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Fixed point sets in digital topology, 2

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Abstract

We continue the work of [10], studying properties of digital images determined by fixed point invariants. We introduce pointed versions of invariants that were introduced in [10]. We introduce freezing sets and cold sets to show how the existence of a fixed point set for a continuous self-map restricts the map on the complement of the fixed point set.

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1. Introduction

As stated in [10]:

> Digital images are often used as mathematical models of real-world objects. A digital model of the notion of a continuous function, borrowed from the study of topology, is often useful for the study of digital images. However, a digital image is typically a finite, discrete point set. Thus, it is often necessary to study digital images using methods not directly derived from topology. In this paper, we examine some properties of digital images concerned with the fixed points of digitally continuous functions; among these properties are discrete measures that are not natural analogues of properties of subsets of $\mathbb{R}^n$.

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In [10], we studied rigidity, pull indices, fixed point spectra for digital images and for digitally continuous functions, and related notions. In the current work, we study pointed versions of notions introduced in [10]. We also study such questions as when a set of fixed points $\mathrm{Fix}(f)$ determines that f is an identity function, or is "approximately" an identity function.

Some of the results in this paper were presented in [6].

2. Preliminaries

Much of this section is quoted or paraphrased from [10].

Let $\mathbb{N}$ denote the set of natural numbers; $\mathbb{N}^* = \{0\} \cup \mathbb{N}$, the set of nonnegative integers; and $\mathbb{Z}$, the set of integers. $\#X$ will be used for the number of members of a set X.

2.1. **Adjacencies.** A digital image is a pair (X, κ) where $X \subset \mathbb{Z}^n$ for some n and κ is an adjacency on X. Thus, (X, κ) is a graph for which X is the vertex set and κ determines the edge set. Usually, X is finite, although there are papers that consider infinite X. Usually, adjacency reflects some type of "closeness" in $\mathbb{Z}^n$ of the adjacent points. When these "usual" conditions are satisfied, one may consider the digital image as a model of a black-and-white "real world" image in which the black points (foreground) are represented by the members of X and the white points (background) by members of $\mathbb{Z}^n \setminus \{X\}$.

We write $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when κ is understood or when it is unnecessary to mention κ, to indicate that x and y are κ-adjacent. Notations $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ is understood, indicate that x and y are κ-adjacent or are equal.

The most commonly used adjacencies are the c_u adjacencies, defined as follows. Let $X \subset \mathbb{Z}^n$ and let $u \in \mathbb{Z}$, $1 \le u \le n$. Then for points

$$x = (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) = y$$

we have $x \leftrightarrow_{c_u} y$ if and only if

- for at most u indices i we have $|x_i - y_i| = 1$, and
- for all indices j, $|x_j - y_j| \neq 1$ implies $x_j = y_j$.

The c_u-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. E.g.,

- in $\mathbb{Z}$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 8-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

The literature also contains several adjacencies to exploit properties of Cartesian products of digital images. These include the following.

Definition 2.1 ([1])**.** Let (X, κ) and (Y, λ) be digital images. The *normal product adjacency* or *strong adjacency* on $X \times Y$, $NP(\kappa, \lambda)$, is defined as follows. Given $x_0, x_1 \in X$, $y_0, y_1 \in Y$ such that

$$p_0 = (x_0, y_0) \neq (x_1, y_1) = p_1,$$

we have $p_0 \leftrightarrow_{NP(\kappa,\lambda)} p_1$ if and only if one of the following is valid:

- $x_0 \leftrightarrow_\kappa x_1$ and $y_0 = y_1$, or
- $x_0 = x_1$ and $y_0 \leftrightarrow_\lambda y_1$, or
- $x_0 \leftrightarrow_\kappa x_1$ and $y_0 \leftrightarrow_\lambda y_1$.

Building on the normal product adjacency, we have the following.

Definition 2.2 ([4])**.** Given $u, v \in \mathbb{N}$, $1 \le u \le v$, and digital images (X_i, κ_i), $1 \le i \le v$, let $X = \Pi_{i=1}^v X_i$. The adjacency $NP_u(\kappa_1, \ldots, \kappa_v)$ for X is defined as follows. Given $x_i, x_i' \in X_i$, let

$$p = (x_1, \ldots, x_v) \neq (x_1', \ldots, x_v') = q.$$

Then $p \leftrightarrow_{NP_u(\kappa_1,\ldots,\kappa_v)} q$ if for at least 1 and at most u indices i we have $x_i \leftrightarrow_{\kappa_i} x_i'$ and for all other indices j we have $x_j = x_j'$.

Notice $NP(\kappa, \lambda) = NP_2(\kappa, \lambda)$ [4].

Let $x \in (X, \kappa)$. We use the notations

$$N(x) = N_\kappa(x) = \{y \in X \mid y \leftrightarrow_\kappa x\}$$

and

$$N^*(x) = N_\kappa^*(x) = N_\kappa(x) \cup \{x\}.$$

2.2. **Digitally continuous functions.** We denote by id or id_X the identity map $\mathrm{id}(x) = x$ for all $x \in X$.

Definition 2.3 ([16, 3])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-*continuous*, or *digitally continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y. If $(X, \kappa) = (Y, \lambda)$, we say a function is κ-*continuous* to abbreviate "(κ, κ)-continuous."

Theorem 2.4 ([3])**.** *A function $f : X \to Y$ between digital images (X, κ) and (Y, λ) is (κ, λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

Theorem 2.5 ([3])**.** *Let $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (Z, \mu)$ be continuous functions between digital images. Then $g \circ f : (X, \kappa) \to (Z, \mu)$ is continuous.*

It is common to use the term *path* with the following distinct but related meanings.

- A path from x to y in a digital image (X, κ) is a set $\{x_i\}_{i=0}^m \subset X$ such that $x_0 = x$, $x_m = y$, and $x_i \leftrightarroweq_\kappa x_{i+1}$ for $i = 0, 1, \ldots, m-1$. If the x_i are distinct, then m is the *length* of this path.
- A path from x to y in a digital image (X, κ) is a $(2, \kappa)$-continuous function $P : [0, m]_\mathbb{Z} \to X$ such that $P(0) = x$ and $P(m) = y$. Notice that in this usage, $\{P(0), \ldots, P(m)\}$ is a path in the previous sense.

Definition 2.6 ([3]; see also [14])**.** Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $h : X \times [0, m]_\mathbb{Z} \to Y$ such that

- for all $x \in X$, $h(x, 0) = f(x)$ and $h(x, m) = g(x)$;
- for all $x \in X$, the induced function $h_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$h_x(t) \;=\; h(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

  is $(2, \kappa')-$continuous. That is, h_x is a path in Y.
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $h_t : X \to Y$ defined by

$$h_t(x) \;=\; h(x, t) \text{ for all } x \in X$$

  is $(\kappa, \kappa')-$continuous.

Then h is a *digital* $(\kappa, \kappa')-$*homotopy between* f and g, and f and g are *digitally* $(\kappa, \kappa')-$*homotopic in* Y, denoted $f \sim_{\kappa,\kappa'} g$ or $f \sim g$ when κ and κ' are understood. If $(X, \kappa) = (Y, \kappa')$, we say f and g are κ-*homotopic* to abbreviate "(κ, κ)-homotopic" and write $f \sim_\kappa g$ to abbreviate "$f \sim_{\kappa,\kappa} g$". If further $h(x, t) = x$ for all $t \in [0, m]_{\mathbb{Z}}$, we say h *holds* x *fixed.*

If there exists $x_0 \in X$ such that $f(x_0) = g(x_0) = y_0 \in Y$ and $h(x_0, t) = y_0$ for all $t \in [0, m]_{\mathbb{Z}}$, then h is a *pointed homotopy* and f and g are *pointed homotopic* [3].

If there exist continuous $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (X, \kappa)$ such that $g \circ f \sim_{\kappa,\kappa} \mathrm{id}_X$ and $f \circ g \sim_{\lambda,\lambda} \mathrm{id}_Y$, then (X, κ) and (Y, λ) are *homotopy equivalent.*

If there is a κ-homotopy between id_X and a constant map, we say X is κ-*contractible*, or just *contractible* when κ is understood.

Theorem 2.7 ([4])**.** *Let (X_i, κ_i) and (Y_i, λ_i) be digital images, $1 \le i \le v$. Let $f_i : X_i \to Y_i$. Then the product map $f : \prod_{i=1}^v X_i \to \prod_{i=1}^v Y_i$ defined by*

$$f(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v))$$

for $x_i \in X_i$, is $(NP_v(\kappa_1, \ldots, \kappa_v), NP_v(\lambda_1, \ldots, \lambda_v))$-continuous if and only if each f_i is (κ_i, λ_i)-continuous.

Definition 2.8. Let $A \subset X$. A κ-continuous function $r : X \to A$ is a *retraction*, and A *is a retract of* X, if $r(a) = a$ for all $a \in A$. If such a map r satisfies $i \circ r \sim_\kappa \mathrm{id}_X$ where $i : A \to X$ is the inclusion map, then A is a κ-*deformation retract* of X.

