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On a metric on the space of idempotent probability measures

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Abstract

In this paper we introduce a metric on the space $I(X)$ of idempotent probability measures on a given compact metric space (X, ρ), which extends the metric ρ. It is proven the introduced metric generates the pointwise convergence topology on $I(X)$.

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1. Introduction

Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i. e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra $\mathbb{R}_{\max}$.

Many authors (S. C. Kleene, S. N. N. Pandit, N. N. Vorobjev, B. A. Carré, R. A. Cuninghame-Green, K. Zimmermann, U. Zimmermann, M. Gondran, F. L. Baccelli, G. Cohen, S. Gaubert, G. J. Olsder, J.-P. Quadrat, and others) used idempotent semirings and matrices over these semirings for solving some applied problems in computer science and discrete mathematics, starting from the classical paper by S. C. Kleene [7].

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The modern idempotent analysis (or idempotent calculus, or idempotent mathematics) was founded by V. P. Maslov and his collaborators [10]. Some preliminary results are due to E. Hopf and G. Choquet, see [2], [5].

Idempotent mathematics can be treated as the result of a dequantization of the traditional mathematics over numerical fields as the Planck constant h tends to zero taking imaginary values. This point of view was presented by G. L. Litvinov and V. P. Maslov [11]. In other words, idempotent mathematics is an asymptotic version of the traditional mathematics over the fields of real and complex numbers.

The basic paradigm is expressed in terms of an idempotent correspondence principle. This principle is closely related to the well-known correspondence principle of N. Bohr in quantum theory. Actually, there exists a heuristic correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i. e., semirings and semifields with idempotent addition).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over fields, its "shadow," idempotent mathematics, appears. This "shadow" stands approximately in the same relation to traditional mathematics as classical physics does to quantum theory.

Recall [10] that a set S equipped with two algebraic operations: addition $\oplus$ and multiplication $\odot$, is said to be a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$:

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z \quad \text{and}$$
$$(x \oplus y) \odot z = x \odot z \oplus y \odot z$$

  for all $x,\ y,\ z \in S$.

A *unit* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of the semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x \oplus \mathbf{0} = x$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A (an idempotent) semiring S with neutral elements $\mathbf{0}$ and $\mathbf{1}$ is called a (an *idempotent*) *semifield* if every nonzero element of S is invertible. Note that dioïds, quantales and inclines are examples of idempotent semirings [10].

Let $\mathbb{R} = (-\infty, +\infty)$ be the field of real numbers and $\mathbb{R}_+ = [0, +\infty)$ be the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). Consider a map $\Phi_h \colon \mathbb{R}_+ \to S = \mathbb{R} \cup \{-\infty\}$ defined by the equality

$$\Phi_h(x) = h \ln x, \qquad h > 0.$$

Let $x, y \in X$ and $u = \Phi_h(x)$, $v = \Phi_h(y)$. Put $u \oplus_h v = \Phi_h(x + y)$ and $u \odot v = \Phi_h(xy)$. The imagine $\Phi_h(0) = -\infty$ of the usual zero 0 is a zero $\mathbf{0}$ and the imagine $\Phi_h(1) = 0$ of the usual unit 1 is a unit $\mathbf{1}$ in S with respect to these new operations. Thus S obtains the structure of a semiring $\mathbb{R}^{(h)}$ isomorphic to $\mathbb{R}_+$.

A direct check shows that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. The convention $-\infty \odot x = -\infty$ allows us to extend $\oplus$ and $\odot$ over S. It can easily be checked that S forms a semiring with respect to the addition $u \oplus v = \max\{u, v\}$ and the multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by $\mathbb{R}_{\max}$; it is idempotent, i. e., $u \oplus u = u$ for all its elements u. The semiring $\mathbb{R}_{\max}$ is actually a semifield. The analogy with quantization is obvious; the parameter h plays the role of the Planck constant, so $\mathbb{R}_+$ can be viewed as a "quantum object" and $\mathbb{R}_{\max}$ as the result of its "dequantization". This passage to $\mathbb{R}_{\max}$ is called the *Maslov dequantization* (for details, see [8], [9], [15]).

The notion of idempotent (Maslov) measure finds important applications in different parts of mathematics, mathematical physics and economics (see the survey article [10] and the bibliography therein). Topological and categorical properties of the functor of idempotent measures were studied in [16], [17]. Although idempotent measures are not additive and the corresponding functionals are not linear, there are some parallels between topological properties of the functor of probability measures and the functor of idempotent measures (see, for example [15], [14], [16]) which are based on existence of natural equiconnectedness structure on both functors.

However, some differences appear when the problem of the metrizability of the space of idempotent probability measures is studied. The problem of the metrizability of the space of the usual probability measures was investigated in [3]. We show that the analog of the metric introduced in [3] (on the space of probability measures) is not a metric on the space of idempotent probability measures. We show the mentioned analog is only a pseudometric.

It is well-known that if (X, ρ) is a compact metric space, then the space $P(X)$ of probability measures can be endowed with the Kantorovich metric. In [17], M. Zarichnyi posed the problem of building a metric on the space of idempotent probability measures. Still the problem of existence of a natural metrization of the space $I(X)$ has been open. In this paper we give a positive answer and introduce a metric on the space of idempotent probability measures.

2. Idempotent probability measures. Preliminaries

Let X be a compact Hausdorff space, $C(X)$ be the algebra of continuous functions on X with the usual algebraic operations. On $C(X)$ the operations $\oplus$ and $\odot$ are determined by $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$ where φ, $\psi \in C(X)$.

Recall [17] that a functional $\mu\colon C(X) \to \mathbb{R}$ is said to be an *idempotent probability measure* on X if it satisfies the following properties:

(1) $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$, where λ_X is a constant function;

(2) $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

(3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

For a compact Hausdorff space X by $I(X)$ we denote the set of all idempotent probability measures on X. Since $I(X) \subset \mathbb{R}^{C(X)}$, we consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$. A family of sets of the form

$$\langle \mu;\ \varphi_1, \dots, \varphi_n;\ \varepsilon \rangle = \{\nu \in I(X) :\ |\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon,\ i = 1, \dots, n\}$$

is a base of open neighbourhoods of a given idempotent probability measure $\mu \in I(X)$ according to the induced topology, where $\varphi_i \in C(X)$, $i = 1, \dots, n$, and $\varepsilon > 0$. It is obvious that the induced topology and the pointwise convergence topology on $I(X)$ coincide.

Let X, Y be compact Hausdorff spaces and $f\colon X \to Y$ be a continuous map. It is easy to check that the map $I(f)\colon I(X) \to I(Y)$ determined by the formula $I(f)(\mu)(\psi) = \mu(\psi \circ f)$ is continuous. The construction I is a normal functor acting in the category of compact Hausdorff spaces and their continuous maps. Therefore, for each idempotent probability measure $\mu \in I(X)$ one may determine its *support*:

$$\mathrm{supp}\mu = \bigcap \left\{ A \subset X : \overline{A} = A,\ \mu \in I(A) \right\}.$$

Consider functions of the type $\lambda : X \to [-\infty, 0]$. On a given set X we determine a max-plus-*characteristic function* ${}^{\oplus}\chi_A\colon X \to \mathbb{R}_{\max}$ of a subset $A \subset X$ by the rule

$$(2.1) \qquad {}^{\oplus}\chi_A(x) = \begin{cases} 0 & \text{at } x \in A, \\ -\infty & \text{at } x \in X \setminus A. \end{cases}$$

For a singleton $\{x\}$ we will write ${}^{\oplus}\chi_x$ instead of ${}^{\oplus}\chi_{\{x\}}$.

