

## Existence results of delay and fractional differential equations via fuzzy weakly contraction mapping principle

REHANA TABASSUM<sup>a</sup>, AKBAR AZAM<sup>a</sup> AND MOHAMMED SHEHU SHAGARI<sup>b</sup>

<sup>a</sup> Department of Mathematics, COMSATS University, Chak Shahzad, Islamabad, 44000, Pakistan (reha.alvi@yahoo.com, akbarazam@yahoo.com)

<sup>b</sup> Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria (ssmohammed@abu.edu.ng)

Communicated by S. Romaguera

### ABSTRACT

---

*The purpose of this article is to extend the results derived through former articles with respect to the notion of weak contraction into intuitionistic fuzzy weak contraction in the context of  $(\mathcal{T}, \mathcal{N}, \alpha)$  – cut set of an intuitionistic fuzzy set. We intend to prove common fixed point theorem for a pair of intuitionistic fuzzy mappings satisfying weakly contractive condition in a complete metric space which generalizes many results existing in the literature. Moreover, concrete results on existence of the solution of a delay differential equation and a system of Riemann-Liouville Cauchy type problems have been derived. In addition, we also present illustrative examples to substantiate the usability of our main result.*

---

2010 MSC: 46S40; 47H10; 54H25.

KEYWORDS: common fixed point; intuitionistic fuzzy set-valued maps;  $(\mathcal{T}, \mathcal{N}, \alpha)$  – cut set; weakly contractive condition; delay differential equation; Riemann-Liouville fractional differential equations.

## 1. INTRODUCTION

Metric fixed point theory is generally based on the Banach Contraction Principle, which has been used to study the existence and uniqueness of fixed points. This principle has been extensively studied in different directions. In 1997, Alber and Guerr-Delabriere [3] proposed the notion of weak contractive mappings on Hilbert spaces and studied the existence of fixed point results in the context of weakly contractive single valued maps on Hilbert spaces as a generalization of Banach Contraction Principle. However, in 2001, Rhoades [30] presented some results of [3] to arbitrary Banach spaces. Later on, Bae [10] established the fixed points of weakly contractive multivalued mappings and Beg and Abbas [19] demonstrated the fixed point results for a pair of single valued mappings one is weakly contractive relative to the other.

On the other hand, there are many complicated practical problems in the domain of real world such as engineering, economics, social sciences, medical science and many other fields that involve data which are not always precise. To overcome these difficulties, classical mathematical notions may not be applied effectively, because there are numerous types of vagueness appear in these domains. However, in response to this fact, Zadeh [39] developed the concept of fuzzy set as an extension of conventional set theory. Over the years, several mathematicians extended this notion in different directions, for instance,  $L$ -fuzzy set, intuitionistic fuzzy set, fuzzy soft set and hesitant fuzzy soft set. Consequently, in 1981, Heilpern [20] initiated the idea of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings as an extension of multivalued mappings of Nadler's contraction principle. Thus, this result motivated several researchers to study and establish the fixed point results satisfying a fuzzy contractive inequalities ( see, [1, 6, 7]).

One of the generalizations of fuzzy set theory [39] is the notion of intuitionistic fuzzy set (IF-set) introduced by Atanassov [5]. Moreover, IF-sets create a valuable mathematical structure to deal with inaccuracy and hesitancy originating from insufficient decision information and as a consequence, it has remarkable applications in various fields like image processing [21], medical diagnosis [18], drug selection [22], decision analysis [26], etc.

Until now, research on IF-set has been very active and many results have been proved with different aspects. Recently, Azam et al. [8] developed new approach to discuss the fixed point theorems using the idea of intuitionistic fuzzy mappings [38] on a complete metric space. Later on, Azam and Rehana [9] presented existence of common coincidence point for three intuitionistic fuzzy set valued maps and they also studied existence results for a system of integral equations.

In this manuscript, the idea of weakly contraction is used for intuitionistic fuzzy mappings in association with  $(\mathcal{T}, \mathcal{N}, \alpha)$ -cut set of an IF-set [27]. On the basis of this concept, an existence result of common fixed point on complete metric space is presented. From an application perspective, we apply our main

result to establish existence theorems of the solution of a delay differential equation and a system of Riemann-Liouville Cauchy type problems.

## 2. PRELIMINARIES

Throughout this paper,  $(U, \rho)$ ,  $(W, \rho_W)$  and  $V(W)$  denote a metric space, a metric linear space and a subcollection of all approximate quantities in  $W$ , respectively. Let

$$CB(U) = \{A^* : A^* \text{ is nonempty, closed and bounded subset of } U\},$$

$$C(U) = \{A^* : A^* \text{ is nonempty compact subset of } U\}.$$

For  $A^*, B^* \in CB(U)$ , define

$$\rho(u, A^*) = \inf_{v \in A^*} \rho(u, v),$$

$$\rho(A^*, B^*) = \inf_{u \in A^*, v \in B^*} \rho(u, v).$$

The Hausdorff metric  $\rho_H$  on  $CB(U)$  induced by  $\rho$  is defined as

$$\rho_H(A^*, B^*) = \max \left\{ \sup_{u \in A^*} \rho(u, B^*), \sup_{v \in B^*} \rho(A^*, v) \right\}.$$

**Definition 2.1** ([3, 17]). Let  $(U, \rho)$  be a metric space and a mapping  $f : U \rightarrow U$  is called a weakly contractive mapping if for  $u, v \in U$ ,

$$\rho(f(u), f(v)) \leq \rho(u, v) - \phi(\rho(u, v)),$$

where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous non-decreasing function with  $\phi(t) = 0$  if and only if  $t = 0$ .