A function $f : (X, \kappa) \to (Y, \lambda)$ is an *isomorphism* (called a *homeomorphism* in [2]) if f is a continuous bijection such that f^{-1} is continuous.

We use the following notation. For a digital image (X, κ),

$$C(X, \kappa) = \{f : X \to X \mid f \text{ is continuous}\}.$$

Given $f \in C(X, \kappa)$, a point $x \in X$ is a *fixed point of* f if $f(x) = x$. We denote by $\mathrm{Fix}(f)$ the set $\{x \in X \mid x \text{ is a fixed point of } f\}$. If $x \in X \setminus \mathrm{Fix}(f)$, we say f *moves* x.

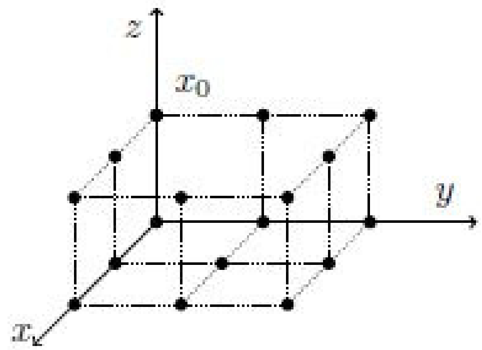

Figure 1. (Figure 1 of [9].) The image X discussed in Example 3.1. The coordinates are ordered according to the axes in this figure.

3. Rigidity and reducibility

A function $f : (X, \kappa) \to (Y, \lambda)$ is *rigid* [10] when no continuous map is homotopic to f except f itself. When the identity map $\mathrm{id} : X \to X$ is rigid, we say X is rigid [12]. If $f : X \to Y$ with $f(x_0) = y_0$, then f is *pointed rigid* [12] if no continuous map is pointed homotopic to f other than f itself. When the identity map $\mathrm{id} : (X, x_0) \to (X, x_0)$ is pointed rigid, we say (X, x_0) is pointed rigid.

Rigid maps and digital images are discussed in [12, 10].

Clearly, a rigid map is pointed rigid, and a rigid digital image is pointed rigid. (Note these assertions may seem counterintuitive as, e.g., pointed homotopic functions are homotopic, but the converse is not always true.) We show in the following that the converses of these assertions are not generally true.

Example 3.1 ([9]). Let $X = ([0, 2]_{\mathbb{Z}}^2 \times [0, 1]_{\mathbb{Z}}) \setminus \{(1, 1, 1)\}$. Let $x_0 = (0, 0, 1) \in X$. See Figure 1. It was shown in [9] that X is 6-contractible (i.e., c_1-contractible) but (X, x_0) is not pointed 6-contractible. The proof of the latter uses an argument that is easily modified to show that any homotopy of id_X that moves some point must move x_0. It follows that id_X is not rigid but is x_0-pointed rigid, i.e., that X is not c_1-rigid but (X, x_0) is c_1-pointed rigid.

Definition 3.2 ([12]). A finite image X is *reducible* if it is homotopy equivalent to an image of fewer points. Otherwise, we say X is *irreducible*.

Lemma 3.3 ([12]). *A finite image X is reducible if and only if id_X is homotopic to a nonsurjective map.*

Let (X, κ) be reducible. By Lemma 3.3, there exist $x \in X$ and $f \in C(X, \kappa)$ such that $\mathrm{id}_X \simeq_\kappa f$ and $x \notin f(X)$. We will call such a point a *reduction point*.

In Lemma 3.4 below, we have changed the notation of [12], since the latter paper uses the notation "$N(x)$" for what we call "$N^*(x)$" or "$N_\kappa^*(x)$".

Lemma 3.4 ([12])**.** *If there exist distinct $x, y \in X$ so that $N^*(x) \subset N^*(y)$, then X is reducible. In particular, x is a reduction point of X, and $X \setminus \{x\}$ is a deformation retract of X.*

Remark 3.5 ([12]). A finite rigid image is irreducible.

Theorem 3.6. *Let (X, c_2) be a digital image in $\mathbb{Z}^2$. Suppose there exists $x_0 \in X$ such that $N_{c_2}(x_0)$ is c_2-connected and $\#N_{c_2}(x_0) \in \{1, 2, 3\}$. Then (X, c_2) is reducible.*

Proof. We first show that in all cases, there exists $y \in N_{c_2}(x_0)$ such that $N^*_{c_2}(x_0) \subset N^*_{c_2}(y)$.

(1) Suppose $\#N_{c_2}(x_0) = 1$. Then there exists $y \in X$ such that $\{y\} = N_{c_2}(x_0)$. Clearly, then, $N^*_{c_2}(x_0) \subset N^*_{c_2}(y)$.
(2) Suppose $\#N_{c_2}(x_0) = 2$. Then there exist distinct $y, y' \in X$ such that $\{y, y'\} = N_{c_2}(x_0)$, which by hypothesis is connected. Therefore, $\{x_0, y'\} \subset N_{c_2}(y)$, so $N^*_{c_2}(x_0) \subset N^*_{c_2}(y)$.
(3) Suppose $\#N_{c_2}(x_0) = 3$. Then there exist distinct $y, y_0, y_1 \in X$ such that $\{y, y_0, y_1\} = N_{c_2}(x_0)$, which by hypothesis is connected. Therefore, one of the members of $N_{c_2}(x_0)$, say, y, is adjacent to the other two. Thus, $\{x_0, y_0, y_1\} \subset N_{c_2}(y)$, so $N^*_{c_2}(x_0) \subset N^*_{c_2}(y)$.

In all cases we have $N^*_{c_2}(x_0) \subset N^*_{c_2}(y)$. The assertion follows from Lemma 3.4. □

Remark 3.7. If instead we use the c_1-adjacency, the analog of the previous theorem is simpler, since if (X, c_1) is a digital image in $\mathbb{Z}^2$ and $x_0 \in X$ such that $N_{c_1}(x_0)$ is nonempty and c_1-connected, then $\#N_{c_1}(x_0) = 1$. This case is similar to the case $\#N_{c_2}(x_0) = 1$ of Theorem 3.6 above, so (X, c_1) is reducible.

4. Pointed homotopy fixed point spectrum

In this section, we define pointed versions of the *homotopy fixed point spectrum* of $f \in C(X, \kappa)$ and the *fixed point spectrum* of a digital image (X, κ).

Definition 4.1. Let (X, κ) be a digital image.

- [10] Given $f \in C(X, \kappa)$, the *homotopy fixed point spectrum of f* is
$$S(f) = \{\# \operatorname{Fix}(g) \mid g \sim_\kappa f\}.$$
- Given $f \in C(X, \kappa)$ and $x_0 \in \operatorname{Fix}(f)$, the *pointed homotopy fixed point spectrum of f* is
$$S(f, x_0) = \{\# \operatorname{Fix}(g) \mid g \sim_\kappa f \text{ holding } x_0 \text{ fixed}\}.$$

Definition 4.2. Let (X, κ) be a digital image.

- [10] The *fixed point spectrum* of (X, κ) is
$$F(X) = F(X, \kappa) = \{\# \operatorname{Fix}(f) \mid f \in C(X, \kappa)\}.$$
- Given $x_0 \in X$, the *pointed fixed point spectrum* of (X, κ, x_0) is
$$F(X, x_0) = F(X, \kappa, x_0) = \{\# \operatorname{Fix}(f) \mid f \in C(X, \kappa), x_0 \in \operatorname{Fix}(f)\}.$$

Theorem 4.3 ([10])**.** *Let A be a retract of (X, κ). Then $F(A) \subseteq F(X)$.*

The argument used to prove Theorem 4.3 is easily modified to yield the following.

Theorem 4.4. *Let (A, κ, x_0) be a retract of (X, κ, x_0). Then $F(A, \kappa, x_0) \subseteq F(X, \kappa, x_0)$.*

Theorem 4.5 ([10])**.** *Let $X = [1, a]_{\mathbb{Z}} \times [1, b]_{\mathbb{Z}}$. Let $\kappa \in \{c_1, c_2\}$. Then*

$$S(\mathrm{id}_X, \kappa) = F(X, \kappa) = \{i\}_{i=0}^{ab}.$$

Example 4.6. Consider the pointed digital image (X, c_1, x_0) of Example 3.1. Since $f \in C(X, c_1)$ and $x_0 \in \mathrm{Fix}(f)$ imply $f = \mathrm{id}_X$,

$$S(\mathrm{id}_X, c_1, x_0) = \{\#X\} = \{17\}.$$

However, (X, c_1) is not rigid. It is easily seen that there is a c_1-deformation retraction of X to $\{(x, y, 0) \in X\}$, which is isomorphic to $[1, 3]_{\mathbb{Z}}^2$. It follows from Theorem 4.3 and Theorem 4.5 that $\{i\}_{i=0}^{9} \subset S(\mathrm{id}_X)$. Since every $f \in C(X, c_1)$ such that $f \simeq_{c_1} \mathrm{id}_X$ and $f \neq \mathrm{id}_X$ moves every point q of X such that $p_3(q) = 1$, it follows easily that

$$S(\mathrm{id}_X, c_1) = F(X, c_1) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 17\}.$$

5. Freezing sets

In this section, we consider subsets of $\mathrm{Fix}(f)$ that determine that $f \in C(X, \kappa)$ must be the identity function id_X. Interesting questions include what properties such sets have, and how small they can be.

