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A note on rank 2 diagonals

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Abstract

We solve two questions regarding spaces with a (G_δ)-diagonal of rank 2. One is a question of Basile, Bella and Ridderbos about weakly Lindelöf spaces with a G_δ-diagonal of rank 2 and the other is a question of Arhangel'skii and Bella asking whether every space with a diagonal of rank 2 and cellularity continuum has cardinality at most continuum.

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1. Introduction

A space is said to have a G_δ-diagonal if its diagonal can be written as the intersection of a countable family of open subsets in the square. This notion is of central importance in metrization theory, ever since Sneider's 1945 theorem [14] stating that every compact Hausdorff space with a G_δ-diagonal is metrizable. Sneider's result was later improved by Chaber [8] who proved that every countably compact space with a G_δ-diagonal is compact and hence metrizable.

Around the same time, Ginsburg and Woods [10] showed the influence of G_δ-diagonals in the theory of cardinal invariants for topological spaces by proving that every space with a G_δ-diagonal without uncountable closed discrete sets has cardinality at most continuum. Their result led them to conjecture that every ccc space with a G_δ-diagonal must have cardinality at most continuum.

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Shakhmatov [13] and Uspenskii [15] gave a pretty strong disproof to this conjecture by constructing Tychonoff ccc spaces with a G_δ-diagonal of arbitrarily large cardinality. However, in the meanwhile, several strengthenings of the notion of a G_δ-diagonal had been introduced, leading several researchers to test Ginsburg and Woods's conjecture against these stronger diagonal properties. That culminated in Buzyakova's surprising result [7] that a ccc space with a regular G_δ-diagonal has cardinality at most continuum. A space has a *regular* G_δ*-diagonal* if there is a countable family of neighbourhoods of the diagonal in the square such that the diagonal is the intersection of their closures.

Another way of strengthening the property of having a G_δ-diagonal is by considering the notion of rank. Recall that given a family $\mathcal{U}$ of subsets of a topological space and a point $x \in X$, $St(x,\mathcal{U}) := \bigcup\{U \in \mathcal{U} : x \in U\}$. The set $St^n(x,\mathcal{U})$ is defined by induction as follows: $St^1(x,\mathcal{U}) = St(x,\mathcal{U})$ and $St^n(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap St^{n-1}(x,\mathcal{U}) \neq \varnothing\}$ for every $n > 1$. A space is said to have a diagonal of rank n if there is a sequence $\{\mathcal{U}_k : k < \omega\}$ of open covers of X such that $\bigcap\{St^n(x,\mathcal{U}_k) : k < \omega\} = \{x\}$, for every $x \in X$. By a well-known characterization, having a diagonal of rank 1 is equivalent to having a G_δ-diagonal. Note also that a space with a G_δ-diagonal of rank 2 is necessarily T_2.

Zenor [17] observed that every space with a diagonal of rank 3 also has a regular G_δ-diagonal so by Buzyakova's result, every ccc space with a diagonal of rank 3 has cardinality at most continuum. In [3], the first author proved the stronger result that every ccc space with a G_δ-diagonal of rank 2 has cardinality at most 2^ω. The following question is still open though:

Question 1.1 (Arhangel'skii and Bella [1])**.** *Is every regular G_δ-diagonal always of rank 2?*

A positive answer would lead to a far-reaching generalization of Buzyakova's cardinal bound.

Arhangel'skii and the first-named author proved in [1] that every space with a diagonal of rank 4 and cellularity $\leq \mathfrak{c}$ has cardinality at most continuum, and leave open whether this is also true for spaces with a diagonal of rank 2 or 3.

Question 1.2. *Let X be a space with a diagonal of rank 2 or 3 and cellularity at most $\mathfrak{c}$. Is it true that $|X| \leq \mathfrak{c}$.*

From Proposition 4.7 of [4] it follows that $|X| \leq c(X)^\omega$ for every space X with a diagonal of rank 3, which in turn that the answer to Arhangel'skii and Bella's question is yes for spaces with a diagonal of rank 3. We show that the answer to their question is no for spaces with a diagonal of rank 2, by constructing a space with a diagonal of rank 2, cellularity $\leq \mathfrak{c}$ and cardinality larger than the continuum. That leads to a complete solution to Arhangel'skii and Bella's question.

Recall that space X is *weakly Lindelöf* provided that every open cover has a countable subfamily whose union is dense in X. This notion is a common generalisation of the Lindelöf property and the countable chain condition (ccc).

In view of the results by Ginsburg-Woods and Bella mentioned above it is natural to consider the following question:

Question 1.3 ([4]). *Let X be a weakly Lindelöf space with a G_δ-diagonal of rank 2. Is it true that $|X| \leq 2^\omega$?*

The above question was explicitly formulated in [4] and two partial positive answers were obtained there under the assumptions that the space is either Baire or has a rank 3 diagonal. Here we will prove that Question 1.3 has a positive answer assuming that the space is normal.

All undefined notions can be found in [12].

