compact subset but not bounded by any compact set.

4. Introduction of *RCC* property and its significance

We start first with the following definition;

Definition 4.1. A space X is said to satisfy ***RCC*** property if the set of points having compact neighbourhood and the set of points having realcompact neighbourhood are same.

As for instance, locally compact spaces satisfy ***RCC*** property.

Theorem 4.2. *Every nearly pseudocompact space satisfies **RCC** property*

Proof. As $cl_{\beta X}(\upsilon X\backslash X)=cl_{\beta X}(\beta X\backslash X)$, $cl_{\beta X}(\upsilon X\backslash X)\cap X=cl_{\beta X}(\beta X\backslash X)\cap X$. But $cl_{\beta X}(\upsilon X\backslash X)\cap X$ and $cl_{\beta X}(\beta X\backslash X)\cap X$ are respectively the set of points in X having no realcompact and compact neighbourhoods. Hence the set of points having compact neighbourhood is identical with that of having realcompact neighbourhood. Hence every nearly pseudocompact space satisfies ***RCC*** property. □

In 2004, Aliabad et. al [14, Corollary 1.2] proved that for a locally compact Hausdorff space X, $C_\infty(X)$ is ideal of $C(X)$ if and only if X is pseudocompact space. In the year 1980, Henriksen and Rayburn, [10, Theorem 3.9], proved that regular closed almost locally compact subset of nearly pseudocompact space is pseudocompact which in turn implies that locally compact nearly pseudocompact space is pseudocompact. So we have the following theorem.

Theorem 4.3. *Under the assumption of locally compactness, ideal, nearly pseudocompact and pseudocompact spaces are identical.*

In the next theorem we shall show that under ***RCC*** condition, nearly pseudocompact and ideal spaces are same.

In [14], Aliabad et. al introduced an ideal $C_{l\sigma}(X):=\{f\in C(X):\operatorname{coz}f\text{is locally compact and }\sigma-compact\}$. $C_{l\sigma}(X)$ is a $z-ideal$ of $C(X)$ and by the result of [14, Proposition 3.2], $C_{l\sigma}(X)$ is the smallest z-ideal of $C(X)$ containing $C_\infty(X)$. Further we note that $C_{l\sigma}(X)\subseteq C_{RC}(X)$.

Theorem 4.4. *A space X is nearly pseudocompact if and only if X satisfies **RCC** property and $C_{l\sigma}(X)\subset C^*(X)$*

Proof. Let X be nearly pseudocompact. Then X satisfies ***RCC*** property by theorem 4.2 . Furthermore $H_\infty(X)\subseteq C^*(X)$, by theorem 2.5 (5). By theorem 3.3, $H_\infty(X)=C_{RC}(X)$. Thus $C_{l\sigma}(X)\subseteq C_{RC}(X)=H_\infty(X)\subseteq C^*(X)$.

Conversely, suppose X is not nearly pseudocompact space. Since X satisfies ***RCC*** property, then the set D_X of points with compact neighbourhood is non-empty, otherwise X would be anti-locally realcompact and hence nearly pseudocompact [10, Corollary 3.5]. Now for all $f \in H_\infty(X)$, each point of $X \backslash Z(f)$ has a realcompact neighbourhood [Theorem 3.3]. As the space satisfies $\boldsymbol{RCC}$ property, $\forall f \in H_\infty(X)$ we have $X \backslash Z(f) \subseteq D_X$. As X is not nearly pseudocompact, by theorem 2.5(5), there exists $f \in H_\infty(X)$ such that f is unbounded on $X \backslash Z(f)$. Thus there exists a copy N of $\mathbb{N}$ in $X \backslash Z(f)$, C-embedded in X, [7, corollary 1.20]. We consider a continuous function h on X so that $h(n) \gneqq n^3$, for all $n \in N$.

As N is C-embedded in X, N, hence any of its subset, is closed in X. So $cl_{\beta X} N \backslash N$ is contained in $\beta X \backslash X$. So $cl_{\beta X}(\beta X \backslash X) \cup \{m \in N : m \neq n\}$ is indeed a closed subsets of βX for each $n \in N$. Due to complete regularity, for each n , there exists a continuous function $\hat{g}_n : \beta X \to \mathbb{R}$, such that

$$\hat{g}_n(x) = \begin{cases} = 0 & \text{when } x \in cl_{\beta X}(\beta X \backslash X) \text{ or } x \in \{m : m \neq n\}. \\ = \frac{n^3}{h(n)} & \text{when } x = n \end{cases}$$

Without loss of generality, we assume $|\hat{g}_n| \leq 1$ as for each n, $\frac{n^3}{h(n)} \lneqq 1$.