In classical topology, given a connected set $X \subset \mathbb{R}^n$ and a continuous self-map f on X, knowledge of a finite subset A of the fixed points of f rarely tells us much about the behavior of f on $X \setminus A$. By contrast, we see in this section that knowledge of a subset of the fixed points of a continuous self-map f on a digital image can completely characterize f as an identity map.

5.1. **Definition and basic properties.**

Definition 5.1. Let (X, κ) be a digital image. We say $A \subset X$ is a *freezing set for X* if given $g \in C(X, \kappa)$, $A \subset \mathrm{Fix}(g)$ implies $g = \mathrm{id}_X$.

Theorem 5.2. *Let (X, κ) be a digital image. Let $A \subset X$. The following are equivalent.*

1. *A is a freezing set for X.*
2. *id_X is the unique extension of id_A to a member of $C(X, \kappa)$.*
3. *For every isomorphism $F : X \to (Y, \lambda)$, if $g : X \to Y$ is (κ, λ)-continuous and $F|_A = g|_A$, then $g = F$.*
4. *Any continuous $g : A \to Y$ has at most one extension to an isomorphism $\bar{g} : X \to Y$.*

Proof. 1) $\Leftrightarrow$ 2): This follows from Definition 5.1.

1) $\Rightarrow$ 3): Suppose A is a freezing set for X. Let $F : X \to Y$ be a (κ, λ)-isomorphism. Let $g : X \to Y$ be (κ, λ)-continuous, such that $g|_A = F|_A$. Then

$$F^{-1} \circ g|_A = F^{-1} \circ F|_A = \mathrm{id}_X|_A = \mathrm{id}_A .$$

Since the composition of digitally continuous functions is continuous, it follows by hypothesis that $F^{-1} \circ g = \mathrm{id}_X$, and therefore that

$$g = F \circ (F^{-1} \circ g) = F \circ \mathrm{id}_X = F.$$

3) $\Rightarrow$ 1): Suppose for every isomorphism $F : X \to (Y, \lambda)$, if $g : X \to Y$ is (κ, λ)-continuous and $F|_A = g|_A$, then $g = F$. For $g \in C(X, \kappa)$, $A \subset \mathrm{Fix}(g)$ implies $g|_A = \mathrm{id}_X|_A$, so since id_X is an isomorphism, $g = \mathrm{id}_X$.

3) $\Rightarrow$ 4): This is elementary.

4) $\Rightarrow$ 2): This follows by taking g to be the inclusion of A into X, which extends to id_X. $\square$

Freezing sets are topological invariants in the sense of the following.

Theorem 5.3. *Let A be a freezing set for the digital image (X, κ) and let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. Then $F(A)$ is a freezing set for (Y, λ).*

Proof. Let $g \in C(Y, \lambda)$ such that $g|_{F(A)} = \mathrm{id}_Y|_{F(A)}$. Then

$$g \circ F|_A = g|_{F(A)} \circ F|_A = \mathrm{id}_Y|_{F(A)} \circ F|_A = F|_A.$$

By Theorem 5.2, $g \circ F = F$. Thus

$$g = (g \circ F) \circ F^{-1} = F \circ F^{-1} = \mathrm{id}_Y .$$

By Definition 5.1, $F(A)$ is a freezing set for (Y, λ). $\square$

We will use the following.

Proposition 5.4 ([10])**.** *Let (X, κ) be a digital image and $f \in C(X, \kappa)$. Suppose $x, x' \in \mathrm{Fix}(f)$ are such that there is a unique shortest κ-path P in X from x to x'. Then $P \subset \mathrm{Fix}(f)$.*

Let $p_i : \mathbb{Z}^n \to \mathbb{Z}$ be the projection to the i^{th} coordinate: $p_i(z_1, \ldots, z_n) = z_i$.

The following assertion can be interpreted to say that in a c_u-adjacency, a continuous function that moves a point p also moves a point that is "behind" p. E.g., in $\mathbb{Z}^2$, if q and q' are c_1- or c_2-adjacent with q left, right, above, or below q', and a continuous function f moves q to the left, right, higher, or lower, respectively, then f also moves q' to the left, right, higher, or lower, respectively.

Lemma 5.5. *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \le u \le n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.*

(1) *If $p_i(f(q)) > p_i(q) > p_i(q')$ then $p_i(f(q')) > p_i(q')$.*
(2) *If $p_i(f(q)) < p_i(q) < p_i(q')$ then $p_i(f(q')) < p_i(q')$.*

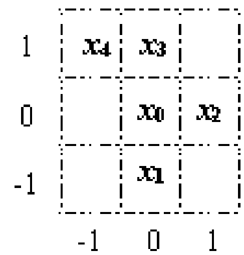

FIGURE 2. Illustration for Example 5.9

Proof. (1) Suppose $p_i(f(q)) > p_i(q) > p_i(q')$. Since $q \leftrightarrow_{c_u} q'$, if $p_i(q) = m$ then $p_i(q') = m - 1$. Then $p_i(f(q)) > m$. By continuity of f, we must have $f(q') \leftrightarroweq_{c_u} f(q)$, so $p_i(f(q')) \geq m > p_i(q')$.

(2) This case is proven similarly. □

Theorem 5.6. *Let (X, κ) be a digital image. Let X' be a proper subset of X that is a retract of X. Then X' does not contain a freezing set for (X, κ).*

Proof. Let $r : X \to X'$ be a retraction. Then $f = i \circ r \in C(X, \kappa)$, where $i : X' \to X$ is the inclusion map. Then $f|_{X'} = \mathrm{id}_{X'}$, but $f \neq \mathrm{id}_X$. The assertion follows. □

Corollary 5.7. *Let (X, κ) be a reducible digital image. Let x be a reduction point for X. Let A be a freezing set for X. Then $x \in A$.*

Proof. Since x is a reduction point for X, by Lemma 3.4, there is a retraction $r : X \to X \setminus \{x\}$. It follows that $X \setminus \{x\}$ does not contain a freezing set for (X, κ). □

Proposition 5.8. *Let (X, c_2) be a connected digital image in $\mathbb{Z}^2$. Suppose $x_0 \in X$ is such that $N_{c_2}(x_0)$ is connected and $\#N_{c_2}(x_0) \in \{1, 2, 3\}$. If A is a freezing set for (X, c_2), then $x_0 \in A$.*

Proof. By the proof of Theorem 3.6, we can use Lemma 3.4 to conclude that x_0 is a reduction point. The assertion follows from Corollary 5.7. □

Proposition 5.8 cannot in general be extended to permit $\#N_{c_2}(x_0) = 4$, as shown in the following.

Example 5.9. Let $X = \{x_i\}_{i=0}^4 \subset \mathbb{Z}^2$, where

$$x_0 = (0, 0),\ x_1 = (0, -1),\ x_2 = (1, 0),\ x_3 = (0, 1),\ x_4 = (-1, 1).$$

See Figure 2. Then $N_{c_2}(x_0)$ is c_2-connected and $\#N_{c_2}(x_0) = 4$. It is easily seen that $X \setminus \{x_0\}$ is a freezing set for (X, c_2).

5.2. **Boundaries and freezing sets.** For any digital image (X,κ), clearly X is a freezing set. An interesting question is how small $A \subset X$ can be for A to be a freezing set for X. We say a freezing set A is *minimal* if no proper subset of A is a freezing set for X.

Definition 5.10. Let $X \subset \mathbb{Z}^n$.

- The *boundary of* X [15] is

  $Bd(X) = \{x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_1} x\}$.
- The *interior of* X is $int(X) = X \setminus Bd(X)$.

Proposition 5.11. *Let $a < b, [a,b]_{\mathbb{Z}} \subset [c,d]_{\mathbb{Z}}$, and let $f : [a,b]_{\mathbb{Z}} \to [c,d]_{\mathbb{Z}}$ be c_1-continuous.*

- *If $\{a,b\} \subset \operatorname{Fix}(f)$, then $[a,b]_{\mathbb{Z}} = \operatorname{Fix}(f)$.*
- *$Bd([a,b]_{\mathbb{Z}}) = \{a,b\}$ is a minimal freezing set for $[a,b]_{\mathbb{Z}}$.*

Proof. If $[a,b]_{\mathbb{Z}} \neq \operatorname{Fix}(f)$, then we have at least one of the following:

- For some smallest t_0 satisfying $a < t_0 < b$, $f(t_0) > t_0$. But then $f(t_0 - 1) \leq t_0 - 1$, so $f(t_0-1) \not\leftrightarrows_{c_1} f(t_0)$, contrary to the continuity of f.
- For some largest t_1 satisfying $a < t_1 < b$, $f(t_1) < t_1$. But then $f(t_1 + 1) \geq t_1 + 1$, so $f(t_1+1) \not\leftrightarrows_{c_1} f(t_1)$, contrary to the continuity of f.

