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Numerical reckoning fixed points via new faster iteration process

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Abstract

In this paper, we propose a new iteration process which is faster than the leading; S [J. Nonlinear Convex Anal. 8, no. 1 (2007), 61–79], Thakur et al. [App. Math. Comp. 275 (2016), 147–155] and M [Filomat 32, no. 1 (2018), 187–196] iterations for numerical reckoning fixed points. Using this new iteration process, some fixed point convergence results for generalized α-nonexpansive mappings in the setting of uniformly convex Banach spaces are proved. At the end of paper, we offer a numerical example to compare the rate of convergence of the proposed iteration process with the leading iteration processes.

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1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$. Let X be a Banach space and M be a nonempty subset of X. A mapping

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$T : M \to M$ is said to nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \text{ for all } x, y \in M.$$

An element $p \in M$ is said to be a fixed point of T if $p = T(p)$. From now on, we will denote the set of all fixed points of T by $F(T)$. A mapping $T : M \to M$ is said to be a quasi-nonexpansive mapping if $F(T) \neq \varnothing$ and $||T(x) - T(p)|| \leq ||x - p||$ for all $x \in M$ and $p \in F(T)$. It is well-known that $F(T)$ is nonempty in the case when X is uniformly convex, T is nonexpansive and M is closed, bounded and convex; see [6, 7, 10]. A number of generalizations of nonexpansive mappings have been considered by some researchers in recent years. Suzuki [17] introduced a new class of mappings known as Suzuki generalized nonexpansive mappings which is a condition on mappings called condition (C) and obtained some convergence and existence results for such mappings. Note that, a mapping $T : M \to M$ is said to satisfy condition (C) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \text{ implies } ||Tx - Ty|| \leq ||x - y||,$$

for each $x, y \in M$.

Aoyama and Kohsaka [4] introduced the class of α-nonexpansive mappings in the framework of Banach spaces and obtained some fixed point results for such mappings. A mapping $T : M \to M$ is said to be α-nonexpansive if there exists a real number $\alpha \in [0, 1)$ such that for all $x, y \in M$,

$$||Tx - Ty||^2 \leq \alpha||Tx - y||^2 + \alpha||x - Ty||^2 + (1 - 2\alpha)||x - y||^2.$$

Ariza-Puiz et al. [5] proved that the concept of α-nonexpansive is trivial for $\alpha < 0$. It is obvious that every nonexpansive mapping is 0-nonexpansive and also every α-nonexpansive mapping with $F(T) \neq \varnothing$ is a quasi-nonexpansive. Note that, in general condition (C) and α-nonexpansive mappings are not continuous (see [17] and [14]).

Recently, Pant and Shukla [14] introduced an interesting class of generalized nonexpansive mappings in Banach spaces known as generalized α-nonexpansive mappings which contains the class of Suzuki generalized nonexpansive mappings. A mapping $T : M \to M$ is said to generalized α-nonexpansive if there exists a real number $\alpha \in [0, 1)$ such that for each $x, y \in M$,

$$\frac{1}{2}||x-Tx|| \leq ||x-y|| \Rightarrow ||Tx-Ty|| \leq \alpha||Tx-y||+\alpha||Ty-x||+(1-2\alpha)||x-y||.$$

Once the existence result of a fixed point for a mapping is established, an algorithm to find the value of the fixed point is desirable. The famous Banach contraction mapping principle uses Picard iteration $x_{n+1} = Tx_n$ for approximation of fixed point. Some other well-known iterations are the Mann [11], Ishikawa [9], S [3], Picard-S [8], Noor [12], Abbas [1], Thakur et al. [19] and so on. Speed of convergence plays an important role for an iteration process to be preferred on another iteration process. Rhoades [15] mentioned that the Mann iteration process for a decreasing function converges faster than the Ishikawa iteration process and for an increasing function the Ishikawa iteration process is better than the Mann iteration process.

The well-known Mann [11] and Ishikawa [9] iteration schemes are respectively defined as:

$$(1.1) \qquad \begin{cases} x_1 \in M, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n \in (0,1)$.

$$(1.2) \qquad \begin{cases} x_1 \in M, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

In 2007, Agarwal et al. [3] introduced the following iteration process known as S iteration:

$$(1.3) \qquad \begin{cases} x_1 \in M, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$. They proved that the rate of convergence of iteration process (1.3) is same to the Picard iteration $x_{n+1} = T x_n$ and faster than the Mann [11] iteration process in the class of contraction mappings.

