APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 21, no. 1 (2020), 53-56
doi:10.4995/agt.2020.11976

Counterexample to theorems on star versions of Hurewicz property

Manoj Bhardwaj

Department of Mathematics, University of Delhi, New Delhi-110007, India (manojmnj27@gmail.com)

Communicated by S. Romaguera

Abstract

In this paper, an example contradicting Theorem 4.5 and Theorem 5.3 [1] is provided and these theorems are proved under some extra hypothesis.

2010 MSC: *54D20; 54B20.*

Keywords: *Hurewicz spac; star-Hurewicz space; strongly star-Hurewicz space.*

1. Introduction

In covering properties, Hurewicz property is one of the most important property. In 1925, Hurewicz [4] (see also [5]) introduced Hurewicz property in topological spaces and studied it. This property is stronger than Lindelöf and weaker than σ-compactness. In 2004, the authors M. Bonanzinga, F. Cammaroto, Lj.D.R. Kočinac [1] introduced the star version of Hurewicz property.

For the terms and symbols that we do not define follow [2]. The basic definitions are given.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

In [6], Kočinac introduced star selection principles in the following way.

The symbol $S_1^\star(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $< \mathcal{U}_n : n \in \omega >$ of elements of $\mathcal{A}$ there exists a sequence $< U_n : n \in \omega >$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

The symbol $S_{fin}^\star(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $< \mathcal{U}_n : n \in \omega >$ of elements of $\mathcal{A}$ there exists a sequence $< \mathcal{V}_n : n \in \omega >$

Received 12 June 2019 – Accepted 24 November 2019

such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\omega}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$

The symbol $U^{\star}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $< \mathcal{U}_n : n \in \omega >$ of elements of $\mathcal{A}$ there exists a sequence $< \mathcal{V}_n : n \in \omega >$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$ or there is some n such that $St(\bigcup \mathcal{V}_n,\mathcal{U}_n) = X$.

Let $\mathcal{K}$ be a family of subsets of X. Then we say that X belongs to the class $SS^{\star}_{\mathcal{K}}(\mathcal{A},\mathcal{B})$ if X satisfies the following selection hypothesis that for every sequence $< \mathcal{U}_n : n \in \omega >$ of elements of $\mathcal{A}$ there exists a sequence $< K_n : n \in \omega >$ of elements of $\mathcal{K}$ such that $\{St(K_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $SS^{\star}_1(\mathcal{A},\mathcal{B})$ [resp., $SS^{\star}_{fin}(\mathcal{A},\mathcal{B})$, $SS^{\star}_{comp}(\mathcal{A},\mathcal{B})$] instead of $SS^{\star}_{\mathcal{K}}(\mathcal{A},\mathcal{B})$.

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

- $\mathcal{O}$: the collection of all open covers of X.
- Ω : the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is an ω-cover [3] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$.
- Γ : the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is a γ-cover [3] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.
- $\mathcal{O}^{gp}$: the collection of groupable open covers. An open cover $\mathcal{U}$ of X is groupable [7] if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n.

2. Main results

In [1], Bonanzinga, Cammaroto and Kočinac introduced the notion of $SH_{\leq n}$ in topological spaces.

A space X is said to have $SH_{\leq n}$ if for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X there is a sequence $< \mathcal{V}_n : n \in \omega >$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ of cardinalilty atmost n and $\{St(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X.

Theorem 2.1 ([1])**.** *Let a space X satisfies $SH_{\leq n}$. Then X satisfies $S^{\star}_1(\mathcal{O},\mathcal{O}^{gp})$.*

According to definition of $SH_{\leq n}$, there does not exist any topological space which satisfies $SH_{\leq n}$, take any topological space X and $\mathcal{U}_n = \{X\}$ for each n. Then for any finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$, $St(\bigcup \mathcal{V}_n,\mathcal{U}_n) = X$ for each n. So $\{X\}$ is not a γ-cover of X since it is finite. To avoid this possibility, without loss of generality if we consider infinite open covers for the Theorem then following example shows that the above theorem is not correct.

Example 2.2. There is a space which satisfies $SH_{\leq n}$ but does not satisfy $S^{\star}_1(\mathcal{O},\mathcal{O}^{gp})$.

Proof. Let $\mathbb{N}$ be set of natural numbers with discrete topology on it. Since $\mathbb{N}$ is countable, it satisfies $SH_{\leq n}$. Now consider a sequence $< \mathcal{U}_n = \{\{n\} : n \in \omega\} >$ of open covers for each n. Then it does not satisfy $S_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$ since $\{\{n\} : n \in \omega\}$ is not groupable. $\square$

For the existence of $SH_{\leq n}$, we consider only infinite covers such that X does not belong to each cover. In order to prove above theorem we need more hypothesis on a space X, that is, we define $CDR^{\star}_{sub}(\mathcal{A}, \mathcal{B})$.