It follows that $f|_{[a,b]_{\mathbb{Z}}}$ is an inclusion function, as asserted.

By taking $[c,d]_{\mathbb{Z}} = [a,b]_{\mathbb{Z}}$ and considering all $f \in C([a,b]_{\mathbb{Z}}, c_1)$ such that $\{a,b\} \subset \operatorname{Fix}(f)$, we conclude that $\{a,b\}$ is a freezing set for $[a,b]_{\mathbb{Z}}$.

To establish minimality, observe that the proper nonempty subsets B of $\{a,b\}$ allow constant functions c that are c_1-continuous non-identities with $c|_B = \mathrm{id}_B$. □

Proposition 5.12. *Let $X \subset \mathbb{Z}^n$ be finite. Let $1 \leq u \leq n$. Let $A \subset X$. Let $f \in C(X, c_u)$. If $Bd(A) \subset \operatorname{Fix}(f)$, then $A \subset \operatorname{Fix}(f)$.*

Proof. By hypothesis, it suffices to show $int(A) \subset \operatorname{Fix}(f)$. Let $x = (x_1, \ldots, x_n) \in int(A)$. Suppose, in order to obtain a contradiction, $x \notin \operatorname{Fix}(f)$. Then for some index j,

$$p_j(f(x)) \neq x_j. \tag{5.1}$$

Since X is finite, there exists a path $P = \{y_i = (x_1, \ldots, x_{j-1}, a_i, x_{j+1} \ldots, x_n)\}_{i=1}^m$ in X such that $a_1 < x_j < a_m$ and $a_{i+1} = a_i + 1$; $y_1, y_m \in Bd(A)$; and $\{y_i\}_{i=2}^{m-1} \subset int(A)$. Note $x \in P$. Now, (5.1) implies either $p_j(f(x)) < x_j$ or $p_j(f(x)) > x_j$. If the former, then by Lemma 5.5, $y_m \notin \operatorname{Fix}(f)$; and if the latter, then by Lemma 5.5, $y_1 \notin \operatorname{Fix}(f)$; so in either case, we have a contradiction. We conclude that $x \in \operatorname{Fix}(f)$. The assertion follows. □

Theorem 5.13. *Let $X \subset \mathbb{Z}^n$ be finite. Then for $1 \leq u \leq n$, $Bd(X)$ is a freezing set for (X, c_u).*

Proof. The assertion follows from Proposition 5.12. □

Without the finiteness condition used in Proposition 5.12 and in Theorem 5.13, the assertions would be false, as shown in the following.

Example 5.14. Let $X = \{(x,y) \in \mathbb{Z}^2 \mid y \geq 0\}$. Consider the function $f: X \to X$ defined by

$$f(x,y) = \begin{cases} (x,0) & \text{if } y = 0; \\ (x+1,y) & \text{if } y > 0. \end{cases}$$

Then $f \in C(X,c_2)$, $Bd(X) = \mathbb{Z} \times \{0\}$, and $f|_{Bd(X)} = \text{id}_{Bd(X)}$, but $X \not\subset \text{Fix}(f)$, so $Bd(X)$ is not a c_2-freezing set for X.

5.3. **Digital cubes and c_1.** In this section, we consider freezing sets for digital cubes using the c_1 adjacency.

Theorem 5.15. *Let $X = \Pi_{i=1}^n [0,m_i]_{\mathbb{Z}}$. Let $A = \Pi_{i=1}^n \{0,m_i\}$.*

- *Let $Y = \Pi_{i=1}^n [a_i,b_i]_{\mathbb{Z}}$ be such that $[0,m_i] \subset [a_i,b_i]_{\mathbb{Z}}$ for all i. Let $f: X \to Y$ be c_1-continuous. If $A \subset \text{Fix}(f)$, then $X \subset \text{Fix}(f)$.*
- *A is a freezing set for (X,c_1); minimal for $n \in \{1,2\}$.*

Proof. The first assertion has been established for $n = 1$ at Proposition 5.11. We can regard this as a base case for an argument based on induction on n, and we now assume the assertion is established for $n \leq k$ where $k \geq 1$.

Now suppose $n = k+1$ and $f: X \to Y$ is c_1-continuous with $A \subset \text{Fix}(f)$. Let

$$X_0 = \Pi_{i=1}^k [0,m_i]_{\mathbb{Z}} \times \{0\}, \quad X_1 = \Pi_{i=1}^k [0,m_i]_{\mathbb{Z}} \times \{m_{k+1}\}.$$

We have that $f|_{X_0}$ and $f|_{X_1}$ are c_1-continuous, $A \cap X_0 \subset \text{Fix}(f|_{X_0})$, and $A \cap X_1 \subset \text{Fix}(f|_{X_1})$. Since X_0 and X_1 are isomorphic to k-dimensional digital cubes, by Theorem 5.3 and the inductive hypothesis, we have

$$\left(\Pi_{i=1}^k [0,m_i]_{\mathbb{Z}} \times \{0\}\right) \cup \left(\Pi_{i=1}^k [0,m_i]_{\mathbb{Z}} \times \{m_n\}\right) \subset \text{Fix}(f).$$

Then given $x = (x_1,\ldots,x_n) \in X$, x is a member of the unique shortest c_1-path $\{(x_1,x_2,\ldots,x_k,t)\}_{t=0}^{m_1}$ from $(x_1,x_2,\ldots,x_k,0) \in A$ to $(x_1,x_2,\ldots,x_k,m_n) \in A$. By Proposition 5.4, $x \in \text{Fix}(f)$. Since x was taken arbitrarily, this completes the induction proof that $X \subset \text{Fix}(f)$.

By taking $Y = X$ and applying the above to all $f \in C(X,c_1)$ such that $A \subset \text{Fix}(f)$, we conclude that A is a freezing set for (X,c_1).

Minimality of A for $n = 1$ was established at Proposition 5.11. To show minimality of A for $n = 2$, consider a proper subset A' of A. Without loss of generality, $(0,0) \in A \setminus A'$, $m_1 > 0$, and $m_2 > 0$. For $x \in X$, let $g: X \to X$ be the function

$$g(x) = \begin{cases} x & \text{if } x \neq (0,0); \\ (1,1) & \text{if } x = (0,0). \end{cases}$$

Suppose $y \in X$ is such that $y \leftrightarrow_{c_1} (0,0)$. Then $y = (1,0)$ or $y = (0,1)$, hence

$$g(y) = y \leftrightarrow_{c_1} (1,1) = g(0,0).$$

Thus $g \in C(X,c_1)$, $A' \subset \text{Fix}(g)$, and $g \neq \text{id}_X$. Therefore, A' is not a freezing set for (X,c_1), so A is minimal. $\square$

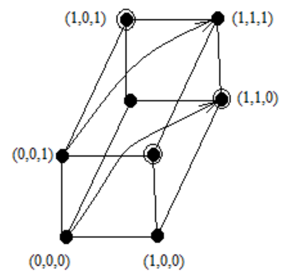

Figure 3. The function g in the proof of Example 5.16. Members of $A\setminus\{(0,0,0)\}$ are circled. Straight line segments indicate c_1 adjacencies. Curved arrows show the mapping for points in $X \setminus \text{Fix}(g)$.

The minimality assertion of Theorem 5.15 does not extend to $n = 3$, as shown in the following.

Example 5.16. Let $X = [0,1]^3_{\mathbb{Z}}$. Let

$$A = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}.$$

See Figure 3. Then A is a minimal freezing set for (X, c_1).

Proof. Note if $x \in X \setminus A$ then for each index $i \in \{1,2,3\}$, x is c_1-adjacent to $y_i \in A$ such that x and y_i differ in the i^{th} coordinate. Therefore, if $f \in C(X,c_1)$ such that $f(x) \neq x$, then c_1-continuity requires that for some i we have $f(y_i) \neq y_i$. It follows that A is a freezing set for (X, c_1).

Minimality is shown as follows. Let A' be a proper subset of A. Without loss of generality, $(0,0,0) \in A \setminus A'$. Let $g : X \to X$ be the function (see Figure 3)

$$g(x) = \begin{cases} (1,1,0) & \text{if } x = (0,0,0); \\ (1,1,1) & \text{if } x = (0,0,1); \\ x & \text{otherwise.} \end{cases}$$

Then $g \in C(X,c_1)$, $g|_{A'} = \text{id}_{A'}$, and $g \neq \text{id}_X$. Therefore, A' is not a freezing set for (X, c_1). $\square$

5.4. **Digital cubes and c_n.** In this section, we consider freezing sets for digital cubes in $\mathbb{Z}^n$, using the c_n adjacency.