In 2016, Thakur et al. [19] introduced the following iteration scheme:

$$(1.4) \qquad \begin{cases} x_1 \in M, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T\left((1-\alpha_n)x_n + \alpha_n z_n\right), \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$. With the help of a numerical example, they proved that (1.4) is faster than the Picard, Mann [11], Ishikawa [9], S [3], Noor [12] and Abbas [1] iteration processes in the class of Suzuki generalized nonexpansive mappings.

Recently in 2018, Ullah and Arshad [20] used a new iteration process known as M iteration:

$$(1.5) \qquad \begin{cases} x_1 \in M, \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n \in (0,1)$. With the help of a numerical example, they proved that (1.5) is faster than S [3], Picard-S [8] and Thakur et al. [19] iteration processes for Suzuki generalized nonexpansive mappings.

Problem 1.1. *Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration process (1.5) ?*

As an answer, we introduce the following new iteration called KF iteration scheme:

$$(1.6)\qquad \begin{cases} x_1 \in M, \\ z_n = T((1-\beta_n)x_n + \beta_n T x_n), \\ y_n = T z_n, \\ x_{n+1} = T((1-\alpha_n)T x_n + \alpha_n T y_n), n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

With the help of numerical example, we compare the rate of convergence of iteration (1.6) with the leading S (1.3), Thakur et al. (1.4) and M (1.5) iteration.

2. Preliminaries

In this section, we give some preliminaries.

Let X be a Banach space and M be a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in M. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} ||x - x_n||.$$

The asymptotic radius of $\{x_n\}$ relative to M is given by

$$r(M, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in M\}.$$

The asymptotic center of $\{x_n\}$ relative to M is the set

$$A(M, \{x_n\}) = \{x \in M : r(x, \{x_n\}) = r(M, \{x_n\})\}.$$

It is well-known that in a uniformly convex Banach space setting, $A(M, x_n)$ consists of exactly one point. Also, $A(M, x_n)$ is nonempty and convex when M is weakly compact and convex (see, [18] and [2]). A Banach space X is said to uniformly convex if for all $\varepsilon > 0$, there is a $\lambda > 0$ such that, for $x, y \in X$ with $||x|| \le 1$, $||y|| \le 1$ and $||x - y|| \le \varepsilon$, $||x + y|| \le 2(1 - \lambda)$ holds. Note that, a Banach space X is said to have Opial's property [13] if for each sequence $\{x_n\}$ in X which weakly converges to $x \in X$ and for every $y \in X$, it follows the following

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||.$$

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$.

We now list some basic facts about generalized α-nonexpansive mappings, which can be found in [14].

Proposition 2.1. *Let X be a Banach space, M be a nonempty subset of X and $T : M \to M$ be a mapping.*

(i) *If T is a Suzuki generalized nonexpansive mapping, then T is a generalized α-nonexpansive mapping.*
(ii) *If T is a generalized α-nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii) *If T is a generalized α-nonexpansive mapping. Then $F(T)$ is closed. Moreover, if X is strictly convex and M is convex, then $F(T)$ is also convex.*

(iv) *If T is a generalized α-nonexpansive mapping. Then for each $x, y \in M$,*

$$||x - Ty|| \leq \left(\frac{3+\alpha}{1-\alpha}\right) ||x - Tx|| + ||x - y||.$$

(v) *If X has Opial property, T is generalized α-nonexpansive, $\{x_n\}$ converges weakly to a point v and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$, then $v \in F(T)$.*

Lemma 2.2 ([16]). *Let X be a uniformly convex Banach space and $0 < p \leq \alpha_n \leq q < 1$ for every $n \in \mathbb{N}$. If $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n\to\infty} ||x_n|| \leq t$, $\limsup_{n\to\infty} ||y_n|| \leq t$ and $\lim_{n\to\infty} ||\alpha_n x_n + (1 - \alpha_n)y_n|| = t$ for some $t \geq 0$ then, $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

3. Main Results

We open this section with the following important lemma.