Let $F_1, F_2, \dots, F_n$ be a disjoint system of closed sets of a space X, and $a_1, a_2, \dots, a_n$ be non-positive real numbers. A function

$$(2.2) \qquad {}^{\oplus}\chi^{a_1, \dots, a_n}_{F_1, \dots, F_n}(x) = \begin{cases} a_1 & \text{at } x \in F_1, \\ \dots, & \\ a_n & \text{at } x \in F_n, \\ -\infty & \text{at } x \in X \setminus \bigcup\limits_{i=1}^{n} F_n \end{cases}$$

we call the max-plus-*step-function* defined by the sets $F_1, F_2, \dots, F_n$ and the numbers $a_1, a_2, \dots, a_n$.

Note that

$${}^{\oplus}\chi^a_A(x) = a \odot {}^{\oplus}\chi_A(x) = \begin{cases} 0 \odot a & \text{at } x \in A, \\ -\infty & \text{at } x \in X \setminus A \end{cases} = \begin{cases} a & \text{at } x \in A, \\ -\infty & \text{at } x \in X \setminus A \end{cases}$$

for a set A in X and a non-positive number a. Consequently, for a disjoint system of closed sets $F_1, F_2, \dots, F_n$ in a space X, and non-positive real numbers $a_1, a_2, \dots, a_n$ we have

$$^{\oplus}\chi_{F_1,\dots,F_n}^{a_1,\dots,a_n}(x) = {}^{\oplus}\chi_{F_1}^{a_1}(x) \oplus {}^{\oplus}\chi_{F_2}^{a_2}(x) \oplus \dots \oplus {}^{\oplus}\chi_{F_n}^{a_n}(x).$$

In the case when $F_1, F_2, \dots, F_n$ are singletons, say $F_i = \{x_i\}$, $i = 1, \dots, n$, we have

$$^{\oplus}\chi_{\{x_1\},\dots,\{x_n\}}^{a_1,\dots,a_n} = {}^{\oplus}\chi_{\{x_1\}}^{a_1} \oplus {}^{\oplus}\chi_{\{x_2\}}^{a_2} \oplus \dots \oplus {}^{\oplus}\chi_{\{x_n\}}^{a_n}. \tag{2.3}$$

The notion of density for an idempotent measure was introduced in [8], where the main result on the existence on densities for arbitrary measures was proved. A more detailed exposition is given in [9] – the first systematic monograph on the idempotent analysis. Later the paper [1] appeared, where further investigations of densities were done. Let $\mu \in I(X)$. Then we can define a function $d_\mu \colon X \to [-\infty, 0]$ by the formula

$$d_\mu(x) = \inf\{\mu(\varphi) : \varphi \in C(X) \text{ such that } \varphi \le 0 \text{ and } \varphi(x) = 0\}, \qquad x \in X. \tag{2.4}$$

The function d_μ is upper semicontinuous and is called the *density* of μ. Conversely, each upper semicontinuous function $f \colon X \to [-\infty, 0]$ with $\max\{f(x) : x \in X\} = 0$ determines an idempotent measure ν_f by the formula

$$\nu_f(\varphi) = \bigoplus_{x \in X} f(x) \odot \varphi(x), \qquad \varphi \in C(X). \tag{2.5}$$

Note that a function $f \colon X \to \mathbb{R}$ is said to be *upper semicontinuous* if for each $x \in X$, and for every real number r which satisfies $f(x) < r$, there exists an open neighbourhood $U \subset X$ of x such that $f(x') < r$ for all $x' \in U$. It is easy to see that functions defined by (2.1) or by (2.2) are upper semicontinuous.

Put

$$U_S(X) = \big\{\lambda \colon X \to [-\infty, 0] \,\big|\, \lambda \text{ is upper semicontinuous and there exists a } x_0 \in X \text{ such that } \lambda(x_0) = 0\big\}.$$

Then we have

$$I(X) = \left\{\bigoplus_{x \in X} \lambda(x) \odot \delta_x : \lambda \in U_S(X)\right\}.$$

Obviously that $\bigoplus_{x \in X} {}^{\oplus}\chi_{x_0}(x) \odot \delta_x = \delta_{x_0}$, i. e. for a max-plus-characteristic function ${}^{\oplus}\chi_{x_0}$ formula (2.5) defines the Dirac measure δ_{x_0} supported on the singleton $\{x_0\}$. A set of all Dirac measures on a Hausdorff compact space X we denote by $\delta(X)$. It is easy to notice that the space X and the subspace $\delta(X) \subset I(X)$ are homeomorphic. This phenomenon gives us the opportunity to consider X as subspace of $I(X)$.

Let A be a closed subset of a compact Hausdorf space X. It is easy to check that $\nu \in I(A)$ iff $\{x \in X : d_\nu(x) > -\infty\} \subset A$. Hence,

$$\operatorname{supp} \nu = \{x \in X : d_\nu(x) > -\infty\}.$$

It is evident that $\operatorname{supp}\nu = \{x_1, \ldots, x_n\}$ if and only if the density d_ν of ν has the shape (2.3) for singletons $\{x_1\}, \ldots, \{x_n\}$ and for some non-negative numbers $a_1, \ldots, a_n$ with $a_i > -\infty$, $i = 1, \ldots, n$, and $\max\{a_1, \ldots, a_n\} = 0$. In this case ν is said to be an *idempotent probability measure with finite support*. A subset of $I(X)$ consisting of all idempotent probability measures with finite support we denote by $I_\omega(X)$.

Consider an idempotent probability measure $\mu = \bigoplus\limits_{x\in X} \lambda(x) \odot \delta_x \in I(X)$ and a finite system $\{U_1, \ldots, U_n\}$ of open sets $U_i \subset X$ such that $\operatorname{supp}\mu \cap U_i \neq \varnothing$, $i = 1, \ldots, n$, and $\operatorname{supp}\mu \subset \bigcup\limits_{i=1}^{n} U_i$. Define a set

$$(2.6)\quad \langle \mu; U_1 \ldots, U_n; \varepsilon\rangle = \Big\{\nu = \bigoplus_{x\in X} \gamma(x) \odot \delta_x \in I(X) :$$
$$\operatorname{supp}\nu \cap U_i \neq \varnothing,\ \operatorname{supp}\nu \subset \bigcup_{i=1}^{n} U_i, \text{ and } |\lambda(x) - \gamma(y)| < \varepsilon$$
$$\text{at the points } x \in \operatorname{supp}\mu \cap U_i \text{ and } y \in \operatorname{supp}\nu \cap U_i,\ i = 1, \ldots, n, \Big\}.$$