Theorem 5.17. *Let $X = \prod_{i=1}^n [0,m_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$, where $m_i > 1$ for all i. Then $Bd(X)$ is a minimal freezing set for (X, c_n).*

Proof. That $Bd(X)$ is a freezing set for (X, c_n) follows from Theorem 5.13.

To show $Bd(X)$ is a minimal freezing set, it suffices to show that if A is a proper subset of $Bd(X)$ then A is not a freezing set for (X, c_n). We must show that there exists

$$f \in C(X,c_n) \text{ such that } f|_A = \text{id}_A \text{ but } f \neq \text{id}_X\,. \tag{5.2}$$

By hypothesis, there exists $y = (y_1, \ldots, y_n) \in Bd(X) \setminus A$.

Since $y \in Bd(X)$, for some index j, $y_j \in \{0, m_j\}$.

- If $y_j = 0$ the function $f : X \to X$ defined by
$$f(y) = (y_1, \ldots, y_{j-1}, 1, y_{j+1}, \ldots, y_n), \quad f(x) = x \text{ for } x \neq y,$$
satisfies (5.2).
- If $y_j = m_j$ the function $f : X \to X$ defined by
$$f(y) = (y_1, \ldots, y_{j-1}, m_j - 1, y_{j+1}, \ldots, y_n), \quad f(x) = x \text{ for } x \neq y,$$
satisfies (5.2).

The assertion follows. □

5.5. **Freezing sets and the normal product adjacency.** In the following, $p_j : \prod_{i=1}^{v} X_i \to X_j$ is the map
$$p_j(x_1, \ldots, x_v) = x_j \text{ where } x_i \in X_i.$$

Theorem 5.18. *Let (X_i, κ_i) be a digital image, $i \in [1, v]_{\mathbb{Z}}$. Let $X = \prod_{i=1}^{v} X_i$. Let $A \subset X$. Suppose A is a freezing set for $(X, NP_v(\kappa_1, \ldots, \kappa_v))$. Then for each $i \in [1, v]_{\mathbb{Z}}$, $p_i(A)$ is a freezing set for (X_i, κ_i).*

Proof. Let $f_i \in C(X_i, \kappa_i)$. Let $F : X \to X$ be defined by
$$F(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v)).$$
Then by Theorem 2.7, $F \in C(X, NP_v(\kappa_1, \ldots, \kappa_v))$.

Suppose for all $a = (a_1, \ldots, a_v) \in A$, $F(a) = a$, hence $f_i(a_i) = a_i$ for all $a_i \in p_i(A)$. Since A is a freezing set of X, we have that $F = \mathrm{id}_X$, and therefore, $f_i = \mathrm{id}_{X_i}$. The assertion follows. □

5.6. **Cycles.** A *cycle* or *digital simple closed curve* of n distinct points is a digital image (C_n, κ) with $C_n = \{x_i\}_{i=0}^{n-1}$ such that $x_i \leftrightarrow_\kappa x_j$ if and only if $j = i + 1 \mod n$ or $j = i - 1 \mod n$.

Given indices $i < j$, there are two distinct paths determined by x_i and x_j in C_n, consisting of the sets $P_{i,j} = \{x_k\}_{k=i}^{j}$ and $P'_{i,j} = C_n \setminus \{x_k\}_{k=i+1}^{j-1}$. If one of these has length less than $n/2$, it is the *shorter path* from p_i to p_j and the other is the *longer path*; otherwise, both have length $n/2$, and each is a *shorter path* and a *longer path* from p_i to p_j.

In this section, we consider minimal fixed point sets for $f \in C(C_n)$ that force f to be an identity map.

Theorem 5.19. *Let $n > 4$. Let x_i, x_j, x_k be distinct members of C_n be such that C_n is a union of unique shorter paths determined by these points. Let $f \in C(C_n, \kappa)$. Then $f = \mathrm{id}_{C_n}$ if and only if $\{x_i, x_j, x_k\} \subset \mathrm{Fix}(f)$; i.e., $\{x_i, x_j, x_k\}$ is a freezing set for C_n. Further, this freezing set is minimal.*

Proof. Clearly $f = \mathrm{id}_{C_n}$ implies $\{x_i, x_j, x_k\} \subset \mathrm{Fix}(f)$.

Suppose $\{x_i, x_j, x_k\} \subset \mathrm{Fix}(f)$. By hypothesis, there are unique shorter paths P_0 from x_i to x_j, P_1 from x_j to x_k, and P_2 from x_k to x_i, in C_n. By

Proposition 5.4, each of P_0, P_1, and P_2 is contained in $\mathrm{Fix}(f)$. By hypothesis $C_n = P_0 \cup P_1 \cup P_2$, so $f = \mathrm{id}_{C_n}$. Hence $\{x_i, x_j, x_k\}$ is a freezing set.

For any distinct pair $x_i, x_j \in C_n$, there is a non-identity continuous self-map on C_n that takes a longer path determined by x_i and x_j to a shorter path determined by x_i and x_j. Thus, $\{x_i, x_j\}$ is not a freezing set for C_n, so the set $\{x_i, x_j, x_k\}$ discussed above is minimal. □

Remark 5.20. In Theorem 5.19, we need the assumption that $n > 4$, as there is a continuous self-map f on C_4 with 3 fixed points such that $f \neq \mathrm{id}_{C_4}$ [10].

5.7. **Wedges.** Let $(X, \kappa) \subset \mathbb{Z}^n$ be such that $X = X_0 \cup X_1$, where $X_0 \cap X_1 = \{x_0\}$; and if $x \in X_0$, $y \in X_1$, and $x \leftrightarrow_\kappa y$, then $x_0 \in \{x, y\}$. We say X is the *wedge of* X_0 *and* X_1, denoted $X = X_0 \vee X_1$. We say x_0 is the *wedge point*.

Theorem 5.21. *Let A be a freezing set for (X, κ), where $X = X_0 \vee X_1 \subset \mathbb{Z}^n$, $\#X_0 > 1$, and $\#X_1 > 1$. Let $X_0 \cap X_1 = \{x_0\}$. Then A must include points of $X_0 \setminus \{x_0\}$ and $X_1 \setminus \{x_0\}$.*

Proof. Otherwise, either $A \subset X_0$ or $A \subset X_1$.

Suppose $A \subset X_0$. Then the function $f : X \to X$ given by

$$f(x) = \begin{cases} x & \text{if } x \in X_0; \\ x_0 & \text{if } x \in X_1, \end{cases}$$

belongs to $C(X, \kappa)$ and satisfies $f|_A = \mathrm{id}_A$, but $f \neq \mathrm{id}_X$. Thus A is not a freezing set for (X, κ).

The case $A \subset X_1$ is argued similarly. □

Example 5.22. The wedge of two digital intervals is (isomorphic to) a digital interval. It follows from Theorem 5.3 and Proposition 5.11 that a freezing set for a wedge need not include the wedge point.

Theorem 5.23. *Let C_m and C_n be cycles, with $m > 4$, $n > 4$. Let x_0 be the wedge point of $X = C_m \vee C_n$. Let $x_i, x_j \in C_m$ and $x'_k, x'_p \in C_n$ be such that C_m is the union of unique shorter paths determined by x_i, x_j, x_0 and C_n is the union of unique shorter paths determined by x'_k, x'_p, x_0. Then $A = \{x_i, x_j, x'_k, x'_p\}$ is a freezing set for X.*

Proof. Let $f \in C(X, \kappa)$ be such that $A \subset \mathrm{Fix}(f)$. Let P_0 be the unique shorter path in C_m from x_i to x_j; let P_1 be the unique shorter path in C_m from x_j to x_0; let P_2 be the unique shorter path in C_m from x_0 to x_i; let P'_0 be the unique shorter path in C_n from x'_k to x'_p; let P'_1 be the unique shorter path in C_n from x'_p to x_0; let P'_2 be the unique shorter path in C_n from x_0 to x'_k.

By Proposition 5.4, each of the following paths is contained in $\mathrm{Fix}(f)$: P_0, $P_1 \cup P'_1$ (from x_j to x_0 to x'_p), $P_2 \cup P'_2$ (from x_i to x_0 to x'_k), and P'_0. Since

$$X = P_0 \cup (P_1 \cup P'_1) \cup (P_2 \cup P'_2) \cup P'_0 \subset \mathrm{Fix}(f),$$

the assertion follows. □

5.8. **Trees.** A *tree* is an acyclic graph (X, κ) that is connected, i.e., lacking any subgraph isomorphic to C_n for $n > 2$. The *degree* of a vertex x in X is the number of distinct vertices $y \in X$ such that $x \leftrightarrow y$. A vertex of a tree may be designated as the *root*. We have the following.