**Definition 2.2** ([10]). Let  $(U, \rho)$  be a metric space. A mapping  $f : U \rightarrow C(U)$  is said to be a weakly contractive multivalued mapping, if there exists a continuous non-decreasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  and  $\phi(t) > 0$  for all  $t > 0$ , such that

$$\rho_H(f(u), f(v)) \leq \rho(u, v) - \phi(\rho(u, v)),$$

for all  $u, v \in U$ .

**Definition 2.3** ([20, 39]). Let  $Z$  be a universal set. A fuzzy set in  $U$  is an object of the form

$$A^* = \{(z, A^*(z)) : z \in Z\},$$

where  $A^*(z)$  denotes the membership values of  $z$  in  $A^*$ .

**Definition 2.4** ([20, 39]). Let  $A^*$  be a fuzzy set of universe  $Z$ . The  $\alpha$ -cut set of  $A^*$  denoted by  $[A^*]_\alpha$  is a crisp subset of  $Z$  whose membership value in  $A^*$  is greater than or equal to some specific value of  $\alpha$ , i.e.

$$[A^*]_\alpha = \{z \in Z : A^*(z) \geq \alpha\} \quad \text{if } \alpha \in (0, 1].$$

**Definition 2.5** ([20]). A fuzzy set  $A^*$  in a metric linear space  $W$  is said to be an approximate quantity if and only if only if  $[A^*]_\alpha$  is compact and convex in  $W$  for each  $\alpha \in (0, 1]$  with  $\sup_{w \in W} A^*(w) = 1$ .

**Definition 2.6** ([20]). Let  $Z$  be an arbitrary set and  $U$  be a metric space. A mapping from  $Z$  into  $I^U$  is called a fuzzy mapping.

**Definition 2.7** ([7]). An element  $u^* \in U$  is called a fuzzy fixed point of a fuzzy mapping  $S : U \rightarrow I^U$  if there exists  $\alpha \in (0, 1]$  such that  $u^* \in [S(u^*)]_\alpha$ .

Let  $I^U$  be the collection of all fuzzy sets in  $U$  and

$$\mathcal{F}(U) = \{A^* \in I^U : [A^*]_\alpha \in C(U) \text{ for all } \alpha \in [0, 1]\}.$$

For  $A^*, B^* \in I^U$ , if there exists an  $\alpha \in [0, 1]$  such that  $[A^*]_\alpha, [B^*]_\alpha \in C(U)$ , then define

$$D_\alpha(A^*, B^*) = \rho_H([A^*]_\alpha, [B^*]_\alpha),$$

$$D(A^*, B^*) = \sup_\alpha D_\alpha([A^*]_\alpha, [B^*]_\alpha),$$

where  $D$  is a metric on  $\mathcal{F}(U)$  and the completeness of  $(U, \rho)$  implies  $(C(U), H_\rho)$  and  $(\mathcal{F}(U), D)$  are complete.

**Lemma 2.8** ([28]). Let  $(U, \rho)$  be a metric space and  $A^*, B^* \in C(U)$ , then for each  $u \in A^*$ , there exists an element  $v \in B^*$  such that  $\rho(u, v) \leq \rho_H(A^*, B^*)$ .

**Lemma 2.9** ([28]). Let  $(U, \rho)$  be a metric space and  $A^*, B^* \in CB(U)$ . If  $u \in A^*$ , then  $\rho(u, B^*) \leq \rho_H(A^*, B^*)$ .

**Definition 2.10** ([5]). Let  $Z$  be a fixed set. Then an IF-set  $E$  in  $Z$  is a set of ordered triples given by

$$E = \{ \langle z, \mu_E(z), \nu_E(z) \rangle : z \in Z \},$$

where  $\mu_E : Z \rightarrow [0, 1]$  and  $\nu_E : Z \rightarrow [0, 1]$  define the degree of membership and the degree of non-membership respectively, of the elements  $z$  in  $E$  and satisfying  $0 \leq \mu_E(z) + \nu_E(z) \leq 1$ , for each element  $z \in Z$ . In addition, the degree of hesitancy of  $z$  to  $E$  is defined by

$$\pi_E(z) = 1 - \mu_E(z) - \nu_E(z).$$

Particularly, If  $\pi_E(z) = 0$ , for all  $z \in Z$ , then an IF-set  $E$  is reduced to a fuzzy set  $A^*$ .

**Example 2.11.** Consider an IF-set  $E$  of high-experienced and low-experienced employees of a company  $Z$ , whose degrees of membership  $\mu_E(z)$  and non-membership  $\nu_E(z)$  are depicted in Fig. 1.