Theorem 2.1. *The sets of the type (2.6) form a base of the pointwise convergence topology in $I(X)$.*

Proof. Let $\langle \mu; \varphi; \varepsilon\rangle$ be a prebase element, where $\varphi \in C(X)$, $\varepsilon > 0$ and $\mu = \bigoplus\limits_{x\in X} \lambda(x) \odot \delta_x \in I(X)$. As φ is continuous, for each point $x \in \operatorname{supp}\mu$ there is its open neighbourhood U_x in X such that for any point $y \in U_x$ the inequality $|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2}$ holds. From the open cover $\{U_x : x \in \operatorname{supp}\mu\}$ in X of $\operatorname{supp}\mu$ by owing to compactness of $\operatorname{supp}\mu$ one can choose a finite subcover $\{U_i : i = 1, \ldots, n\}$. Further, for every $\nu = \bigoplus\limits_{x\in X} \gamma(x) \odot \delta_x \in \langle \mu; U_1, \ldots, U_n; \frac{\varepsilon}{2}\rangle$ we have $|\lambda(x) - \gamma(y)| < \frac{\varepsilon}{2}$ at $x \in \operatorname{supp}\mu \cap U_i$ and $y \in \operatorname{supp}\nu \cap U_i$. Let us estimate the following absolute value $|\mu(\varphi) - \nu(\varphi)| = \left|\bigoplus\limits_{x\in X} \lambda(x) \odot \varphi(x) - \bigoplus\limits_{x\in X} \gamma(x) \odot \varphi(x)\right| = a$.

Two cases are possible:

Case 1: $\bigoplus\limits_{x\in X} \lambda(x) \odot \varphi(x) \geq \bigoplus\limits_{x\in X} \gamma(x) \odot \varphi(x)$. Let $\bigoplus\limits_{x\in X} \lambda(x) \odot \varphi(x) = \lambda(x') \odot \varphi(x')$. Then $x' \in U_i$ for some i, and

$$a = \bigoplus_{x\in X} \lambda(x) \odot \varphi(x) - \bigoplus_{x\in X} \gamma(x) \odot \varphi(x) = \lambda(x') \odot \varphi(x') - \bigoplus_{x\in X} \gamma(x) \odot \varphi(x) \leq$$
$$\leq (\text{for every } y \in \operatorname{supp}\nu \cap U_i) \leq$$
$$\leq \lambda(x') \odot \varphi(x') - \gamma(y) \odot \varphi(y) = |\lambda(x') \odot \varphi(x') - \gamma(y) \odot \varphi(y)| \leq$$
$$\leq |\lambda(x') - \gamma(y)| + |\varphi(x') - \varphi(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Case 2: $\bigoplus_{x\in X} \lambda(x)\odot\varphi(x) \le \bigoplus_{x\in X} \gamma(x)\odot\varphi(x)$. Let $\bigoplus_{x\in X} \gamma(x)\odot\varphi(x) = \gamma(x')\odot \varphi(x')$. Then $x' \in U_i$ for some i, and

$$a = \bigoplus_{x\in X} \gamma(x)\odot\varphi(x) - \bigoplus_{x\in X} \lambda(x)\odot\varphi(x) = \gamma(x')\odot\varphi(x') - \bigoplus_{x\in X} \lambda(x)\odot\varphi(x) \le$$
$$\le (\text{for every } y \in \operatorname{supp}\mu \cap U_i) \le$$
$$\le \gamma(x')\odot\varphi(x') - \lambda(y)\odot\varphi(y) = |\gamma(x')\odot\varphi(x') - \lambda(y)\odot\varphi(y)| \le$$
$$\le |\lambda(x') - \gamma(y)| + |\varphi(x') - \varphi(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $|\mu(\varphi) - \nu(\varphi)| < \varepsilon$. From here $\nu \in \langle \mu; \varphi; \varepsilon\rangle$, in other words,

$$\left\langle \mu;\, U_1,\, \dots,\, U_n;\, \frac{\varepsilon}{2} \right\rangle \subset \langle \mu;\, \varphi;\, \varepsilon\rangle.$$

□

Theorem 2.1 immediately yields the following statement.

Corollary 2.2. *The subset $I_\omega(X)$ is everywhere dense in $I(X)$ with respect to the pointwise convergence topology.*

We recall some concepts from [13] and if necessary, modify them for the max-plus case . Let X and Y be compact Hausdorff spaces, $f\colon X \to Y$ be a map, $f^\circ\colon C(Y) \to C(X)$ be the *induced operator* defined by equality $f^\circ(\varphi) = \varphi\circ f$, $\varphi \in C(Y)$. We say that an operator $u\colon C(X) \to C(Y)$ is a max-plus-*linear operator* provided $u(\alpha\odot\varphi \oplus \beta\odot\psi) = \alpha\odot u(\varphi) \oplus \beta\odot u(\psi)$ for every pair of functions φ, $\psi \in C(X)$, where $-\infty \le \alpha$, $\beta \le 0$, $\alpha\oplus\beta = 0$. A max-plus-linear operator $u\colon C(X) \to C(Y)$ is max-plus-*regular* provided $\|u\| = \sup\{\|u(\varphi)\| : \varphi \in C(X),\, \|\varphi\| \le 1\} = 1$ and $u(1_X) = 1_Y$. A max-plus-linear operator $u\colon C(X) \to C(Y)$ is said to be a max-plus-*linear exave* for f provided $f^\circ \circ u$ is the identity on $f^\circ(C(Y))$ or equivalently $f^\circ\circ u\circ f^\circ = f^\circ$. A max-plus-*regular exave* is a max-plus-linear exave which is a regular operator. If f is a homeomorphic embedding, then a max-plus-linear exave (max-plus-regular exave) for f is called max-plus-*linear extension operator* (max-plus-*regular extension operator*). If f is a surjective map, then a max-plus-linear exave (max-plus-regular exave) for f is called max-plus-*linear averaging operator* (max-plus-*regular averaging operator*).

Remind, in category theory a *monomorphism* (an *epimorphism*) is a left-cancellative (respectively, right-cancellative) morphism, that is, a morphism $f\colon Z \to X$ (respectively, $f\colon X \to Y$) such that, for each pair of morphisms g_1, $g_2\colon Y \to Z$ the following implication holds

$$f\circ g_1 = f\circ g_2 \Rightarrow g_1 = g_2 \qquad (\text{respectively, } g_1\circ f = g_2\circ f \Rightarrow g_1 = g_2).$$

If u is an exave for $f\colon X \to Y$ and $y \in f(X)$, then for every function $\varphi \in C(Y)$ we have

$$(u\circ f^\circ)(\varphi)(y) = \varphi(y).$$

Proposition 2.3. *Let $f\colon X \to Y$ be a map. A* max-plus-*regular operator $u\colon C(X) \to C(Y)$ is a* max-plus-*regular extension (respectively, averaging) operator if and only if $f^\circ \circ u = \mathrm{id}_{C(X)}$ (respectively, $u \circ f^\circ = \mathrm{id}_{C(Y)}$).*

Proof. Let u be a max-plus-regular extension (respectively, averaging) operator. Then the induced operator $f^\circ\colon C(Y) \to C(X)$ is an epimorphism (respectively, monomorphism). That is why equalities $f^\circ \circ u \circ f^\circ = f^\circ = \mathrm{id}_{C(X)} \circ f^\circ$ imply $f^\circ \circ u = \mathrm{id}_{C(X)}$ (respectively, $f^\circ \circ u \circ f^\circ = f^\circ = f^\circ \circ \mathrm{id}_{C(Y)}$ imply $u \circ f^\circ = \mathrm{id}_{C(Y)}$).