2. Spaces with a diagonal of rank 2

Recall that a neighbourhood assignment for a space X is a function ϕ from X to its topology such that $x \in \phi(x)$ for every $x \in X$. A set $Y \subseteq X$ is a *kernel* for ϕ if $X = \bigcup\{\phi(y) : y \in Y\}$. Following [11], we say that a space X is dually $\mathcal{P}$ if every neighbourhood assignment in X has a kernel Y satisfying the property $\mathcal{P}$. Of course, $\mathcal{P}$ implies dually $\mathcal{P}$. A dually ccc space may fail to be even weakly Lindelöf.

Here we need the countable version of a well-known result of Erdös and Rado:

Lemma 2.1. *Let X be a set with $|X| > 2^\omega$. If $[X]^2 = \bigcup\{P_n : n < \omega\}$, then there exist an uncountable set $S \subseteq X$ and an integer $n_0 \in \omega$ such that $[S]^2 \subseteq P_{n_0}$.*

Theorem 2.2. *If X is a dually weakly Lindelöf normal space with a G_δ-diagonal of rank 2, then $|X| \leq 2^\omega$.*

Proof. Let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers of X such that $\{x\} = \bigcap\{St^2(x, \mathcal{U}_n) : n < \omega\}$ for each $x \in X$. Assume by contradiction that $|X| > 2^\omega$ and for any $n < \omega$ put $P_n = \{\{x, y\} \in [X]^2 : St(x, \mathcal{U}_n) \cap St(y, \mathcal{U}_n) = \varnothing\}$. The assumption that the sequence $\{\mathcal{U}_n : n < \omega\}$ has rank 2 implies that $[X]^2 = \bigcup\{P_n : n < \omega\}$. By Lemma 2.1 there exists an uncountable set $S \subseteq X$ and an integer n_0 such that $[S]^2 \subseteq P_{n_0}$. The collection $\{St(x, \mathcal{U}_{n_0}) : x \in S\}$ consists of pairwise disjoint open sets. From that it follows that, for any $z \in X$, the set $St(z, \mathcal{U}_{n_0})$ cannot meet S in two distinct points, which implies that the set S is closed and discrete.

We define a neighbourhood assignment ϕ for X as follows: if $x \in S$ let $\phi(x) = St(x, \mathcal{U}_{n_0})$ and if $x \in X \setminus S$ let $\phi(x) = X \setminus S$. Since X is dually weakly Lindelöf, there exists a weakly Lindelöf subspace Y such that $X = \bigcup\{\phi(y) : y \in Y\}$. By the way ϕ is defined, it follows that $S \subseteq \bigcup\{\phi(y) : y \in Y \cap S\}$ and hence $S \subseteq Y$. As X is normal, we may pick an open set V such that $S \subseteq V$ and $\overline{V} \subseteq \bigcup\{St(x, \mathcal{U}_{n_0}) : x \in S\}$. The trace on Y of the open cover $\{St(x, \mathcal{U}_{n_0}) : x \in X\} \cup \{X \setminus \overline{V}\}$ witnesses the failure of the weak Lindelöf property on Y. This is a contradiction and we are done. □

Related results for the classes of dually ccc spaces and for that of cellular-Lindelöf spaces were proved in [16] and [6].

Finally we will construct a space with a diagonal of rank 2, cellularity at most continuum and cardinality larger than the continuum, thus solving Problem 2 from [1]. Recall that a κ-Suslin Line L is a continuous linear order (endowed with the order topology) such that $c(L) \leq \kappa < d(L)$. The existence of a κ-Suslin Line for every $\kappa \geq \omega$ is consistent with ZFC (Jensen proved that it follows from $V = L$).

Theorem 2.3. *($V = L$) There is a space X with a diagonal of rank 2 such that $c(X) \leq \mathfrak{c}$ and $|X| \geq \mathfrak{c}^+$.*

Proof. Let T be an ω_1-Suslin Line. Let S be the set of all points of L which have countable cofinality. Since T is a continuous linear order the set S is dense in T and hence $d(S) > \aleph_1$. In particular, $|S| > \aleph_1$. Let τ be the topology on S generated by intervals of the form $(x, y]$, where $x < y \in S$. Note that $c((S, \tau)) = \aleph_1$ and that the space (S, τ) is first-countable and regular. So applying Mike Reed's *Moore Machine* (see, for example [9]) to (S, τ) we obtain a Moore space $\mathcal{M}(S)$ such that $|\mathcal{M}(S)| = |S| > \aleph_1$ and $c(\mathcal{M}(S)) \leq \aleph_1$. Recalling that Moore spaces have a diagonal of rank 2 (see Proposition 1.1 of [2]) and that $\mathfrak{c} = \aleph_1$ under $V = L$, we see that $X = \mathcal{M}(S)$ satisfies the statement of the theorem. □

The above theorem also shows that the assumption that the space is Baire is essential in Proposition 4.5 from [4], thus solving a question asked by the authors of [4] (see the paragraph after the proof of Lemma 4.6).

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