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A new topology over the primary-like spectrum of a module

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Abstract

Let R be a commutative ring with identity and M a unitary R-module. The primary-like spectrum $Spec_L(M)$ is the collection of all primary-like submodules Q of M, the recent generalization of primary ideals, such that M/Q is a primeful R-module. In this article, we topologies $Spec_L(M)$ with the patch-like topology, and show that when, $Spec_L(M)$ with the patch-like topology is a quasi-compact, Hausdorff, totally disconnected space.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. For a submodule N of M, we let $(N : M)$ denote the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $Ann(M)$, is the ideal $(0 : M)$. By a prime submodule (or a p-prime submodule) of M, we mean a proper submodule P with $p = (P : M)$ such that $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in p$. The prime spectrum (or simply, the spectrum) of M, denoted by $Spec(M)$, is the set of all prime submodules of M [1, 2, 3, 5]. The intersection of all prime submodules of M containing N is called the radical of N and denoted by $radN$. If there is no prime submodule containing N, then we define $radN = M$. As a new generalization of a primary ideal on the one hand and a generalization of a prime submodule on the other

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hand, a proper submodule Q of M is said to be primary-like if $rm \in Q$ implies $r \in (Q : M)$ or $m \in radQ$ ([6]). We say that a submodule N of an R-module M satisfies the primeful property if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P containing N such that $(P : M) = p$. If the zero submodule of M satisfies the primeful property, then M is called primeful. For instance finitely generated modules, projective modules over domains and (finite and infinite dimensional) vector spaces are primeful (see [9]). It is easy to see that, if Q is a primary-like submodule satisfying the primeful property, then $p = \sqrt{(Q : M)}$ is a prime ideal of R and so in this case, Q is called a p-primary-like submodule. The primary-like spectrum $Spec_L(M)$ is defined to be the set of all primary-like submodules of M satisfying the primeful property. If $Q \in Spec_L(M)$, since Q satisfies the primeful property, there exists a maximal ideal m of R and a prime submodule P containing Q such that $(P : M) = m$ and so $radQ \neq M$.

For any submodule N of M, let

$$\nu(N) = \{Q \in Spec_L(M) | \sqrt{(Q : M)} \supseteq \sqrt{(N : M)}\}.$$

Then we have the following lemma.

Lemma 1.1. *Let M be an R-module. Let N, N' and $\{N_i | i \in I\}$ be submodules of M. Then the following hold.*

1. $\nu(M) = \varnothing$.
2. $\nu(0) = Spec_L(M)$.
3. *If* $N \subseteq N'$, *then* $\nu(N') \subseteq \nu(N)$.
4. $\bigcap_{i \in I} \nu(N_i) = \nu(\sum_{i \in I} (N_i : M)M)$.
5. $\nu(N) \cup \nu(N') = \nu(N \cap N')$.
6. $\nu(radN) \subseteq \nu(N)$.
7. *If* $\sqrt{(N : M)} = \sqrt{(N' : M)}$, *then* $\nu(N) = \nu(N')$. *The converse is also true if both* N *and* N' *are primary-like.*
8. $\nu(N) = \nu(\sqrt{(N : M)}M)$.

Also, for each submodule N of M we denote the complement of $\nu(N)$ in $Spec_L(M)$ by $\mathcal{U}(N)$. From (1), (2), (4) and (5) above, the family $\eta(M) = \{\mathcal{U}(N) | N \leq M\}$ is closed under finite intersections and arbitrary unions. Moreover, we have $\mathcal{U}(M) = Spec_L(M)$ and $\mathcal{U}(0) = \varnothing$. Therefore, $\eta(M)$, as the family of all open sets, satisfy the axioms of a topology $\mathcal{T}$ on $Spec_L(M)$, called the Zariski topology on M.
In Section 2, we topologies $Spec_L(M)$ with a patch-like topology, and show that, if M is a Noetherian multiplication R-module and $(N : M)$ is a radical ideal for every submodule N of M, then $Spec_L(M)$ with the patch-like topology is a quasi-compact, Hausdorff, totally disconnected space (Corollary 2.16).

2. Main Results

We need to recall the patch topology (see [7, 8], for definition and more details). Let X be topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and Hausdorff (see [8]). Also, the patch topology associated to the Zariski topology of a ring R (not necessarily commutative) with ACC on ideals is compact and Hausdorff (see [7, Proposition 16.1]).