Let u be a max-plus-regular operator and $f^\circ \circ u = \mathrm{id}_{C(X)}$. It requires to show $f\colon X \to Y$ is an embedding. Suppose $f(x_1) = f(x_2)$, $x_1, x_2 \in X$. Assume there exists a function $\varphi \in C(X)$ such that $\varphi(x_1) \neq \varphi(x_2)$. Conversely, we have $\varphi(x_1) = f^\circ \circ u(\varphi)(x_1) = u(\varphi)(f(x_1)) = u(\varphi)(f(x_2)) = f^\circ \circ u(\varphi)(x_2) = \varphi(x_2)$. We get a contradiction. So, $x_1 = x_2$.

Let u be a max-plus-regular operator and $u \circ f^\circ = \mathrm{id}_{C(Y)}$. We should show that $f\colon X \to Y$ is a surjective map. Suppose f is not so. Then $Y \setminus f(X) \neq \varnothing$ and for every $y \in Y \setminus f(X)$, since the image $f(X)$ is a compact Hausdorff space, any $\varphi\colon f(X) \to \mathbb{R}$ has different extensions $\varphi_1, \varphi_2\colon Y \to \mathbb{R}$ such $\varphi_1(y) \neq \varphi_2(y)$. Hence, $\varphi_1 \neq \varphi_2$. On the other hand $\varphi_1 = u \circ f^\circ(\varphi_1) = u \circ f^\circ(\varphi_2) = \varphi_2$. The obtained contradiction finishes the proof.

□

An epimorphism $f\colon X \to Y$ is said to be a max-plus-*Milutin epimorphism* provided it permits a max-plus-regular averaging operator. A compact Hausdorff space X is a max-plus-*Milutin space* if there exists a max-plus-Milutin epimorphism $f\colon D^\tau \to X$ [13]. Every compact metrizable space is a Milutin space ([4], Corollary VIII.4.6.). Analogously, every compact metrizable space is a max-plus-Milutin space.

3. An analog of the Kantorovich metric

It is well-known (see, for example [4]) that every zero-dimensional space of the weight $\mathfrak{m} \geq \aleph_0$ embeds into Cantor cube $D^{\mathfrak{m}}$. Consequently, a zero-dimensional compact metrizable space is a max-plus-Milutin space.

Let $\mu_i = \bigoplus\limits_{x \in X} \lambda_i(x) \odot \delta_x \in I(X)$, $i = 1, 2$. Put

$$\Lambda_{1\,2} = \Lambda(\mu_1, \mu_2) = \{\xi \in I(X^2) : I(\pi_i)(\xi) = \mu_i,\ i = 1, 2\},$$

where $\pi_i\colon X \times X \to X$ is the projection onto i-th factor, $i = 1, 2$. We will show the set $\Lambda(\mu_1, \mu_2)$ is nonempty. Let $x_{i\,0} \in \operatorname{supp} \mu_i$ be points such that $\lambda_i(x_{i\,0}) = 0$, $i = 1, 2$. Then the direct checking shows that $I(\pi_i)(\xi) = \mu_i$, $i = 1, 2$, for all $\xi \in I(X^2)$ of the form $\xi = \xi^0 \oplus R(\mu_1, \mu_2)$. Here

$$\xi^0 = 0 \odot \delta_{(x_{1\,0},\, x_{2\,0})} \bigoplus_{x \in X \setminus \{x_{1\,0}\}} \lambda_2(x) \odot \delta_{(x_{1\,0},\, x)} \oplus \bigoplus_{x \in X \setminus \{x_{2\,0}\}} \lambda_1(x) \odot \delta_{(x,\, x_{2\,0})}$$

is an idempotent probability measure on X^2, and

$$R(\mu_1,\, \mu_2) = \bigoplus_{\substack{x\in X\setminus\{x_{1\,0}\}\\ y\in X\setminus\{x_{2\,0}\}}} \gamma(x,\, y)\odot \delta_{(x,\, y)}$$

is some functional on $C(X)$ where

$$-\infty \le \gamma(x,\, y) \le \min\{\lambda_1(x),\ \lambda_2(y)\}, \qquad x\in M, \qquad y \in N,$$
$$M\subset X\setminus\{x_{1\,0}\}, \qquad N\subset X\setminus\{x_{2\,0}\}.$$

Thus $\xi \in \Lambda(\mu_1,\, \mu_2)$, i. e. $\Lambda(\mu_1,\, \mu_2)\neq\varnothing$. In fact, here more is proved: it is easy to see if $|X|\ge 2$ and $|Y|\ge 2$ then quantity of the numbers $\gamma(x,\, y)$ is uncountable. From here one concludes that the potency of the set $\Lambda(\mu_1,\, \mu_2)$ is no less than continuum potency as soon as each of the supports $\operatorname{supp}\mu_i$, $i=1,\, 2$, contains no less than two points.

Note that $\xi=\xi^0$ if one takes empty set as K and M.

Idempotent probability measures $\xi\in I(X^2)$ with $I(\pi_i)(\xi)=\mu_i$, $i=1,\,2$ we will call as a *coupling* of μ_1 and μ_2.

The following statement is rather evident.

Proposition 3.1. *Let* $\mu_i=\bigoplus_{x\in X}\lambda_i(x)\odot\delta_x$, $i=1,\,2$, *be idempotent probability measures. Then every their coupling* $\xi=\bigoplus_{(x,\, y)\in X^2}\lambda_{1\,2}(x,\, y)\odot\delta_{(x,\, y)}\in I(X^2)$ *satisfies the following equalities:*

$$\lambda_1(x)=\bigoplus_{y\in X}\lambda_{1\,2}(x,\, y), \quad x\in X, \quad \text{and} \quad \lambda_2(y)=\bigoplus_{x\in X}\lambda_{1\,2}(x,\, y), \quad y\in X.$$

Consider a compact metrizable space $(X,\, \rho)$. We define a function $\rho_0\colon I(X)\times I(X)\to\mathbb{R}$ by the formula

$$\rho_0(\mu_1,\, \mu_2)=\inf\{\xi(\rho):\, \xi\in\Lambda_{1\,2}\}.$$

This function was offered by V. V. Uspenskii and in [3] it was proved that it is a metric on the space $P(X)$ of probability measures. Its analog for idempotent probability measures is not a metric on the space of idempotent probability measures.

L. V. Kantorovich, G. Sh. Rubinshtein offer another metric on the space of all measures [6]. For the space of probability measures their metric has the form

$$\rho_K(\mu_1,\, \mu_2)=\inf\{\xi(\rho):\, \xi\in P(X\times X),\ P(\pi_1)(\xi)-P(\pi_2)(\xi)=\mu_1-\mu_2\}.$$

In [12] it was shown that on the space of all probability measures the above metrics ρ_0 and ρ_K coincide.