Lemma 3.1. *Let M be a nonempty closed convex subset of a Banach space X and $T : M \to M$ be a generalized α-nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by (1.6), then $\lim_{n\to\infty} ||x_n - p||$ exists for each $p \in F(T)$.*

Proof. Let $p \in F(T)$. By Proposition 2.1 part (ii), we have

$$\begin{aligned} ||z_n - p|| &= ||T((1-\beta_n)x_n + \beta_n Tx_n) - p|| \\ &\leq ||(1-\beta_n)x_n + \beta_n Tx_n - p|| \\ &\leq (1-\beta_n)||x_n - p|| + \beta_n||Tx_n - p|| \\ &\leq (1-\beta_n)||x_n - p|| + \beta_n||x_n - p|| \\ &\leq ||x_n - p||, \end{aligned}$$

and

$$\begin{aligned} ||y_n - p|| &= ||Tz_n - p|| \\ &\leq ||z_n - p||. \end{aligned}$$

They imply that,

$$\begin{aligned} ||x_{n+1} - p|| &= ||T((1-\alpha_n)Tx_n + \alpha_n Ty_n) - p|| \\ &\leq ||(1-\alpha_n)Tx_n + \alpha_n Ty_n - p|| \\ &\leq (1-\alpha_n)||Tx_n - p|| + \alpha_n||Ty_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||y_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||z_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned}$$

Thus $\{||x_n - p||\}$ is bounded and nonincreasing, which implies that $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F(T)$. $\square$

The following theorem is necessary for the next results.

Theorem 3.2. *Let M be a nonempty closed convex subset of a uniformly convex Banach space X and $T : M \to M$ a generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by (1.6). Then, $F(T) \neq \varnothing$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$.*

Proof. Suppose that $F(T) \neq \varnothing$ and $p \in F(T)$. Then, by Lemma 3.1, $\lim_{n\to\infty} ||x_n - p||$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} ||x_n - p|| = t. \tag{3.1}$$

In view of the proof of Lemma 3.1 together with (3.1), we have

$$\limsup_{n\to\infty} ||z_n - p|| \le \limsup_{n\to\infty} ||x_n - p|| = t. \tag{3.2}$$

By Proposition 2.1 part (ii), we have

$$\limsup_{n\to\infty} ||Tx_n - p|| \le \limsup_{n\to\infty} ||x_n - p|| = t. \tag{3.3}$$

Again by the proof of Lemma 3.1, we have

$$||x_{n+1} - p|| \le (1 - \alpha_n)||x_n - p|| + \alpha_n ||z_n - p||.$$

It follows that,

$$||x_{n+1} - p|| - ||x_n - p|| \le \frac{||x_{n+1} - p|| - ||x_n - p||}{\alpha_n} \le ||z_n - p|| - ||x_n - p||.$$

So, we can get $||x_{n+1} - p|| \le ||z_n - p||$ and from (3.1), we have

$$t \le \liminf_{n\to\infty} ||z_n - p||. \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$t = \lim_{n\to\infty} ||z_n - p||. \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
t &= \lim_{n\to\infty} ||z_n - p|| \\
&= \lim_{n\to\infty} ||T((1 - \beta_n)x_n + \beta_n Tx_n) - p|| \\
&\le \lim_{n\to\infty} ||(1 - \beta_n)x_n + \beta_n Tx_n - p|| \\
&= \lim_{n\to\infty} ||(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)|| \\
&\le \lim_{n\to\infty} (1 - \beta_n)||x_n - p|| + \lim_{n\to\infty} \beta_n ||Tx_n - p|| \\
&\le \lim_{n\to\infty} (1 - \beta_n)||x_n - p|| + \lim_{n\to\infty} \beta_n ||x_n - p|| \\
&\le t.
\end{aligned}$$

Hence,

$$t = \lim_{n\to\infty} ||(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)||. \tag{3.6}$$

Now from (3.1), (3.3) and (3.6) together with Lemma 2.2, we obtain

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0.$$

Conversely, we assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. Let $p \in A(M, \{x_n\})$. By proposition 2.1 part (iv), we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty} ||x_n - Tp|| \\ &\le \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{n\to\infty} ||Tx_n - x_n|| + \limsup_{n\to\infty} ||x_n - p|| \\ &= \limsup_{n\to\infty} ||x_n - p|| \\ &= r(p, \{x_n\}). \end{aligned}$$

Hence, we conclude that $Tp \in A(M, \{x_n\}$. Since X is uniformly convex, $A(M, \{x_n\})$ consist of a unique element. Thus, we have $p = T(p)$. □

First we prove our weak convergence result.