Definition 2.1. Let M be an R-module, and let $\omega(M)$ be the family of all subsets of $Spec_L(M)$ of the form $\nu(N)\cup\mathcal{U}(K)$ where $\nu(N)$ is any Zariski-closed subset of $Spec_L(M)$ and $\mathcal{U}(K)$ is a Zariski-quasi-compact subset of $Spec_L(M)$. Clearly $\omega(M)$ is closed under finite unions and contains $Spec_L(M)$ and the empty set, since $Spec_L(M)$ equals $\nu(0)\cup\mathcal{U}(0)$ and the empty set equals $\nu(M)\cup\mathcal{U}(0)$. Therefore $\omega(M)$ is basis for the family of closed sets of a topology on $Spec_L(M)$, and call it patch-like topology of M. Thus

$$\omega(M) = \{\nu(N)\cup\mathcal{U}(K) | N, K \leq M, \mathcal{U}(K) \text{ is Zariski-quasi-compact}\},$$

and hence we obtain the family

$$\Omega(M) = \{\nu(N)\cap\mathcal{U}(K) | N, K \leq M, \mathcal{U}(K) \text{ is Zariski-quasi-compact}\},$$

which is a basis for the open sets of the patch-like topology, i.e., the patch-like-open subsets of $Spec_L(M)$ are precisely the unions of sets from $\Omega(M)$. We denote the patch-like topology of $Spec_L(M)$ by $\mathcal{T}_p(M)$.

Definition 2.2. Let M be an R-module, and let $\tilde{\Omega}(M)$ be the family of all subsets of $Spec_L(M)$ of the form $\nu(N)\cap\mathcal{U}(K)$ where $N, K \leq M$. Clearly $\tilde{\Omega}(M)$ contains $Spec_L(M)$ and the empty set, since $Spec_L(M)$ equals $\nu(0)\cap\mathcal{U}(M)$ and the empty set equals $\nu(M)\cap\mathcal{U}(0)$. Let $\tilde{\mathcal{T}_p}(M)$ to be the collection $\tilde{U}$ of all unions of elements of $\tilde{\Omega}(M)$. Then $\tilde{\mathcal{T}_p}(M)$ is a topology on $Spec_L(M)$ and it is called the finer patch-like topology (in fact, $\tilde{\Omega}(M)$ is a basis for the finer patch-like topology of M).

We will use $\mathcal{X}$ to represent $Spec_L(M)$.

Lemma 2.3. *Let M be an R-module and $Q \in \mathcal{X}$. Then for each finer patch-like-neighborhood $\mathcal{W}$ of Q, there exists a submodule L of M such that $\sqrt{(Q:M)} \subseteq \sqrt{(L:M)}$ and $Q \in \nu(Q)\cap\mathcal{U}(L) \subseteq \mathcal{W}$.*

Proof. Since $Q \in \mathcal{W}$, there exists a neighborhood of the form $\nu(K)\cap\mathcal{U}(N) \subseteq \mathcal{W}$ such that $Q \in \nu(K)\cap\mathcal{U}(N)$ where $\sqrt{(Q:M)} \supseteq \sqrt{(K:M)}$ and $\sqrt{(Q:M)} \not\supseteq \sqrt{(N:M)}$. Since $Q \in \nu(Q)$ and $\nu(Q) \subseteq \nu(K)$, we may replace $\nu(K)$ by $\nu(Q)$. Now we claim that $\nu(Q)\cap\mathcal{U}(N) = \nu(Q)\cap\mathcal{U}((I+p)M)$, where $p = \sqrt{(Q:M)}$ and $I = \sqrt{(N:M)}$. Since $\mathcal{U}(IM) \subseteq \mathcal{U}((I+p)M)$,

$$\nu(Q)\cap\mathcal{U}(N) = \nu(Q)\cap\mathcal{U}(IM) \subseteq \nu(Q)\cap\mathcal{U}((I+p)M).$$

Suppose that $Q' \in \nu(Q)\cap\mathcal{U}((I+p)M)$, then $Q' \notin \mathcal{U}(Q)$. On the other hand $Q' \in \mathcal{U}((I+p)M) = \mathcal{U}(N)\cup\mathcal{U}(Q)$. This follows that $Q' \in \mathcal{U}(N)$. Thus $\nu(Q)\cap\mathcal{U}(N) = \nu(Q)\cap\mathcal{U}((I+p)M)$. Now let $L = (I+p)M$. Then $p \subseteq I+p \subseteq \sqrt{(L:M)}$ and $Q \in \nu(Q)\cap\mathcal{U}(L) \subseteq \mathcal{W}$. ☐

Let $\mathcal{Y}$ be a subset of $\mathcal{Y}$ for a module M. We will denote the closure of $\mathcal{Y}$ in $\mathcal{X}$ with finer patch-like topology by $\overline{\mathcal{Y}}$.