Proposition 3.2. *For every pair* $\mu_1,\, \mu_2\in I(X)$ *there exists a coupling* $\xi\in I(X^2)$ *of* μ_1 *and* μ_2 *such that*

$$\rho_0(\mu_1,\, \mu_2)=\xi(\rho).$$

Proof. Consider a sequence $\{\xi_n\}$ of couplings of μ_1 and μ_2 such that $\xi_n(\rho) \longrightarrow \rho_0(\mu_1, \mu_2)$. Passing in the case of need to a subsequence, owing to the compactness of $I(X^2)$, it is possible to assume that $\{\xi_n\}$ tends to some $\xi \in I(X^2)$. Since the projections $I(\pi_i)$ are continuous, ξ is a coupling of μ_1 and μ_2. Further, for an arbitrary $\varepsilon > 0$ there exists n_0 such that $\xi_n \in \langle \xi;\, \rho;\, \varepsilon \rangle$ for all $n \geq n_0$, where $\langle \xi;\, \rho;\, \varepsilon \rangle$ is a prebase neighbourhood of ξ in the pointwise convergence topology on $I(X^2)$. So, $|\xi(\rho) - \xi_n(\rho)| < \varepsilon$. Consequently, $\rho_0(\mu_1, \mu_2) = \xi(\rho)$.

□

Proposition 3.3. *The function ρ_0 is a pseudometric on $I(X)$.*

Proof. Since each $\xi \in I(X^2)$ is order-preserving then the inequality $\rho \geq 0$ immediately implies $\rho_0 \geq 0$. So, ρ_0 is nonnegative. Obviously, ρ_0 is symmetric.

Let $\mu_1 = \mu_2 = \mu$. There exists $\lambda \in U_S(X)$ such that $\mu = \bigoplus_{x \in X} \lambda(x) \odot \delta_x$. Then $\xi_\mu = \bigoplus_{x \in X} \lambda(x) \odot \delta_{(x,\, x)}$ is a coupling of μ_1 and μ_2, and

$$0 \leq \rho_0(\mu_1, \mu_2) = \inf\{\xi(\rho) : \xi \in \Lambda_{1\,2}\} \leq \xi_\mu(\rho) = \bigoplus_{x \in X} \lambda(x) = 0,$$

i. e. $\rho_0(\mu_1, \mu_2) = 0$.

Let us show that the triangle inequality is true as well. Take arbitrary triple $\mu_i \in I(X)$, $i = 1, 2, 3$. Let $\mu_{1\,2}, \mu_{2\,3} \in I(X^2)$ be couplings of μ_1 and μ_2, and μ_2 and μ_3, respectively, such that $\rho_0(\mu_1, \mu_2) = \mu_{1\,2}(\rho)$ and $\rho_0(\mu_2, \mu_3) = \mu_{2\,3}(\rho)$, respectively. For a compact Hausdorff space X we put

$$X_1 = X_2 = X_3 = X, \qquad X_{1\,2\,3} = X^3 = X_1 \times X_2 \times X_3,$$
$$X_{i\,j} = X^2 = X_i \times X_j,$$

and let

$$\pi^{1\,2\,3}_{i\,j} \colon X_{1\,2\,3} \to X_{i\,j}, \qquad \pi^{i\,j}_k \colon X_{i\,j} \to X_k, \qquad 1 \leq i < j \leq 3, \qquad k \in \{i, j\},$$

be corresponding projections.

According to Corollary 4.3 [17] the functor I is bicommutative. Using this fact one can similarly to Lemma 4 [3] show that for idempotent probability measures

$$\mu_2 \in I(X_2), \qquad \mu_{1\,2} \in I(X_{1\,2}), \qquad \mu_{2\,3} \in I(X_{2\,3})$$

such that

$$I(\pi^{1\,2}_2)(\mu_{1\,2}) = \mu_2 = I(\pi^{2\,3}_2)(\mu_{2\,3}),$$

there exists $\mu_{1\,2\,3} \in I(X_{1\,2\,3})$ which satisfies the equalities

$$I(\pi^{1\,2\,3}_{1\,2})(\mu_{1\,2\,3}) = \mu_{1\,2} \qquad \text{and} \qquad I(\pi^{1\,2\,3}_{2\,3})(\mu_{1\,2\,3}) = \mu_{2\,3}.$$

Put

$$\mu_{1\,3} = I(\pi^{1\,2\,3}_{1\,3})(\mu_{1\,2\,3}). \tag{3.1}$$

Then according to Proposition 3.1 $\mu_{1\,3}$ is a coupling of μ_1 and μ_3. Using Proposition 3.1, we obtain

$$\rho_0(\mu_1,\, \mu_2) + \rho_0(\mu_2,\, \mu_3) = \mu_{1\,2}(\rho) + \mu_{2\,3}(\rho) =$$

$$= \bigoplus_{(x_1,\, x_2)\in X_{1\,2}} d_{\mu_{1\,2}}(x_1,\, x_2) \odot \rho(x_1,\, x_2) + \bigoplus_{(x_2,\, x_3)\in X_{2\,3}} d_{\mu_{2\,3}}(x_2,\, x_3) \odot \rho(x_2,\, x_3) =$$

$$= \bigoplus_{(x_1,\, x_2,\, x_3)\in X_{1\,2\,3}} d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot \rho(x_1,\, x_2) +$$

$$+ \bigoplus_{(x_1,\, x_2,\, x_3)\in X_{1\,2\,3}} d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot \rho(x_2,\, x_3) \geq$$

$$\geq \bigoplus_{(x_1,\, x_2,\, x_3)\in X_{1\,2\,3}} (d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot \rho(x_1,\, x_2) +$$

$$+ d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot \rho(x_2,\, x_3)) =$$

$$= \bigoplus_{(x_1,\, x_2,\, x_3)\in X_{1\,2\,3}} d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot (\rho(x_1,\, x_2) + \rho(x_2,\, x_3)) \geq$$

$$\geq \bigoplus_{(x_1,\, x_2,\, x_3)\in X_{1\,2\,3}} d_{\mu_{1\,2\,3}}(x_1,\, x_2,\, x_3) \odot \rho(x_1,\, x_3) =$$

$$= \bigoplus_{(x_1,\, x_3)\in X_{1\,3}} d_{\mu_{1\,3}}(x_1,\, x_3) \odot \rho(x_1,\, x_3) = \mu_{1\,3}(\rho) \geq \rho_0(\mu_1,\, \mu_3),$$

i. e. $\rho_0(\mu_1,\, \mu_3) \leq \rho_0(\mu_1,\, \mu_2) + \rho_0(\mu_2,\, \mu_3)$. Here d_ν is the density function of the corresponding measure ν ((2.4), see page 39).

□

Unlike usual probability measures, the function ρ_0 is not a metric on $I(X)$.