Theorem 3.3. *Let X be a uniformly Banach space with Opial property, M a nonempty closed convex subset of X and $T : M \to M$ be generalized α-nonexpansive mapping with $F(T) \neq \varnothing$. Then, $\{x_n\}$ generated by (1.6) converges weakly to an element of $F(T)$.*

Proof. By Theorem 3.2, $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. Since X is uniformly convex, X is reflexive. So, a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ exists such that $\{x_{n_i}\}$ converges weakly to some $v_1 \in M$. By Proposition 2.1 part (v), we have $v_1 \in F(T)$. It is sufficient to show that $\{x_n\}$ converges weakly to v_1. In fact, if $\{x_n\}$ does not converges weakly to v_1. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $v_2 \in M$ such that $\{x_{n_j}\}$ converges weakly to v_2 and $v_2 \neq v_1$. Again by Proposition 2.1 part (v), $v_2 \in F(T)$. By Lemma 3.1 together with Opial property, we have

$$\begin{aligned} \lim_{n\to\infty} ||x_n - v_1|| &= \lim_{i\to\infty} ||x_{n_i} - v_1|| \\ &< \lim_{i\to\infty} ||x_{n_i} - v_2|| \\ &= \lim_{n\to\infty} ||x_n - v_2|| \\ &= \lim_{j\to\infty} ||x_{n_j} - v_2|| \\ &< \lim_{j\to\infty} ||x_{n_j} - v_1|| \\ &= \lim_{n\to\infty} ||x_n - v_1||. \end{aligned}$$

This is a contradiction, so, $v_1 = v_2$. Thus, $\{x_n\}$ converges weakly to $v_1 \in F(T)$. □

We now prove our strong convergence result.

Theorem 3.4. *Let M be a nonempty closed convex subset of a uniformly convex Banach space X and $T : M \to M$ be a generalized α-nonexpansive mapping. If $F(T) \neq \varnothing$ and $\liminf_{n\to\infty} dist(x_n, F(T)) = 0$ (where $dist(x, F(T)) = \inf\{||x - p|| : p \in F(T)\}$). Then, $\{x_n\}$ generated by (1.6) converges strongly to an element of $F(T)$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} ||x_n - p||$ exists, for each $p \in F(T)$. So, $\lim_{n\to\infty} dist(x_n, F(T))$ exists, thus

$$\lim_{n\to\infty} dist(x_n, F(T)) = 0.$$

Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{v_k\}$ in F_T such that $||x_{n_k} - v_k|| \leq \frac{1}{2^k}$ for each $k \in \mathbb{N}$. By the proof of Lemma 3.1, $\{x_n\}$ is nonincreasing, so

$$||x_{n_{k+1}} - v_k|| \leq ||x_{n_k} - v_k|| \leq \frac{1}{2^k}.$$

Therefore,

$$\begin{aligned} ||v_{k+1} - v_k|| &\leq ||v_{k+1} - x_{n_{k+1}}|| + ||x_{n_{k+1}} - v_k|| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &\leq \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

Hence, $\{v_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to some p. Since, by Proposition 2.1 part (iii), $F(T)$ is closed, we have $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} ||x_n - p||$ exists, hence $\{x_n\}$ converges strongly to $p \in F(T)$. □

4. Example

We compare rate of convergence of our new KF iteration (1.6) with leading S (1.3), M (1.5) Thakur et al. (1.4) in slightly general setting using Example 4.1, in which T is generalized α-nonexpansive but not Suzuki generalized nonexpansive.

Example 4.1. Let $M = [0, \infty)$ with absolute valued norm. Define a mapping $T : M \to M$ by

$$Tx = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{5000}\right) \\ \frac{x}{2} & \text{if } x \in \left[\frac{1}{5000}, \infty\right). \end{cases}$$

Choose $x = \frac{1}{8000}$ and $y = \frac{1}{5000}$. We see that, $\frac{1}{2}|x - Tx| < |x - y|$ but $|Tx - Ty| > |x - y|$. Thus, T does not satisfy condition (C) and so T is not Suzuki generalized nonexpansive. On the other hand, T is a generalized α-nonexpansive mapping. In fact, for $\alpha = \frac{1}{3}$, we have:

Case I: When $x, y \in \left[0, \frac{1}{5000}\right)$, then clearly

$$\frac{1}{3}|Tx - y| + \frac{1}{3}|x - Ty| + \frac{1}{3}|x - y| \geq 0 = |Tx - Ty|.$$