Proposition 2.4. *Let M be an R-module and $\mathcal{Y} \subseteq \mathcal{X}$ be a finite set. If $Q \in \mathcal{Y}$ with finer patch-like topology, then there exists $\mathcal{A} \subseteq \mathcal{Y}$ such that $\nu(Q) = \nu(\bigcap_{Q' \in \mathcal{A}} Q')$.*

Proof. Suppose $Q \in \overline{\mathcal{Y}}$. If $Q \in \mathcal{Y}$, then we are thorough. Thus we can assume that $Q \notin \mathcal{Y}$. Let $\mathcal{A} = \{Q' \in \mathcal{Y} | \sqrt{(Q:M)} \subset \sqrt{(Q':M)}\}$. Since $Q \in \mathcal{U}(M) \cap \nu(Q)$, there exists $Q'' \in \mathcal{Y}$ such that $Q'' \in \mathcal{U}(M) \cap \nu(Q)$. Since $Q \notin \mathcal{Y}$, $\sqrt{(Q:M)} \subset \sqrt{(Q'':M)}$ and hence $\mathcal{A} \neq \varnothing$. Since $\sqrt{(Q:M)} \subset \sqrt{(Q':M)}$ for each $Q' \in \mathcal{A}$,

$$\sqrt{(Q:M)} \subset \cap_{Q' \in \mathcal{A}} \sqrt{(Q':M)} = \sqrt{(\cap_{Q' \in \mathcal{A}} Q' : M)}.$$

If $\cap_{Q' \in \mathcal{A}} \sqrt{(Q':M)} \nsubseteq \sqrt{(Q:M)}$, then $Q \in \mathcal{U}(\cap_{Q' \in \mathcal{A}} Q') \cap \nu(Q)$. Since $Q \in \overline{\mathcal{Y}}$, there exists $Q'' \in \mathcal{Y}$ such that $Q'' \in \mathcal{U}(\cap_{Q' \in \mathcal{A}} Q') \cap \nu(Q)$. Therefore $Q'' \in \nu(Q)$ and hence $Q'' \in \mathcal{A}$. But $\bigcap_{Q' \in \mathcal{A}} \sqrt{(Q':M)} = \sqrt{(\bigcap_{Q' \in \mathcal{A}} Q' : M)} \subseteq \sqrt{(Q'':M)}$. Thus $Q'' \notin \mathcal{U}(\bigcap_{Q' \in \mathcal{A}} Q')$, a contradiction. Thus $\bigcap_{Q' \in \mathcal{A}} \sqrt{(Q':M)} \subseteq \sqrt{(Q:M)}$, and hence

$$\nu(Q) = \nu(\bigcap_{Q' \in \mathcal{A}} Q').$$

□

A module M over a commutative ring R is called a multiplication module if each submodule of M has the form IM for some ideal I of R [4]. In this case we can take $I = (N : M)$.

Proposition 2.5. *Let M be a multiplication R-module such that $(Q : M)$ is a radical ideal for every $Q \in \mathcal{X}$. Then $\mathcal{X}$ with the finer patch-like topology is Hausdorff. Moreover, $\mathcal{X}$ with this topology is totally disconnected.*

Proof. Assume $Q, Q' \in \mathcal{X}$ are distinct points. Since $Q \neq Q'$, $(Q : M) \neq (Q' : M)$. Thus either $(Q : M) \nsubseteq (Q' : M)$ or $(Q' : M) \nsubseteq (Q : M)$. Suppose that $(Q : M) \nsubseteq (Q' : M)$. By Definition 2.2, $\mathcal{U}_1 := \mathcal{U}(M) \cap \nu(Q)$ is a finer patch-like-neighborhood of Q and since $(Q : M) \nsubseteq (Q' : M)$, $\mathcal{U}_2 := \mathcal{U}(Q) \cap \nu(Q')$ is a finer patch-like-neighborhood of Q'. Clearly $\mathcal{U}(Q) \cap \nu(Q) = \varnothing$ and hence $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Thus $\mathcal{X}$ is a Hausdorff space. On the other hand for every submodule N of M, observer that the sets $\mathcal{U}(N)$ and $\nu(N)$ are open in finer patch-like topology, since $\nu(N) = \mathcal{U}(M) \cap \nu(N)$ and $\mathcal{U}(N) = \mathcal{U}(N) \cap \nu(0)$. Since $\mathcal{U}(N)$ and $\nu(N)$ are complement of each other, they are both finer both-closed as well. Therefore the finer patch-like topology on $\mathcal{X}$ has a basis of open sets which are also closed, and hence $\mathcal{X}$ is totally disconnected in this topology. □

The following example shows that the condition multiplication in Proposition 2.5 is necessary.