We take as usual, $\hat{g} = \sum_n \frac{g_n}{n^2}$. Then $\hat{g} \in C(\beta X)$. Let $g := \hat{g}|_X$. Clearly $\hat{g}$ is the Stone-extension of g. As $\hat{g}$ vanishes everywhere on $\beta X \backslash X$, closure of $\{x \in X : |g(x)| \geq \frac{1}{n}\}$ in βX must not intersect in $\beta X \backslash X$. Hence $\{x \in X : |g(x)| \geq \frac{1}{n}\} = cl_{\beta X}\{x \in X : |g(x)| \geq \frac{1}{n}\}$ and is therefore compact. Thus $g \in C_\infty(X)$.

Now as $g \in C_\infty(X)$, $g \in C_{l\sigma}(X)$. Then $hg \in C_{l\sigma}(X)$. Moreover $hg(n) = n, \forall n$ and hence is unbounded. □

Corollary 4.5. *A space is nearly pseudocompact if and only if it is an ideal space satisfying* $\boldsymbol{RCC}$ *property.*

Proof. Every nearly pseudocompact space is ideal [Theorem 3.4] and satisfies $\boldsymbol{RCC}$ property [Theorem 4.2]. Conversely suppose X is an ideal space having $\boldsymbol{RCC}$ property, then let $f \in C_{l\sigma}(X)$. Then $X \backslash Z(f)$ is locally compact and σ-compact and hence relatively pseudocompact by theorem 3.2 and hence $C_{l\sigma}(X) \subset C^*(X)$. By the above theorem, X is nearly pseudocompact. □

Remark 4.6. Although it is evident from Corollary 4.5 that we can not drop the condition $\boldsymbol{RCC}$. However in support of the above corollary, we do refer the space given in example 3.5 that does not satisfy $\boldsymbol{RCC}$ property. But it is an ideal space which is not nearly pseudocompact. $(0,1) \times [0, \omega_1]$ is precisely the set of points which have compact neighbourhood. But each point of $X_1 \times [0, \omega_1]$ has realcompact neighbourhood X_1 is realcompact and $[0, \omega_1]$ is locally compact. Hence the space does not satisfy $\boldsymbol{RCC}$.

In the year 2001, Azarpanah and Soundararajan [Proposition 2.4, [2]], proved the following result.

Theorem 4.7 (Proposition 2.4, [2])**.** *For any space* X*, let* $C_\psi(X)$ *be the family of all real-valued continuous functions over* X *with pseudocompact support.*

Then $C_\infty(X)$ is subset of $C_\psi(X)$ if and only if every open locally compact subset of X is relatively pseudocompact.

In the year 2005, Henriksen and Mitra proved the following lemma.

Theorem 4.8 (Lemma 2.10, [9])**.** *A function $f \in C(X)$ is in $C_\psi(X)$ if and only if $fg \in C^*(X)$ whenever $g \in C(X)$.*

The following lemma directly follows as a corollary from theorems 3.2, 4.7 and 4.8 above.

Lemma 4.9. *A space X is ideal if and only if $\forall f \in C(X)$ and $\forall g \in C_\infty(X)$, $fg \in C^*(X)$.*

Proof. Suppose X is ideal. By theorem 3.2, every open locally compact subset of X is relatively pseudocompact. By theorem 4.7, $C_\infty(X) \subseteq C_\psi(X)$. Hence by theorem 4.8, for all $f \in C(X)$ and for all $g \in C_\infty(X)$, $fg \in C^*(X)$. Conversely suppose $\forall g \in C_\infty(X)$, $fg \in C^*(X)$. By theorem 4.8, we conclude that $g \in C_\psi(X)$. So $C_\infty(X) \subseteq C_\psi(X)$. By theorem 4.7, every open locally compact subset of X is relatively pseudocompact and hence by theorem 3.2, X is ideal. □

In the year 1990, Blair and Swardson [Proposition 2.6, [3]] proved the following result.

Theorem 4.10 (Proposition 2.6, [3])**.** *A subset A of X is relatively pseudocompact if and only $cl_{\upsilon X}\ A$ is compact.*

W.W. Comfort in his paper [4, Theorem 4.1] included the following result proved by Hager [16].

Theorem 4.11. *(Hager-Johnson) Let U be open subset of X. If $cl_{\upsilon X}U$ is compact, then $cl_X U$ is pseudocompact.*

The following lemma again trivially follows from theorems 4.10 and 4.11.