Case II: When $x \in \left[\frac{1}{5000}, \infty\right)$ and $y \in \left[0, \frac{1}{5000}\right)$, we have

$$\begin{aligned}
\frac{1}{3}|Tx-y| + \frac{1}{3}|x-Ty| + \frac{1}{3}|x-y| &= \frac{1}{3}\left|\frac{x}{2}-y\right| + \frac{1}{3}|x-0| + \frac{1}{3}|x-y| \\
&\geq \frac{1}{3}\left|\left(\frac{x}{2}-y\right)-(x-y)\right| + \frac{1}{3}|x| \\
&= \frac{1}{3}\left|\frac{x}{2}\right| + \frac{1}{3}|x| \\
&\geq \frac{1}{3}\left|\frac{x}{2}+x\right| \\
&= \frac{1}{2}|x| \\
&= |Tx-Ty|.
\end{aligned}$$

Case III: When $x, y \in [\frac{1}{5000}, \infty)$, we have

$$\begin{aligned}
\frac{1}{3}|Tx-y| + \frac{1}{3}|x-Ty| + \frac{1}{3}|x-y| &= \frac{1}{3}\left|\frac{x}{2}-y\right| + \frac{1}{3}\left|x-\frac{y}{2}\right| + \frac{1}{3}|x-y| \\
&\geq \frac{1}{3}\left|\left(\frac{x}{2}-y\right)+\left(x-\frac{y}{2}\right)\right| + \frac{1}{3}|x-y| \\
&= \frac{1}{2}|x-y| + \frac{1}{3}|x-y| \\
&\geq \frac{1}{2}|x-y| \\
&= |Tx-Ty|.
\end{aligned}$$

Hence, T is a generalized α-nonexpansive mapping with $F(T) = \{0\}$. Take $\alpha_n = 0.70$ and $\beta_n = 0.65$. The iterative values for $x_1 = 10$ are given in Table 1. Figure 1 shows the convergence behaviors of different iterative schemes. Clearly the new KF iteration process is moving fast to the fixed point of T as compared to other iteration processes.

Table 1. Sequences generated by KF (1.6), M (1.5), Thakur et al. (1.4) and S (1.3) iteration schemes for mapping T of Example 4.1.

	KF (1.6)	M (1.5)	Thakur et al. (1.4)	S (1.3)
x_1	10	10	10	10
x_2	1.0453120000	1.62500000000	1.9312500000	3.8625000000
x_3	0.1092678222	0.26406250000	0.3729726562	1.4918906250
x_4	0.0114219020	0.04291015625	0.0720303442	0.5762427539
x_5	0.0011939456	0.00697290039	0.0139108602	0.2225737636
x_6	0.0001248046	0.00113309631	0.0026865348	0.0859691162
x_7	0	0.00018412815	0.0005188370	0.0332055711
x_8	0	0	0.0001002004	0.0128256518
x_9	0	0	0	0.0049539080
x_{10}	0	0	0	0.0019134469
x_{11}	0	0	0	0.0007390688
x_{12}	0	0	0	0.0002854653

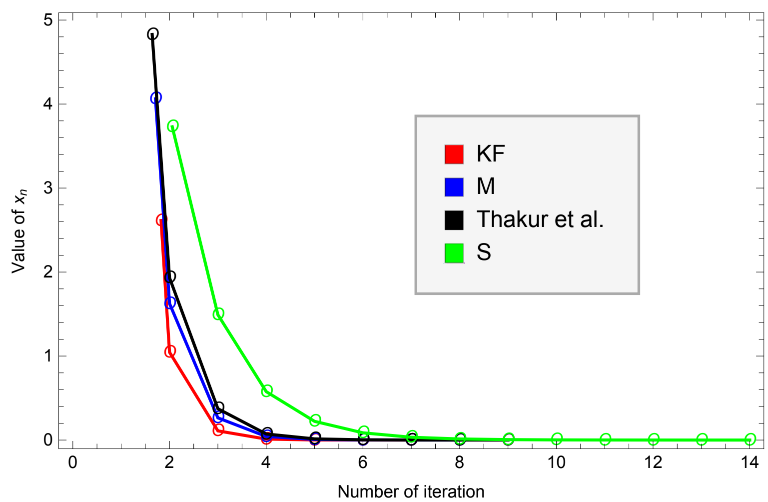

Figure 1. Convergence behaviors of KF, M, Thakur et al. and S iteration processes to the fixed point of the mapping defined in Example 4.1 where $x_1 = 10$.

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