Example 3.4. Let $(X,\, \rho)$ be a metric space, $x,\, y \in X$ be points such that $\rho(x,\, y) = 1$. Consider idempotent probability measures $\mu_1 = 0 \odot \delta_x \oplus (-2) \odot \delta_y$ and $\mu_2 = 0 \odot \delta_x \oplus (-4) \odot \delta_y$. One can directly check that the idempotent probability measure $\xi = 0 \odot \delta_{(x,\, x)} \oplus (-2) \odot \delta_{(y,\, x)} \oplus (-4) \odot \delta_{(x,\, y)}$ is a coupling of μ_1 and μ_2, and $\xi(\rho) = 0$. That is why $\rho_0(\mu_1,\, \mu_2) = 0$, though $\mu_1 \neq \mu_2$.

Example 3.4 shows that the functors P of probability measures and I of idempotent probability measures are not isomorphic.

4. On a metric on the space of idempotent probability measures

Let $(X,\, \rho)$ be a metric compact space. We define distance functions $\rho_1\colon I(X)\times I(X) \to \mathbb{R}$ and $\rho_2\colon I_\omega(X) \times I_\omega(X) \to \mathbb{R}$ as follows

$$\rho_1(\mu_1,\, \mu_2) = \inf\{\sup\{\rho(x,\, y) : (x,\, y) \in \operatorname{supp}\xi\} : \xi \in \Lambda_{1\,2}\},$$

where $\mu_1, \mu_2 \in I(X)$, and

$$\rho_2(\mu_1,\, \mu_2) = \inf \left\{ \frac{\sum\limits_{(x,\, y) \in \mathrm{supp}\, \xi} \mathrm{e}^{\lambda_1(x)+\lambda_2(y)} \cdot \rho(x,\, y)}{\sum\limits_{x \in \mathrm{supp}\, \mu_1} \mathrm{e}^{\lambda_1(x)} \cdot \sum\limits_{y \in \mathrm{supp}\, \mu_2} \mathrm{e}^{\lambda_2(y)}} : \xi \in \Lambda_{1\,2} \right\},$$

where $\mu_i = \bigoplus\limits_{x \in X} \lambda_i(x) \odot \delta_x \in I_\omega(X)$, $i = 1,\, 2$.

It is easy to notice that $\rho_2 \le \rho_1$ on $I_\omega(X)$.

The following statement has technical character and its proof consists of labour-intensive calculations (similarly calculations were done in [16]).

Lemma 4.1. *For every pair* $\mu_i = \bigoplus\limits_{x \in X} \lambda_i(x) \odot \delta_x \in I_\omega(X)$*,* $i = 1,\, 2$*, and for a coupling* $\xi \in I_\omega(X^2)$ *of* μ_1 *and* μ_2 *we have*

$$\rho_2(\mu_1,\, \mu_2) = \frac{\sum\limits_{(x,\, y) \in \mathrm{supp}\, \xi} \mathrm{e}^{\lambda_1(x)+\lambda_2(y)} \cdot \rho(x,\, y)}{\sum\limits_{x \in \mathrm{supp}\, \mu_1} \mathrm{e}^{\lambda_1(x)} \cdot \sum\limits_{y \in \mathrm{supp}\, \mu_2} \mathrm{e}^{\lambda_2(y)}}$$

$$\text{if and only if} \qquad \rho_0(\mu_1,\, \mu_2) = \xi(\rho).$$

Theorem 4.2. *The function* ρ_1 *is a metric on* $I(X)$ *which is an extension of the metric* ρ*.*

Proof. Obviously, ρ_1 is nonnegative and symmetric. If $\mu_1 = \mu_2$ then similarly to the proof of Proposition 3.3 one can show that $\rho_1(\mu_1, \mu_2) = 0$. Inversely, let $\rho_1(\mu_1, \mu_2) = 0$. Then there exists a coupling $\xi \in \Lambda_{1\,2}$ such that $\rho(x,\, y) = 0$ for all $(x,\, y) \in \mathrm{supp}\, \xi$. Consequently $\mathrm{supp}\, \xi$ must lie in the diagonal $\Delta(X) = \{(x,\, x) : x \in X\}$. Applying Proposition 3.1, we have $d_{\mu_1} = d_{\mu_2}$, which implies $\mu_1 = \mu_2$. Now, it remains to check the triangle axiom. But the checking consists only of the repeating of procedure at the proof of Proposition 3.3.

For every pair of Dirac measures $\delta_x,\ \delta_y,\ x,\, y \in X$, the uniqueness of a coupling $\xi \in I(X^2)$ of δ_x and δ_y, $\xi = 0 \odot \delta_{(x,\, y)}$, implies that

$$\rho_1(\delta_x,\, \delta_y) = \xi(\rho) \oplus \rho(x,\, y) = 0 \odot \delta_{(x,\, y)}(\rho) \oplus \rho(x,\, y) = \rho(x,\, y).$$

From here we get that ρ_1 is an extension of ρ.

□

Lemma 4.3. $\mathrm{diam}(I(X),\, \rho_1) = \mathrm{diam}(X,\, \rho)$.

Proof. Indeed, since we may consider X as a subspace of $I(X)$ we get $\mathrm{diam}(X,\, \rho)$ $\le \mathrm{diam}(I(X),\, \rho_1)$. On the other hand, by construction we have

$$\rho_1(\mu_1,\, \mu_2) = \inf\{\sup\{\rho(x,\, y) : (x,\, y) \in \mathrm{supp}\, \xi\} : \xi \in \Lambda_{1\,2}\} \le$$
$$\le \sup\{\rho(x,\, y) : (x,\, y) \in \mathrm{supp}\, \xi\} \le \sup\{\rho(x,\, y) : (x,\, y) \in X \times X\} = \mathrm{diam}(X,\, \rho)$$

for an arbitrary pair $\mu_1, \mu_2 \in I(X)$. Consequently, $\mathrm{diam}(I(X),\, \rho_1) \le \mathrm{diam}(X,\, \rho)$.
□

Theorem 4.4. *The function ρ_2 is a metric on $I_\omega(X)$ which is an extension of the metric ρ.*

Proof. By construction ρ_2 is non-negative. It is clear that ρ_2 is symmetric. The above noticed inequality $\rho_2 \leq \rho_1$ on $I_\omega(X)$ implies that ρ_2 satisfies the identity axiom, i. e. $\rho_2(\mu, \nu) = 0$ if and only if $\mu = \nu$. By definition we have $\rho_2(\delta_x, \delta_y) = \rho(x, y)$.

For a triple $\mu_i \in I_\omega(X)$, $i = 1, 2, 3$, let $\mu_{1\,2}, \mu_{2\,3} \in I_\omega(X^2)$ be couplings of μ_1 and μ_2, and μ_2 and μ_3, respectively, satisfying Proposition 3.2. Let $\mu_{1\,3} \in I_\omega(X^2)$ be an idempotent probability measures, defined by (3.1). Then Proposition 3.1 yields that $\mu_{1\,3}$ is a coupling of μ_1 and μ_3.