Lemma 5.24 ([10]). *Let (X, κ) be a digital image that is a tree in which the root vertex has at least 2 child vertices. Then $f \in C(X, \kappa)$ implies* $\operatorname{Fix}(f)$ *is κ-connected.*

Theorem 5.25. *Let (X, κ) be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let E be the set of vertices of G that have degree 1. Then E is a minimal freezing set for G.*

Proof. First consider the case that each vertex has degree 1. Since X is a tree, it follows that $X = \{x_0, x_1\} = E$, and E is a freezing set. E must be minimal, since X admits constant functions that are identities on their restrictions to proper subsets of E.

Otherwise, there exists $x_0 \in X$ such that x_0 has degree of at least 2 in G. This implies $\#X > 2$, and since G is finite and acyclic, $\#E > 0$. Since G is acyclic, removal of any member of $X \setminus E$ would disconnect X. If we take x_0 to be the root vertex, it follows from Lemma 5.24 that E is a freezing set.

Since $\#E > 0$, for any $y \in E$ there exists $y' \in X \setminus E$ such that $y' \leftrightarrow y$. Then the function $f : X \to X$ defined by

$$f(x) = \begin{cases} y' & \text{if } x = y; \\ x & \text{if } x \neq y, \end{cases}$$

satisfies $f \in C(X, \kappa)$, $f|_{E \setminus \{y\}} = \operatorname{id}_{E \setminus \{y\}}$, and $f \neq \operatorname{id}_X$. Thus $E \setminus \{y\}$ is not a freezing set. Since y was arbitrarily chosen, E is minimal. □

6. s-Cold sets

In this section, we generalize our focus from fixed points to approximate fixed points and, more generally, to points constrained in the amount they can be moved by continuous self-maps in the presence of fixed point sets. We obtain some analogues of our previous results for freezing sets.

6.1. **Definition and basic properties.** In the following, we use the path-length metric d for connected digital images (X, κ), defined [13] as

$$d(x, y) = \min\{\ell \mid \ell \text{ is the length of a } \kappa\text{-path in } X \text{ from } x \text{ to } y\}.$$

If X is finite and κ-connected, the *diameter* of (X, κ) is

$$diam(X, \kappa) = \max\{d(x, y) \mid x, y \in X\}.$$

We introduce the following generalization of a freezing set.

Definition 6.1. Given $s \in \mathbb{N}^*$, we say $A \subset X$ is an *s-cold set* for the connected digital image (X, κ) if given $g \in C(X, \kappa)$ such that $g|_A = \operatorname{id}_A$, then for all $x \in X$, $d(x, g(x)) \leq s$. A *cold set* is a 1-cold set.

Note a 0-cold set is a freezing set.

Theorem 6.2. *Let (X, κ) be a connected digital image. Let $A \subset X$. Then $A \subset Fix(g)$ is an s-cold set for (X, κ) if and only if for every isomorphism $F : (X, \kappa) \to (Y, \lambda)$, if $g : X \to Y$ is (κ, λ)-continuous and $F|_A = g|_A$, then for all $x \in X$, $d(F(x), g(x)) \leq s$.*

Proof. Suppose A is an s-cold set for (X, κ). Then for all $f \in C(X, \kappa)$ such that $f|_A = \mathrm{id}_A$ and all $x \in X$, we have $d(x, f(x)) \leq s$. Let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. Let $g : X \to Y$ be (κ, λ)-continuous with $F|_A = g|_A$. Then

$$\mathrm{id}_A = F^{-1} \circ F|_A = F^{-1} \circ g|_A.$$

Let $x \in X$. Then $d(x, F^{-1} \circ g(x)) \leq s$, i.e., there is a κ-path P in X of length at most s from x to $F^{-1} \circ g(x)$. Therefore, $F(P)$ is a λ-path in Y of length at most s from $F(x)$ to $F \circ F^{-1} \circ g(x) = g(x)$, i.e., $d(F(x), g(x)) \leq s$.

Suppose $A \subset X$ and for every isomorphism $F : (X, \kappa) \to (Y, \lambda)$, if $g : X \to Y$ is (κ, λ)-continuous and $F|_A = g|_A$, then for all $x \in X$, $d(F(x), g(x)) \leq s$. Let $f \in C(X, \kappa)$ with $f|_A = \mathrm{id}_A$. Since id_X is an isomorphism, for all $x \in X$, $d(x, f(x)) \leq s$. Thus, A is an s-cold set for (X, κ). $\square$

Given a digital image (X, κ) and $f \in C(X, \kappa)$, a point $x \in X$ is an *almost fixed point of f* [16] or an *approximate fixed point of f* [7] if $f(x) \leftrightarroweq_\kappa x$.

Remark 6.3. The following are easily observed.

- If $A \subset A' \subset X$ and A is an s-cold set for (X, κ), then A' is an s-cold set for (X, κ).
- A is a cold set (i.e., a 1-cold set) for (X, κ) if and only if given $f \in C(X, \kappa)$ such that $f|_A = \mathrm{id}_A$, every $x \in X$ is an approximate fixed point of f.
- In a finite connected digital image (X, κ), every nonempty subset of X is a $diam(X)$-cold set.
- If $s_0 < s_1$ and A is an s_0-cold set for (X, κ), then A is an s_1-cold set for (X, κ).

Note a freezing set is a cold set, but the converse is not generally true, as shown in the following.

Example 6.4. It follows from Definition 6.1 that $\{0\}$ is a cold set, but not a freezing set, for $X = [0, 1]_{\mathbb{Z}}$, since the constant function g with value 0 satisfies $g|_{\{0\}} = \mathrm{id}_{\{0\}}$, and $g(1) = 0 \leftrightarrow_{c_1} 1$.

s-cold sets are invariant in the sense of the following.

Theorem 6.5. *Let (X, κ) be a connected digital image, let A be an s-cold set for (X, κ), and let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. Then $F(A)$ is an s-cold set for (Y, λ).*

Proof. Let $f \in C(Y, \lambda)$ such that $f|_{F(A)} = \mathrm{id}_{F(A)}$. Then

$$f \circ F|_A = f|_{F(A)} \circ F|_A = \mathrm{id}_{F(A)} \circ F|_A = F|_A.$$

By Theorem 6.2, for all $x \in X$, $d(f \circ F(x), F(x)) \leq s$. Substituting $y = F(x)$, we have that $y \in Y$ implies $d(f(y), y) \leq s$. By Definition 6.1, $F(A)$ is a cold set for (Y, λ). $\square$

A is a *κ-dominating set* (or a *dominating set* when κ is understood) for (X, κ) if for every $x \in X$ there exists $a \in A$ such that $x \leftrightarrows_\kappa a$ [11]. This notion is somewhat analogous to that of a dense set in a topological space, and the following is somewhat analogous to the fact that in topological spaces, a continuous function is uniquely determined by its values on a dense subset of the domain.

Theorem 6.6. *Let (X, κ) be a digital image and let A be κ-dominating in X. Then A is 2-cold in (X, κ).*

Proof. Let $f \in C(X, \kappa)$ such that $f|_A = \mathrm{id}_A$. Since A is κ-dominating, for every $x \in X$ there is an $a \in A$ such that $x \leftrightarrows a$. Then $f(x) \leftrightarrows f(a) = a$. Thus, we have the path $\{x, a, f(x)\} \subset X$ from x to $f(x)$ of length at most 2. The assertion follows. $\square$

Theorem 6.7. *Let (X, κ) be rigid. If A is a cold set for X, then A is a freezing set for X.*

Proof. Let $f \in C(X, \kappa)$ be such that $f|_A = \mathrm{id}_A$. Since A is cold, $f(x) \leftrightarrows x$ for all $x \in X$. Therefore, the map $H : X \times [0,1]_\mathbb{Z} \to X$ defined by $H(x, 0) = x$, $H(x, 1) = f(x)$, is a homotopy. Since X is rigid, $f = \mathrm{id}_X$. The assertion follows. $\square$

6.2. **Cold sets for cubes.** In this section, we consider cold sets for digital cubes in $\mathbb{Z}^n$. Note the hypotheses of Proposition 6.8 imply A is c_1- and c_2-dominating in $Bd(X)$.

Proposition 6.8. *Let $m, n \in \mathbb{N}$. Let $X = [0, m]_\mathbb{Z} \times [0, n]_\mathbb{Z}$. Let $A \subset Bd(X)$ be such that no pair of c_1-adjacent members of $Bd(X)$ belong to $Bd(X) \setminus A$. Then A is a cold set for (X, c_2). Further, for all $f \in C(X, c_2)$, if $f|_A = \mathrm{id}_A$ then $f|_{Int(X)} = \mathrm{id}|_{Int(X)}$.*

Proof. Let $x = (x_0, y_0) \in X$. Let $f \in C(X, c_2)$ such that $f|_A = \mathrm{id}_A$. Consider the following.