Lemma 4.12. *If U is an open relatively pseudocompact subset of X, then $cl_X U$ is pseudocompact.*

Theorem 4.13. *A space is ideal if and only if the closure of its local compactness part is pseudocompact.*

Proof. If X is ideal, then, the set D_X of points which have compact neighbourhoods is open and locally compact and hence relatively pseudocompact by [Theorem 1.3,[2]]. So $cl_X D_X$ is pseudocompact. Conversely, let U be an open locally compact subset of X. Then $U \subseteq D_X \subseteq cl_X D_X$. As $cl_X D_X$ is pseudocompact, U is relatively pseudocompact. Hence X is ideal as follows from theorem 3.2. □

We already mentioned in the introductory section that Henriksen and Rayburn in [10] did not give any example of nearly pseudocompact, which is not anti-locally realcompact. Here we shall produce an example of nearly pseudocompact space which is not anti-locally realcompact.

Example 4.14. We take any anti-locally realcompact space X. Attach $(0,0)$ of the Tychonoff plank with a point y (say) in X. The resulting space is not anti-locally realcompact as its local compact part is $T\setminus\{(0,0)\}$ which is also locally realcompact part; that is the space satisfies $\boldsymbol{RCC}$ property. Moreover its almost local compact part is T which is pseudocompact and hence an ideal space. This space is not even ∞-compact. But the above theorem 4.5 tells that this space is nearly pseudocompact.

5. Few Properties of ideal spaces

Theorem 5.1. *Regular closed subspace of an ideal space is ideal.*

Definition 5.2 (A.H. Stone, [15])**.** A space X is called feebly compact if every pairwise disjoint locally finite family of open sets of X is finite.

I. Glicksberg in [8] proved that in a completely regular space, the notion of feebly compact and pseudocompact are identical. In fact he proved that pseudocompact completely regular space is feebly compact and a feebly compact space is pseudocompact. In the same paper he has further shown that in a feebly compact space, closure of any open set is also feebly compact. So through chronological arguments, we conclude that within the class of completely regular spaces, regular closed subspace of a pseudocompact space is pseudocompact and hence,in particular, in a completely regular pseudocompact space, the closure of an open set is also pseudocompact.

Proof. Let X be an ideal space. A be a regular closed subspace of X. So $cl_X(int_X A) = A$. Let D_A be the set of points in A having compact neighborhood in A. So D_A is open subset of A. As $int_X A$ is dense in A, $D_A \cap int_X A \neq \varnothing$. But D_A being open in A, $D_A \cap int_X A$ open in $int_X A$ and hence it is open in X. Let $x \in D_A \cap int_X A$. As $x \in D_A$, $x \in U_x \subseteq K_x$,where U_x is open in A, $K_x \subseteq A$ is compact. Again $x \in int_X A$. So there exists W_x open in X such that $x \in W_x \subseteq A$. So $x \in U_x \cap W_x \subseteq U_x \subseteq K_x$. Now $U_x \cap W_x$ is open in X as W_x is open in X. So K_x is a compact neighbourhood of x in X. Thus $x \in D_X$, the set of all points in X having compact neighbourhood in X. So $D_A \cap int_X A \subseteq D_X$. As $int_X A$ is dense in A and D_A is open in A. So $D_A \subseteq cl_A(D_A \cap int_X A)$. But $cl_A(D_A \cap int_X A) = cl_X(D_A \cap int_X A)$. So $D_A \subseteq cl_X(D_A \cap int_X A) \subseteq cl_X D_X$.

By theorem 4.13, as X is ideal, $cl_X(D_X)$ is pseudocompact. We denote Ω for $cl_X D_X$. As $(D_A \cap int_X A)$ is open in X, $D_A \cap int_X A$ is open in Ω. Then $cl_\Omega(D_A \cap int_X A)$ is pseudocompact. But $cl_\Omega(D_A \cap int_X A) = cl_X(D_A \cap int_X A)$. So $cl_X(D_A \cap int_X A)$ is pseudocompact. Again $D_A \subseteq cl_X(D_A \cap int_X A) \subseteq A$. Now we track down the same argument again.