Applying Proposition 3.1, Lemma 4.1 and Theorem 2, we have

$$\rho_2(\mu_1, \mu_2) + \rho_2(\mu_2, \mu_3) \geq \rho_2(\mu_1, \mu_3).$$

□

Let $\mu, \nu \in I(X)$.Corollary 2.2 implies the existence of sequences $\{\mu_n\}$, $\{\nu_n\} \subset I_\omega(X)$ converging to μ and ν respectively. We have $0 \leq \rho_2(\mu_n, \nu_n) \leq \rho_1(\mu_n, \nu_n) \leq \mathrm{diam}(X, \rho)$. Therefore there exists a limit of the sequence $\{\rho_2(\mu_n, \nu_n)\}$. Put

$$\rho_I(\mu, \nu) = \lim_{n\to\infty} \rho_2(\mu_n, \nu_n).$$

Now Theorem 4.4 gives the following result.

Corollary 4.5. *The function ρ_I is a metric on $I(X)$ which is an extension of the metric ρ.*

Note that $\rho_I \leq \rho_1$. For this reason from Lemma 4.3 we obtain the following statement.

Corollary 4.6. $\mathrm{diam}(I(X), \rho_I) = \mathrm{diam}(X, \rho)$.

Proposition 4.7. *Let X be a compact metrizable space and a sequence $\{\mu_n\} \subset I(X)$ converges to $\mu_0 \in I(X)$ with respect to point-wise convergence topology. Then for every open neighbourhood U of the diagonal $\Delta(X) = \{(x, x) : x \in X\}$ there exist a positive integer n and a coupling $\mu_{0\,n} \in I(X^2)$ of μ_0 and μ_n such that*

$$\bigoplus_{(x,\,y)\in X^2\setminus U} d_{\mu_{0\,n}}(x, y) \odot \rho(x, y) = -\infty. \tag{4.1}$$

Proof. At first we consider the case of zero-dimensional compact metrizable space X. There exists a disjoint clopen cover $\{V_1, \ldots, V_n\}$ of X (i. e. a cover, which consists of open-closed sets of X) such that $V_i \times V_i \subset U$ for each $i = 1, \ldots, n$. As $\mu_n \to \mu$ there exists n such that $\mu_n \in \langle \mu; {}^{\oplus}\chi_{V_1}, {}^{\oplus}\chi_{V_2}, \ldots, {}^{\oplus}\chi_{V_n}; \varepsilon\rangle$. We will construct a coupling $\mu_{0\,n} \in I(X^2)$ of μ_0 and μ_n.

There exists a base of the compact metrizable space X consisting of clopen sets

$$V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}, \qquad 1 \leq i \leq s, \qquad \varepsilon_k \in \{0, 1\}, \qquad 1 \leq k < \infty,$$

such that

1) $V_i^0 \cup V_i^1 = V_i$;
2) $V_i^0 \cap V_i^1 = \varnothing$;
3) $V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k 0} \cup V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k 1} = V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}$;
4) $V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k 0} \cap V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k 1} = \varnothing$.

The sets $V_i^{\varepsilon_1\varepsilon_2\ldots\varepsilon_k} \times V_{i'}^{\varepsilon'_1\varepsilon'_2\ldots\varepsilon'_k}$ form a base of the compact metrizable space $X_{1\,2}$. To determine $\mu_{0\,n}$ it is enough to construct its density function. Let $\mu_0 = \bigoplus\limits_{x\in X} \lambda_0(x) \odot \delta_x$, $\mu_n = \bigoplus\limits_{x\in X} \lambda_n(x) \odot \delta_x$. We set

$$\lambda_{i\,i'}^{\varepsilon_1\ldots\varepsilon_k,\,\varepsilon'_1\ldots\varepsilon'_k} = \bigoplus_{(x,\,y)\in X\times X} (\lambda_0(x) \odot \lambda_n(y)) \odot \delta_{(x,\,y)}({}^{\oplus}\chi_{V_i^{\varepsilon_1\ldots\varepsilon_k}\times V_{i'}^{\varepsilon'_1\ldots\varepsilon'_k}}),$$

i. e.

$$\lambda_{i\,i'}^{\varepsilon_1\ldots\varepsilon_k,\,\varepsilon'_1\ldots\varepsilon'_k} = \bigoplus_{(x,\,y)\in V_i^{\varepsilon_1\ldots\varepsilon_k}\times V_{i'}^{\varepsilon'_1\ldots\varepsilon'_k}} \lambda_0(x) \odot \lambda_n(y).$$

It is clear that

$$\lambda_{i'}^{\varepsilon'_1\ldots\varepsilon'_k} = \bigoplus_{i=1}^{s} \lambda_{i\,i'}^{\varepsilon_1\ldots\varepsilon_k,\,\varepsilon'_1\ldots\varepsilon'_k} \qquad \text{and} \qquad \lambda_{i}^{\varepsilon_1\ldots\varepsilon_k} = \bigoplus_{i'=1}^{s} \lambda_{i\,i'}^{\varepsilon_1\ldots\varepsilon_k,\,\varepsilon'_1\ldots\varepsilon'_k},$$

where

$$\lambda_i^{\varepsilon_1\ldots\varepsilon_k} = \bigoplus_{x\in X} \lambda_0(x) \odot \delta_x({}^{\oplus}\chi_{V_i^{\varepsilon_1\ldots\varepsilon_k}}) = \bigoplus_{x\in V_i^{\varepsilon_1\ldots\varepsilon_k}} \lambda_0(x)$$

and

$$\lambda_{i'}^{\varepsilon'_1\ldots\varepsilon'_k} = \bigoplus_{x\in X} \lambda_n(x) \odot \delta_x({}^{\oplus}\chi_{V_i^{\varepsilon'_1\ldots\varepsilon'_k}}) = \bigoplus_{x\in V_{i'}^{\varepsilon'_1\ldots\varepsilon'_k}} \lambda_n(x).$$

Put

$$d_{\mu_{0\,n}} = \lim_{s\to\infty} \bigoplus_{i,\,i'=1}^{s} {}^{\oplus}\chi^{\lambda_{i\,i'}^{\varepsilon_1\ldots\varepsilon_k,\,\varepsilon'_1\ldots\varepsilon'_k}}_{V_i^{\varepsilon_1\ldots\varepsilon_k}\times V_{i'}^{\varepsilon'_1\ldots\varepsilon'_k}}.$$

Then $d_{\mu_{0\,n}}$ is an upper semicontinuous function on X^2 and $\mu_{0,\,n} = \bigoplus\limits_{(x,\,y)\in X^2} d_{\mu_{0\,n}}(x,\,y) \odot \delta_{(x,\,y)}$ is a coupling of μ_0 and μ_n with $\operatorname{supp}\mu_{0,\,n} \subset U$. Consequently, $\bigoplus\limits_{(x,\,y)\in X^2\setminus U} d_{\mu_{0\,n}}(x,\,y) = -\infty$ and, the equation (4.1) is proved for the zero-dimensional case.