- If $x \in A$ then $f(x) = x$.
- If $x \in Bd(X) \setminus A$ then both of the c_1-neighbors of x in $Bd(X)$ belong to A. We will show $f(x) \leftrightarrows_{c_2} x$.
  Let $K = \{(0,0), (0,n), (m,0), (m,n)\} \subset Bd(X)$.
  - For $x \in K$, consider the case $x = (0,0)$. Then $\{(0,1), (1,0)\} \subset A$, so we must have
  $$f(x) \in N^*_{c_2}((0,1)) \cap N^*_{c_2}((1,0)) \subset N^*_{c_2}(x).$$
  For other $x \in K$, we similarly find $f(x) \leftrightarrows_{c_2} x$.

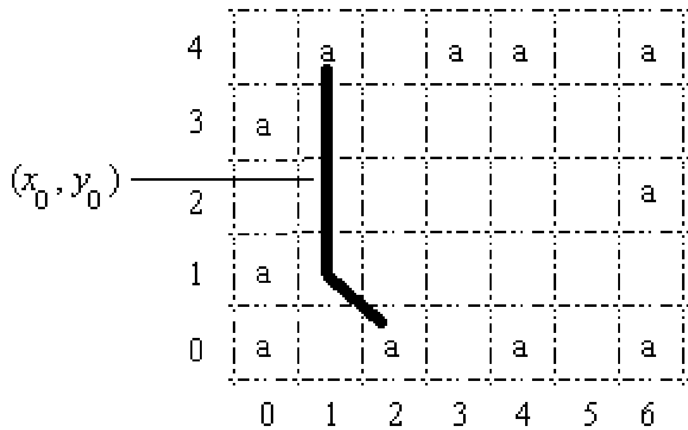

Figure 4. Illustration of the proof of Proposition 6.8 for the case $(x_0, y_0) \in Int(X)$. $X = [0,6]_{\mathbb{Z}} \times [0,4]_{\mathbb{Z}}$. Members of the set $A \subset Bd(X)$ are marked "a". Corner points such as $(0,4)$ need not belong to A; also, although we cannot have c_1-adjacent members of $Bd(X)$ in $Bd(X) \setminus A$, we can have c_2-adjacent members of $Bd(X)$ in $Bd(X) \setminus A$, e.g., $(5,4)$ and $(6,3)$. The heavy polygonal line illustrates a c_2-path P of length $n = 4$: $P(0) = q_L = (2,0)$, $P(1) = (1,1)$, $P(2) = (x_0, y_0) = (1,2)$, $P(3) = (1,3)$, $P(4) = q_U = (1,4)$.

- For $x \in Bd(X) \setminus K$, consider the case $x = (t, 0)$. For this case, $\{(t-1,0),(t+1,0)\} \subset A$, so

$$(t-1,0) = f(t-1,0) \leftrightarrows_{c_2} f(x) \leftrightarrows_{c_2} f(t+1,0) = (t+1,0).$$

  Therefore, $f(x) \in \{x, (t,1)\}$, so $f(x) \leftrightarrows_{c_2} x$.
  For other $x \in Bd(X) \setminus K$, we similarly find $f(x) \leftrightarrows_{c_2} x$.

- If $x \in Int(X)$, let $L = \{(z,0)\}_{z=x_0-1}^{x_0+1}$ and $U = \{(z,n)\}_{z=x_0-1}^{x_0+1}$. We have

$$L \cap A \neq \varnothing \neq U \cap A.$$

  Since no pair of c_1-adjacent members of $Bd(X)$ belong to $Bd(X) \setminus A$, there exist $q_L \in L \cap A$, $q_U \in U \cap A$ such that

$$|p_1(q_L) - x_0| \leq 1 \text{ and } |p_1(q_U) - x_0| \leq 1.$$

  Thus, there is an injective c_2-path $P : [0,n]_{\mathbb{Z}} \to X$ such that $P([0,y_0)]_{\mathbb{Z}})$ runs from q_L to x and $P([y_0,n]_{\mathbb{Z}})$ runs from x to q_U (note since we use c_2-adjacency, there can be steps of the path that change both coordinates - see Figure 4). Therefore, $f \circ P$ is a path from $f(q_L) = q_L$ to $f(x)$ to $f(q_U) = q_U$, and $p_2 \circ f \circ P$ is a path from $p_2(q_L) = 0$ to $p_2(f(x))$ to $p_2(q_U) = n$.

If $y' = p_2(f(x)) > y_0$, then $p_2 \circ f \circ P|_{[0,y_0]_{\mathbb{Z}}}$ is a c_2-path of length y_0 from 0 to y', which is impossible. Similarly, if $y' < y_0$, then $p_2 \circ f \circ P|_{[y_0,n]_{\mathbb{Z}}}$ is a c_2-path of length $n - y_0$ from y' to n, which is impossible. Therefore, we must have

$$p_2 \circ f(x) = y_0. \tag{6.1}$$

Similarly, by replacing the neighborhoods of the projections of x on the lower and upper edges of the cube, L and U, by the neighborhoods of the projections of x on the the left and right edges of the cube, $L' = \{(0,z)\}_{z=y_0-1}^{y_0+1}$ and $R = \{(m,z)\}_{z=y_0-1}^{y_0+1}$, and using an argument similar to that used to obtain (6.1), we conclude that

$$p_1 \circ f(x) = x_0. \tag{6.2}$$

It follows from (6.2) and (6.1) that $f(x) = x$.

Thus, in all cases, $f(x) \leftrightarrows_{c_2} x$, and $f|_{Int(X)} = \mathrm{id}_{Int(X)}$. □

Proposition 6.9. *Let $m, n \in \mathbb{N}$. Let $X = [0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}$. Let $A \subset Bd(X)$ be c_1-dominating in $Bd(X)$. Then A is a 2-cold set for (X, c_2). Further, for all $f \in C(X, c_2)$, if $f|_A = \mathrm{id}_A$ then $f|_{Int(X)} = \mathrm{id}|_{Int(X)}$.*

Proof. Our argument is similar to that of Proposition 6.8. Let $x = (x_0, y_0) \in X$. Let $f \in C(X, c_2)$ such that $f|_A = \mathrm{id}_A$. Consider the following.

- If $x \in A$ then $f(x) = x$.
- If $x \in Bd(X) \setminus A$ then for some $a \in A$, $x \leftrightarrows_{c_1} a$. Therefore, $f(x) \leftrightarrows_{c_1} f(a) = a$. Thus, $\{x, a, f(x)\}$ is a path in X from x to $f(x)$ of length at most 2.
- If $x \in Int(X)$, then as in the proof of Proposition 6.8 we have that $f(x) = x$.

Thus, in all cases, $d(f(x), x) \leq 2$, and $f|_{Int(X)} = \mathrm{id}_{Int(X)}$. □

An example of a 2-cold set A that is not a 1-cold set, such that A is as in Proposition 6.9, is given in the following.

Example 6.10. Let $X = [0,2]_{\mathbb{Z}}^2$. Let

$$A = \{(0,2), (1,0), (2,2)\} \subset X.$$

Then A is c_1-dominating in $Bd(X)$, so by Proposition 6.9, is a 2-cold set for (X, c_2). Let $f : X \to X$ be the function $f(0,0) = (2,0)$, $f(0,1) = (1,1)$, and $f(x) = x$ for all $x \in X \setminus \{(0,0), (0,1)\}$. Then $f \in C(X, c_2)$ but $d((0,0), f(0,0)) = 2$, so A is not a 1-cold set.

Proposition 6.11. *Let $X = \prod_{i=1}^{n} [0, m_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$, where $m_i > 1$ for all i. Let $A \subset Bd(X)$ be such that A is not c_n-dominating in $Bd(X)$. Then A is not a cold set for (X, c_n).*

Proof. By hypothesis, there exists $y = (y_1, \ldots, y_n) \in Bd(X) \setminus A$ such that $N(y, c_n) \cap A = \varnothing$.

Since $y \in Bd(X)$, for some index j we have $y_j \in \{0, m_j\}$. Let $x = (x_1, \ldots, x_n) \in X$, for $x_i \in [0, m_i]_{\mathbb{Z}}$.

- If $y_j = 0$, let $f : X \to X$ be defined as follows.
$$f(x) = \begin{cases} (x_1, \ldots, x_{j-1}, 2, x_{j+1}, \ldots, x_n) & \text{if } x = y; \\ (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n) & \text{if } x \in N_{c_n}(y); \\ x & \text{otherwise.} \end{cases}$$
If $u, v \in X$, $u \leftrightarrows_{c_n} v$, then u and v differ by at most 1 in every coordinate. Consider the following cases.
  - If $u = y$, then $v \in N_{c_n}(y)$, and clearly $f(u)$ and $f(v)$ differ by at most 1 in every coordinate, hence are c_n-adjacent. Similarly if $v = y$.
  - If $u, v \in N_{c_n}(y)$, then clearly $f(u)$ and $f(v)$ differ by at most 1 in every coordinate, hence are c_n-adjacent.
  - If $u \in N_{c_n}(y)$ and $v \notin N_{c_n}(y)$, then $p_j(u) \in \{0, 1\}$, so $p_j(f(v)) = p_j(v) \in \{0, 1, 2\}$, and $p_j(f(u)) = 1$. It follows easily that $f(u)$ and $f(v)$ differ by at most 1 in every coordinate, hence are c_n-adjacent. Similarly if $v \in N_{c_n}(y)$ and $u \notin N_{c_n}(y)$
  - Otherwise, $\{u, v\} \cap N^*_{c_n}(y) = \varnothing$, so $f(u) = u \leftrightarrows_{c_n} v = f(v)$.