Example 2.6. Let V be a vector space over a field F with $\dim_F V > 1$. It is evident that $\mathcal{X}$ and $Spec(V)$ are the set of all proper vector subspaces of V. Now, $\sqrt{(Q : M)} = \sqrt{(Q' : M)}$ for all distinct subspaces $Q, Q' \in \mathcal{X}$. If $(Q : M)$ is a radical ideal for every $Q \in \mathcal{X}$, then $\mathcal{X}$ with the finer patch-like topology is not Hausdorff.

Definition 2.7. An R-module M is called p-module if for each prime ideal p of R such that $(pM : M) = p$, there exists $Q \in \mathcal{X}$ such that $\sqrt{(Q : M)} = p$.

For example every finitely generated faithful module is a p-module. Now we show that every Noetherian R-module M is also a p-module.
Let p be a prime ideal of a ring R, M an R-module, and $N \leq M$. By the saturation of N with respect to p, we mean the contraction of N_p in M and designate it by $S_p(N)$. It is also known that $S_p(N) = \{m \in M | rm \in N$ for some $r \in R\backslash p\}$. Saturations of submodules were investigated in detail in [10].

Lemma 2.8. *Let M be a Notherian R-module. Then M is a p-module.*

Proof. Assume M is a Notherian R-module. Hence M is finitely generated. By [11, Proposition 1.8], for each prime ideal p of R, $S_p(pM)$ is a prime submodule of M such that $(pM : M) = p$. Thus $S_p(pM) \in \mathcal{X}$. □

Theorem 2.9. *Let R be a ring and M be a p-module such that $R/Ann(M)$ has ACC on ideals. If $(N : M)$ is a radical ideal for every submodule N of M, then $\mathcal{X}$ with the finer patch-like topology is a quasi-compact space.*

Proof. Suppose M is a p-module and $R/Ann(M)$ has ACC on ideals. Assume $\mathcal{A}$ is a family of finer patch-like-open sets covering $\mathcal{X}$ and suppose that no finite subfamily of $\mathcal{A}$ covers $\mathcal{X}$. Suppose

$$\mathcal{S} = \{I | I \text{ is an ideal of } R \text{ such that } Ann(M) \subseteq I \text{ and no finite subfamily of } \mathcal{A} \text{ covers } \nu(IM)\}.$$

Since $\nu(Ann(M)M) = \nu(0) = \mathcal{X}$, $\mathcal{S} \neq \varnothing$. We may use the ACC on ideals of $R/Ann(M)$ to choose an ideal m of R maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $\nu(mM)$ (i.e., m is a maximal element of $\mathcal{S}$). It is clear that $mM \neq M$. We claim that m is a prime ideal of R, for if not, suppose that I and J are two ideals of R properly containing m and $IJ \subseteq m$. Then $\nu(IM)$ and $\nu(JM)$ covered by finite subfamily of $\mathcal{A}$. Suppose $Q \in \nu(IJM)$, then $IJ \subseteq p := \sqrt{(Q : M)}$. Since p is prime, either $I \subseteq p$ or $J \subseteq p$, and hence either $Q \in \nu(IM)$ or $Q \in \nu(JM)$. Thus $\nu(IJM)$ covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subseteq m$, then $\nu(mM) \subseteq \nu(IJM)$. Thus $\nu(mM)$ covered by finite subfamily of $\mathcal{A}$, a contradiction. Thus m is a prime ideal of R. We claim that $(mM : M) = m$, for if not, then there exists an ideal m_1 of R such that $m_1 = (mM : M)$ and $m \subset m_1$. This follows that $mM = m_1M$ and so no finite subfamily of $\mathcal{A}$ covers $\nu(m_1M)$, contrary to maximality of m. Therefore $(mM : M) = m$ and since M is p-module, there exists $Q' \in \mathcal{X}$ such that $\sqrt{(Q' : M)} = m$. Let $U \in \mathcal{A}$ such that $Q' \in U$. By Lemma 2.3, there exists a submodule K of M such that $m = \sqrt{(Q' : M)} \subseteq \sqrt{(K : M)}$ and