Let $W = cl_X(D_A \cap int_X A) \subseteq A$. As W is pseudocompact, D_A being open in A and $D_A \subseteq W$, D_A is open in W also. So $cl_W D_A$ is also pseudocompact. But $cl_W D_A = cl_A D_A$. So $cl_A D_A$ is pseudocompact. So by theorem 4.13, A is ideal. ☐

Theorem 5.3. *Every open C-embedded subspace of an ideal space is ideal*

Proof. Let U be a open C-embedded subset of X. Let $p \in D_U \subseteq U$. So p has a compact neighbourhood K in U. As U is open in X, K turns out to be compact neighbourhood of p in X. So $p \in D_X \cap U$. Conversely as $p \in D_X \cap U$, there exists a compact set K such that $p \in int_X K \subseteq K$. As $p \in U$, $p \in int_X K \cap U$, which is also open in X. Due to regularity, there exists an open set W in X such that $p \in W \subseteq cl_X W \subseteq int_X K \cap U$. Now W is also open in U and $cl_X W \subset K$ is also compact subset of U and is therefore compact neighbourhood of p in U. So $p \in D_U$. So $D_U = D_X \cap U$. Now as X is ideal, $cl_X D_X$ is pseudocompact. As any subset of pseudocompact space is relatively pseudocompact, D_U being subset of $cl_X D_X$ is relatively pseudocompact subset of X. As U is C-embedded in X, D_U is relatively pseudocompact subset of U also. Again D_U is open in U. Hence $cl_U D_U$ is pseudocompact, by above lemma 4.12. So U is ideal by theorem 4.13. □

Theorem 5.4. *Product of ideal and compact space is ideal. Conversely if $X \times Y$ is ideal, where Y compact, then X is ideal.*

Proof. Suppose X is ideal and Y is compact. Then $D_X \times Y = D_{X\times Y}$. So $cl_{X\times Y}(D_X \times Y) = cl_X D_X \times Y$ and hence is pseudocompact as $cl_X D_X$ and product of compact and pseudocompact space is pseudocompact. Second part follows immediately from the next theorem 5.6 and from the result that the projection on X from $X \times Y$, where Y is compact, is a perfect map. □

The following theorem is immediate.

Theorem 5.5. *Finite co-product of ideal spaces is ideal.*

Proof. Let X and Y be two ideal spaces. Then $cl_{X\coprod Y} D_{X\coprod Y} = cl_X D_X \cup cl_Y D_Y$ and hence is pseudocompact as $cl_X D_X$ and $cl_Y D_Y$ are pseudocompact. Hence $X \coprod Y$ is ideal. □

But the result may not be true for arbitrary co-product. For that we take a very simple example, say $\mathbb{N}$, the space of natural numbers with usual topology. Then $\mathbb{N}$ is indeed countable co-product of singletons. Every singleton is compact and hence ideal. But $\mathbb{N}$ is popularly known to be non-ideal space.

Theorem 5.6. *Let $f : X \to Y$ be a perfect map. If X is ideal, then Y is also ideal.*

Proof. We first note that $f^{-1}D_Y \subseteq D_X \subseteq cl_X D_X$. Hence $D_Y \subseteq f(cl_X D_X)$. Now X is ideal, $cl_X D_X$ is pseudocompact. As f is closed and preserves pseudocompactness, $f(cl_X D_X)$ is also pseudocompact. $cl_Y D_Y$ being regular closed subset of $f(cl_X D_X)$ is also pseudocompact. Hence by theorem 4.13, Y is ideal. □

Corollary 5.7. *If a space is not ideal, then so is its absolute.*

Proof. The corollary directly follows from the above theorem 5.6 as there always exist a perfect irreducible map from the absolute of a space onto the space itself. □

The next theorem 5.9 follows trivially from the following theorem 5.8 by Henriksen and Rayburn [10, Theorem 3.17].

Theorem 5.8. *If X and Y are nearly pseudocompact, then $X \times Y$ is nearly pseudocompact if and only if $cl_X D_X \times cl_Y D_Y$ is pseudocompact.*

Theorem 5.9. *If X and Y are nearly pseudocompact spaces, then $X \times Y$ is nearly pseudocompact if and only if $X \times Y$ is ideal.*

Proof. As $X \times Y$ is nearly pseudocompact, $X \times Y$ is also ideal. Conversely, if $X \times Y$ is ideal, then $cl_{X\times Y} D_{X\times Y}$, is pseudocompact. But $cl_{X\times Y} D_{X\times Y} = cl_X D_X \times cl_Y D_Y$. So $cl_X D_X \times cl_Y D_Y$ is pseudocompact. Hence $X \times Y$ is nearly pseudocompact. □

Theorem 5.10. *A realcompact, ideal space is ∞-compact.*

Proof. Since X is ideal, $cl_X D_X$ is pseudocompact. Since X is realcompact, $cl_X D_X$ is also realcompact. Hence it is compact. So clearly every open locally compact subset is bounded by a compact set. Hence X is ∞-compact. □

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