  Therefore, $f \in C(X, c_n)$.
- If $y_j = m_j$, let $f : X \to X$ be defined by
$$f(x) = \begin{cases} (x_1, \ldots, x_{j-1}, m_j - 2, x_{j+1}, \ldots, x_n) & \text{if } x = y; \\ (x_1, \ldots, x_{j-1}, m_j - 1, x_{j+1}, \ldots, x_n) & \text{if } x \in N_{c_n}(y); \\ x & \text{otherwise.} \end{cases}$$
By an argument similar to that of the case $y_j = 0$, we conclude that $f \in C(X, c_n)$.

Further, in both cases, $f|_A = \mathrm{id}_A$, and $f(y) \not\leftrightarrows_{c_n} y$. The assertion follows. □

6.3. **s-cold sets for rectangles in $\mathbb{Z}^2$.** The following generalizes the case $n = 2$ of Theorem 5.15.

Proposition 6.12. *Let $X = [-m, m]_{\mathbb{Z}} \times [-n, n]_{\mathbb{Z}} \subset \mathbb{Z}^2$, $s \in \mathbb{N}^*$, where $s \le \min\{m, n\}$. Let*
$$A = \{(-m+s, -n+s), (-m+s, n-s), (m-s, -n+s), (m-s, n-s)\}.$$
Then A is a $4s$-cold set for (X, c_1).

Proof. Let $f \in C(X, c_1)$ such that $f|_A = \mathrm{id}_A$. Let
$$A' = [-m+s, m-s]_{\mathbb{Z}} \times [-n+s, n-s]_{\mathbb{Z}}.$$
By Proposition 5.4, $Bd(A') \subset \mathrm{Fix}(f)$. It follows from Proposition 5.12 that $A' \subset \mathrm{Fix}(f)$.

Thus it remains to show that $x \in X \setminus A'$ implies $d(x, f(x)) \le 4s$. This is seen as follows. For $x \in X \setminus A'$, there exists a c_1-path P of length at most $2s$ from x to some $y \in Bd(A')$. Then $f(P)$ is a c_1-path from $f(x)$ to $f(y) = y$ of length at most $2s$. Therefore, $P \cup f(P)$ contains a path from x to y to $f(x)$ of length at most $4s$. The assertion follows. □

The following generalizes Proposition 6.8.

Proposition 6.13. *Let* $X = [-m, m]_{\mathbb{Z}} \times [-n, n]_{\mathbb{Z}} \subset \mathbb{Z}^2$*,* $s \in \mathbb{N}^*$*, where* $m - s \geq 0$*,* $n - s \geq 0$*. Let*

$$A = [-m + s, m - s]_{\mathbb{Z}} \times [-n + s, n - s]_{\mathbb{Z}} \subset X.$$

Let $A' \subset Bd(A)$ *such that no pair of* c_1*-adjacent members of* $Bd(A)$ *belongs to* $Bd(A) \setminus A'$*. Then* A' *is a* $2s$*-cold set for* (X, c_2)*. Further, if* $f \in C(X, c_2)$ *and* $f|_{A'} = \mathrm{id}_{A'}$*, then* $f|_A = \mathrm{id}_A$*.*

Proof. Let $f \in C(X, c_2)$ be such that $f|_{A'} = \mathrm{id}_{A'}$. As in the proof of Proposition 6.8, $f|_A = \mathrm{id}_A$.

Now consider $x \in X \setminus A$. There is a c_2-path P in X from x to some $y \in A'$ of length at most s. Then $f(P)$ is a c_2-path in X from $f(x)$ to $f(y) = y$ of length at most s. Therefore, $P \cup f(P)$ contains a c_2-path in X from x to y to $f(x)$ of length at most $2s$. The assertion follows. □

6.4. **s-cold sets for Cartesian products.** We modify the proof of Theorem 5.18 to obtain the following.

Theorem 6.14. *Let* (X_i, κ_i) *be a digital image,* $i \in [1, v]_{\mathbb{Z}}$*. Let* $X = \prod_{i=1}^{v} X_i$*. Let* $s \in \mathbb{N}^*$*. Let* $A \subset X$*. Suppose* A *is an* s*-cold set for* $(X, NP_v(\kappa_1, \ldots, \kappa_v))$*. Then for each* $i \in [1, v]_{\mathbb{Z}}$*,* $p_i(A)$ *is an* s*-cold set for* (X_i, κ_i)*.*

Proof. Let $f_i \in C(X_i, \kappa_i)$. Let $F : X \to X$ be defined by

$$F(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v)).$$

Then by Theorem 2.7, $F \in C(X, NP_v(\kappa_1, \ldots, \kappa_v))$.

Suppose for all i, $a_i \in p_i(A)$, we have $f_i(a_i) = a_i$. Note this implies, for $a = (a_1, \ldots, a_v)$, that $F(a) = a$. Since a is an arbitrary member of the s-cold set A of X, we have that $d(F(x), x) \leq s$, for all $x = (x_1, \ldots, x_v) \in X$, $x_i \in X_i$, and therefore, $d(f_i(x_i), x_i) \leq s$. The assertion follows. □

6.5. **s-cold sets for infinite digital images.** In this section, we obtain properties of s-cold sets for some infinite digital images.

Theorem 6.15. *Let* $(\mathbb{Z}^n, c_u)$ *be a digital image,* $1 \leq u \leq n$*. Let* $A \subset \mathbb{Z}^n$*. Let* $s \in \mathbb{N}^*$*. If* A *is an* s*-cold set for* $(\mathbb{Z}^n, c_u)$*, then for every index* i*,* $p_i(A)$ *is an infinite set, with sequences of members tending both to* ∞ *and to* $-\infty$*.*

Proof. Suppose otherwise. Then for some i, there exist m or M in $\mathbb{Z}$ such that

$$m = \min\{p_i(a) \mid a \in A\} \quad \text{or} \quad M = \max\{p_i(a) \mid a \in A\}.$$

If the former, then for $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$, define $f : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$f(z) = \begin{cases} (z_1, \ldots, z_{i-1}, m, z_{i+1}, \ldots, z_n) & \text{if } z_i \leq m; \\ z & \text{otherwise.} \end{cases}$$

Then $f \in C(\mathbb{Z}^n, c_u)$ and $f|_A = \mathrm{id}_A$, but $f \neq \mathrm{id}_{\mathbb{Z}^n}$. Thus, A is not an s-cold set.

Similarly, if $M < \infty$ as above exists, we conclude A is not an s-cold set. □

Corollary 6.16. $A \subset \mathbb{Z}$ *is a freezing set for* $(\mathbb{Z}, c_1)$ *if and only if* A *contains sequences* $\{a_i\}_{i=1}^{\infty}$ *and* $\{a'_i\}_{i=1}^{\infty}$ *such that* $\lim_{i\to\infty} a_i = \infty$ *and* $\lim_{i\to\infty} a'_i = -\infty$*.*

Proof. This follows from Lemma 5.5 and Theorem 6.15. □

The converse of Theorem 6.15 is not generally correct, as shown by the following.

Example 6.17. Let $A = \{(z,z) \mid z \in \mathbb{Z}\} \subset \mathbb{Z}^2$. Then although $p_1(A) = p_2(A) = \mathbb{Z}$ contains sequences tending to ∞ and to $-\infty$, A is not an s-cold set for $(\mathbb{Z}^2, c_2)$, for any s.

Proof. Consider $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by $f(x,y) = (x,x)$. We have $f \in (\mathbb{Z}^2, c_2)$ and $f|_A = \mathrm{id}_A$, but one sees easily that for all s there exist $(x,y) \in \mathbb{Z}^2$ such that $d((x,y), f(x,y)) > s$. □

7. Further remarks

We have continued the work of [10] in studying fixed point invariants and related ideas in digital topology.

We have introduced pointed versions of rigidity and fixed point spectra.

We have introduced the notions of freezing sets and s-cold sets. These show us that although knowledge of the fixed point set $\mathrm{Fix}(f)$ of a continuous self-map f on a connected topological space X generally gives us little information about the nature of $f|_{X \setminus \mathrm{Fix}(f)}$, if $f \in C(X,\kappa)$ then $f|_{X \setminus \mathrm{Fix}(f)}$ may be severely limited if $A \subset \mathrm{Fix}(f)$ is a freezing set or, more generally, an s-cold set for (X,κ).

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