Now let X be an arbitrary compact metrizable space. There exists a zero-dimensional compact metrizable space Z, a max-plus-Milutin epimorphism $f\colon Z \to X$ and a max-plus-regular averaging operator $u\colon C(Z) \to C(X)$ corresponding to this epimorphism. The dual max-plus-map $u^{\oplus}$ which we define by the equality $u^{\oplus}(\mu)(\varphi) = \mu(u(\varphi))$, $\varphi \in C(Z)$, generates an embedding $u^{\oplus}\colon I(X) \to I(Z)$.

For idempotent probability measures $\mu'_0 = u^{\oplus}(\mu_0)$ and $\mu'_n = u^{\oplus}(\mu_n)$ there exists a coupling $\mu'_{0,\,n} = \bigoplus\limits_{(x',\,y')\in Z^2} d_{\mu'_{0\,n}}(x',\,y') \odot \delta_{(x',\,y')} \in I(Z\times Z)$ of μ'_0 and μ'_n such that

$$\bigoplus_{(x',\,y')\in Z^2\setminus(f\times f)^{-1}(U)} d_{\mu'_{0\,n}}(x',\,y') \odot \rho(x',\,y') = -\infty.$$

Put $\mu_{0,\,n} = I(f\times f)(\mu'_{0\,n})$. Then for every $\varphi \in C(X^2)$ we have

$$\begin{aligned} \mu_{0,\,n}(\varphi) &= I(f\times f)(\mu'_{0\,n})(\varphi) = \mu'_{0\,n}(\varphi\circ(f\times f)) = \\ &= \bigoplus_{(x',\,y')\in Z^2} d_{\mu'_{0\,n}}(x',\,y') \odot \varphi\circ(f\times f)(x',\,y') = \\ &= \bigoplus_{(x',\,y')\in Z^2} d_{\mu'_{0\,n}}(x',\,y') \odot \varphi(f(x'),\,f(y')) = \\ &= \bigoplus_{(x,\,y)\in X^2} d_{\mu'_{0\,n}}(x,\,y)) \odot \delta_{(x,\,y)}(\varphi), \end{aligned}$$

i. e. $\mu_{0,\,n} = \bigoplus\limits_{(x,\,y)\in X^2} d_{\mu'_{0\,n}}(x,\,y)) \odot \delta_{(x,\,y)}$. Here

$$d_{\mu'_{0\,n}}(x,\,y) = \bigoplus_{(x',\,y')\in(f\times f)^{-1}(x,\,y)} d_{\mu'_{0\,n}}(x',\,y').$$

That is why

$$\bigoplus_{(x,\,y)\in X^2\setminus U} d_{\mu'_{0\,n}}(x,\,y) \odot \rho(x,\,y) = -\infty.$$

So, $\mu_{0,\,n} = I(f\times f)(\mu'_{0\,n})$ satisfies (4.1). It remains to show that $\mu_{0,\,n}$ is a coupling of μ_0 and μ_n.

A diagram

$$\begin{array}{ccc} Z\times Z & \xrightarrow{f\times f} & X\times X \\ \Big\downarrow{\scriptstyle \theta_1^{12}} & & \Big\downarrow{\scriptstyle \pi_1^{12}} \\ Z & \xrightarrow{f} & X \end{array} \tag{4.2}$$

is commutative, where θ_1^{12}, π_1^{12} are projections onto the first corresponding factors. Then

$$\begin{aligned} I(\pi_1^{12})(\mu_{0\,n}) &= I(\pi_1^{12})\circ I(f\times f)(\mu'_{0\,n}) = I(\pi_1^{12}\circ(f\times f))(\mu'_{0\,n}) = \\ &= (\text{owing to commutativity of the diagram (4.2)}) = \\ &= I(f\circ\theta_1^{12})(\mu'_{0\,n}) = I(f)\circ I(\theta_1^{12})(\mu'_{0,\,n}) = I(f)(\mu'_0) = I(f)(u^{\oplus}(\mu_0)), \end{aligned}$$

i. e. for every $\varphi \in C(X)$ we have

$$\begin{aligned} I(\pi_1^{12})(\mu_{0\,n})(\varphi) &= I(f)(u^{\oplus}(\mu_0))(\varphi) = u^{\oplus}(\mu_0)(\varphi\circ f) = u^{\oplus}(\mu_0)(f^{\circ}(\varphi)) = \\ &= \mu_0(u\circ f^{\circ}(\varphi)) = (\text{with respect to Proposition 2.3}) = \mu_0(\varphi). \end{aligned}$$

Thus, $I(\pi_1^{12})(\mu_{0\,n}) = \mu_0$. Similarly, $I(\pi_2^{12})(\mu_{0\,n}) = \mu_n$. The Proposition is proved.

□

Theorem 4.8. *The metric ρ_I generates pointwise convergence topology on $I(X)$.*

Proof. Let $\{\mu_n\} \subset I(X)$ be a sequence and $\mu_0 \in I(X)$. Suppose the sequence converges to μ_0 with respect to the pointwise convergence topology but not by ρ_I. Passing in the case of need to a subsequence, it is possible to regard that

$$\rho_I(\mu_n,\, \mu_0) \geq a > 0 \qquad \text{for all positive integer} \quad n.$$

Consider an open neighbourhood of the diagonal $\Delta(X)$:

$$U = \left\{(x,\, y) \in X^2 :\, \rho(x,\, y) < \frac{a}{2}\right\}.$$

By virtue of Proposition 4.7 there exist a positive integer n and a coupling $\mu_{0\,n} \in I(X^2)$ of μ_0 and μ_n such that

$$\bigoplus_{(x,\, y)\in X^2\setminus U} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y) = -\infty.$$

Therefore, $\operatorname{supp}\mu_{0\,n} \subset U$, and

$$\rho_I(\mu_n,\, \mu_0) \leq \rho_1(\mu_n,\, \mu_0) \leq \sup_{(z,\, t)\in \operatorname{supp}\mu_{0\,n}} \{\rho(z,\, t)\} = \sup_{(z,\, t)\in \operatorname{supp}\mu_{0\,n}} \{\mu_{0\,n}(\rho) \oplus \rho(z,\, t)\} =$$

$$= \sup_{(z,\, t)\in \operatorname{supp}\mu_{0\,n}} \left\{\Big(\bigoplus_{(x,\, y)\in X^2} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y)\Big) \oplus \rho(z,\, t)\right\} =$$

$$= \sup_{(z,\, t)\in \operatorname{supp}\mu_{0\,n}} \left\{\Big(\bigoplus_{(x,\, y)\in X^2\setminus U} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y) \oplus \sup_{(x,\, y)\in U} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y)\Big) \oplus\right.$$

$$\left. \oplus\, \rho(z,\, t)\right\} = \sup_{(z,\, t)\in \operatorname{supp}\mu_{0\,n}} \left\{\Big(\sup_{(x,\, y)\in U} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y)\Big) \oplus \rho(z,\, t)\right\} \leq$$

$$\leq \sup_{(z,\, t)\in U} \left\{\Big(\sup_{(x,\, y)\in U} d_{\mu_{0\,n}}(x,\, y) \odot \rho(x,\, y)\Big) \oplus \rho(z,\, t)\right\} = \sup_{(z,\, t)\in U} \{\rho(z,\, t)\} \leq \frac{a}{2}.$$

The obtained contradiction finishes the proof.

□

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