$Q' \in \mathcal{U}(K) \cap \nu(Q') \subseteq U$. Suppose $(K : M) = I$. By Lemma 1.1, we know that $\mathcal{U}(K) = \mathcal{U}(IM)$ and $\nu(Q') = \nu(mM)$, and so $Q' \in \mathcal{U}(IM) \cap \nu(mM) \subseteq U$. Since $m \subseteq I$, then $\nu(IM)$ can be covered by some finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But $\nu(mM)\backslash\nu(IM) = \nu(mM)\backslash[\mathcal{U}(IM)]^c = \nu(mM) \cap \mathcal{U}(IM) \subseteq U$ and so $\nu(mM)$ can be covered by $\mathcal{A}' \cup \{U\}$, contrary to our choice of Q'. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers $\mathcal{X}$. Therefore $\mathcal{X}$ is quasi-compact in the finer patch-like topology of M. □

It is well-known that if M is a Noetherian module over a ring R, then $R/Ann(M)$ is a Noetherian ring. Hence we have the following result.

Corollary 2.10. *Let M be a Noetherian R-module. If $(N : M)$ is a radical ideal for every submodule N of M, then $\mathcal{X}$ with the finer patch-like topology is a quasi-compact space.*

Proof. Using Lemma 2.8 and Theorem 2.9. □

We need the following evident lemma.

Lemma 2.11. *Let $\mathcal{T}, \mathcal{T}^*$ be two topology on $\mathcal{X}$ such that $\mathcal{T} \subseteq \mathcal{T}^*$. If $\mathcal{X}$ is quasi-compact in $\mathcal{T}^*$, then $\mathcal{X}$ is also quasi-compact in $\mathcal{T}$.*

Theorem 2.12. *Let M be an R-module. If $\mathcal{X}$ is quasi-compact with the finer patch-like topology, then for each submodule N of M, $\mathcal{U}(N)$ is a quasi-compact subset of $\mathcal{X}$ with the Zariski topology. Consequently, $\mathcal{X}$ with the Zariski topology is quasi-compact.*

Proof. By Definition 2.2, for each submodule N of M, $\nu(N) = \nu(N) \cap \mathcal{U}(M)$ is an open subset of $\mathcal{X}$ with finer patch-like topology, and hence, for each submodule N of $M, \mathcal{U}(N)$ is a closed subset in $\mathcal{X}$ with finer patch-like topology. Since every closed subset of a quasi-compact space is quasi-compact, $\mathcal{U}(N)$ is quasi-compact in $\mathcal{X}$ with finer patch-like topology and so by Lemma 2.11, it is quasi-compact in $\mathcal{X}$ with the Zariski topology. Now, since $\mathcal{X} = \mathcal{U}(M)$, $\mathcal{X}$ is quasi-compact with the Zariski topology. □

Corollary 2.13. *Let M be an R-module. If $\mathcal{X}$ is quasi-compact with finer patch-like topology, then the finer patch-like topology and the patch-like topology of M coincide.*

Proof. By Theorem 2.12, for each submodule K of $M, \mathcal{U}(K)$ is quasi-compact. Therefore for each $N, K \leq M$, $\nu(N) \cap \mathcal{U}(K)$ is an element of the basis $\Omega(M)$ of the patch-like topology on $\mathcal{X}$. □

Corollary 2.14. *Let M be an R-module such that $(N : M)$ is a radical ideal for every submodule N of M. If M is Noetherian or M is a p-module such that $R/Ann(M)$ has ACC on ideals, then the finer patch-like topology and the patch-like topology of M coincide.*

Proof. By Theorem 2.9 and Corollaries 2.10 and 2.13. □

We conclude this section with the following results.

Corollary 2.15. *Let M be a multiplication p-module such that $(N : M)$ is a radical ideal for every submodule N of M and $R/Ann(M)$ has ACC on ideals. Then $\mathcal{X}$ with the Zariski topology is a Hausdorff, quasi-compact, totally disconnected space.*

Proof. By Proposition 2.5, Theorem 2.9 and Corollary 2.13. □

Corollary 2.16. *Let M be a multiplication Noetherian R-module such that $(N : M)$ is a radical ideal for every submodule N of M. Then $\mathcal{X}$ with the Zariski topology is a Hausdorff, quasi-compact, totally disconnected space.*

Proof. By Lemma 2.8, Proposition 2.5, Theorem 2.9 and Corollary 2.13. □

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