Definition 2.3 ([8])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the infinite set S. Then $CDR_{sub}(\mathcal{A}, \mathcal{B})$ denotes the statement that for each sequence $< A_n : n \in \omega >$ of elements of $\mathcal{A}$ there is a sequence $< B_n : n \in \omega >$ such that for each n, $B_n \subseteq A_n$, for $m \neq n$, $B_m \cap B_n = \varnothing$, and each B_n is a member of $\mathcal{B}$.

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the infinite set S. Then $CDR^{\star}_{sub}(\mathcal{A}, \mathcal{B})$ denotes the statement that for each sequence $< A_n : n \in \omega >$ of elements of $\mathcal{A}$ there is a sequence $< B_n : n \in \omega >$ such that for each n, $B_n \subseteq A_n$, for $m \neq n$, $\{St(B, A_m) : B \in B_m\} \cap \{St(B, A_n) : B \in B_n\} = \varnothing$, and each B_n is a member of $\mathcal{B}$.

Theorem 2.5. *Let a space X satisfies $SH_{\leq n}$ and $CDR^{\star}_{sub}(\mathcal{O}, \mathcal{O})$. Then X satisfies $S_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$.*

Proof. The proof is similer as given in [1] with necessary modifications. $\square$

In [1], Bonanzinga, Cammaroto and Kočinac consider the hypothesis : for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each n, V_n has atmost n elements and $\{St(V_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X.

Theorem 2.6 ([1])**.** *Let a space X satisfies the following condition : for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each n, V_n has atmost n elements and $\{St(V_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Then X satisfies $SS_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$.*

According to hypothesis, there does not exist any topological space which satisfies the hypothesis considered in above Theorem, because take any topological space X and $\mathcal{U}_n = \{X\}$ for each n. Then for any finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$, $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$ for each n. So $\{X\}$ is not a γ-cover of X since it is finite. To avoid this possibility, without loss of generality if we consider infinite open covers for the Theorem then the following example shows that the above theorem is not correct.

Example 2.7. There is a space which satisfies following condition : for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each n, V_n has atmost n elements and $\{St(V_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X but does not satisfy $SS_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$.

Proof. Let $\mathbb{N}$ be set of natural numbers with discrete topology on it. Since $\mathbb{N}$ is countable, it satisfies following condition : for each sequence $< \mathcal{U}_n : n \in \omega >$

of open covers of X there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each n, V_n has at most n elements and $\{St(V_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Now consider a sequence $< \mathcal{U}_n = \{\{n\} : n \in \omega\} >$ of open covers for each n. Then it does not satisfy $SS_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$ since $\{\{n\} : n \in \omega\}$ is not groupable. □

For the existence of above hypothesis, we consider only infinite covers such that X does not belong to each cover. In order to prove above Theorem, we need more hypothesis on a space X, that is, we define $CDRF^{\star}_{sub}(\mathcal{A}, \mathcal{B})$.

Definition 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the infinite set S. Then $CDRF^{\star}_{sub}(\mathcal{A}, \mathcal{B})$ denotes the statement that for each sequence $< A_n : n \in \omega >$ of elements of $\mathcal{A}$ there is a sequence $< B_n : n \in \omega >$ such that for each n, $B_n \subseteq A_n$, for $m \neq n$ and for each finite subset F of S, $\{St(x, B_m) : x \in F\} \cap \{St(x, B_n) : x \in F\} = \varnothing$, and each B_n is a member of $\mathcal{B}$.

Theorem 2.9. *Let a space X satisfies $CDRF^{\star}_{sub}(\mathcal{O}, \mathcal{O})$ and the following condition : for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each n, V_n has atmost n elements and $\{St(V_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Then X satisfies $SS_1^{\star}(\mathcal{O}, \mathcal{O}^{gp})$.*

Proof. The proof is similer as given in [1] with necessary modifications. □

Acknowledgements. *The author acknowledges the fellowship grant of University Grants Commission, India.*

References

[1] M. Bonanzinga, F. Cammaroto and Lj. D. R. Kočinac, Star-Hurewicz and related properties, Appl. Gen. Topol. 5, no. 1 (2004), 79–89.

[2] R. Engelking, General Topology, Revised and completed edition, Heldermann Verlag Berlin (1989).

[3] J. Gerlits and Zs. Nagy, Some properties of $C(X), I$, Topology Appl. 14 (1982), 151–161.

[4] W. Hurewicz, Über eine verallgemeinerung des Borelschen Theorems, Math. Z. 24 (1925), 401–421.

[5] W. Hurewicz, Über Folgen stetiger Funktionen, Fund. Math. 9 (1927), 193–204.

[6] Lj. D. Kočinac, Star-Menger and related spaces, Publ. Math. Debrecen 55 (1999), 421–431.

[7] Lj. D. Kočinac and M. Scheepers, Combinatorics of open covers (VII): Groupability, Fund. Math. 179 (2003), 131–155.

[8] M. Scheepers, Combinatorics of open covers (I) : Ramsey theory, Topology Appl. 69 (1996), 31–62.