Table 1. Employees Designations

Designations	Salaries (\$)
Assistant Manager	1,000 – 15,000
Manager	20,000 – 25,000
Senior Manager	20,000 – 25,000
Director	25,000 – 35,000

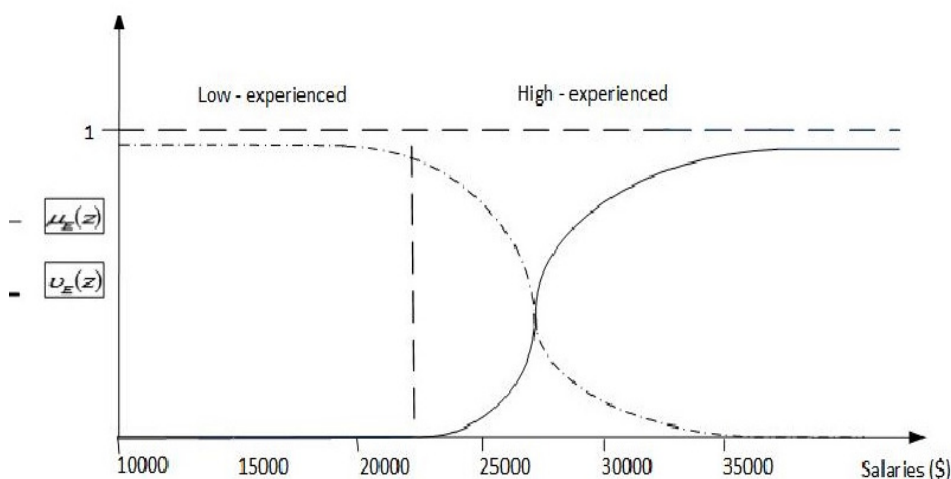


Fig. 1. Representation of an IF-set with Low – experienced and High – experienced employees

**Definition 2.12** ([5]). Let  $E$  be an IF-set of universe  $Z$ . The  $\alpha$  – cut set of  $E$  is a classical subset of elements of  $Z$  denoted by  $[E]_\alpha$  and is defined by

$$[E]_\alpha = \{z \in Z : \mu_E(z) \geq \alpha \text{ and } \nu_E(z) \leq 1 - \alpha\} \quad \text{if } \alpha \in [0, 1].$$

**Definition 2.13** ([27]). A mapping  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  is called a triangular norm (t-norm), if the following conditions are satisfied:

- (i).  $\mathcal{T}(z_1, \mathcal{T}(z_2, z_3)) = \mathcal{T}(\mathcal{T}(z_1, z_2), z_3)$  for all  $z_1, z_2, z_3 \in Z$ .
- (ii).  $\mathcal{T}(z_1, z_2) = \mathcal{T}(z_2, z_1)$  for all  $z_1, z_2 \in Z$ .
- (iii). If  $z_1, z_2, z_3 \in [0, 1]$  and  $z_1 \leq z_2$ , then  $\mathcal{T}(z_1, z_3) \leq \mathcal{T}(z_2, z_3)$ .
- (iv).  $\mathcal{T}(z_1, 1) = z_1$  for all  $z_1 \in Z$ .

Minimum t-norm denoted by  $\mathcal{T}_M$  and is defined by

$$\mathcal{T}_M(z_1, z_2) = \min(z_1, z_2) \quad \text{for all } z_1, z_2 \in [0, 1].$$

**Definition 2.14** ([27]). Fuzzy negation is a non-increasing mapping  $\mathcal{N} : [0, 1] \rightarrow [0, 1]$  such that  $\mathcal{N}(0) = 1, \mathcal{N}(1) = 0$ . If  $\mathcal{N}$  is continuous and strictly decreasing, then it is called strict. Fuzzy negations with  $\mathcal{N}(\mathcal{N}(z)) = z$ , for all

$z \in [0, 1]$ , are called strong fuzzy negations. The example of fuzzy negation is a standard negation defined by  $\mathcal{N}_S(z) = 1 - z$ , for all  $z \in Z$ .

**Definition 2.15** ([27]). Let  $E$  be an IF-set of  $U$ ,  $\mathcal{T}$  and  $\mathcal{N}$  a triangular norm and a fuzzy negation, respectively. Then  $(\mathcal{T}, \mathcal{N}, \alpha)$  – cut set of  $E$  is a crisp set denoted by  $[E]_{(\mathcal{T}, \mathcal{N}, \alpha)}$  and is defined by

$$[E]_{(\mathcal{T}, \mathcal{N}, \alpha)} = \{z \in Z : \mathcal{T}(\mu_E(z), \mathcal{N}(v_E(z))) \geq \alpha\} \text{ if } \alpha \in [0, 1].$$

*Remark 2.16.* If we take  $\mathcal{T} = \mathcal{T}_M$  and  $\mathcal{N} = \mathcal{N}_S$ , then  $(\mathcal{T}, \mathcal{N}, \alpha)$  – cut set is reduced into original definition of a cut set by Atanassov [5].

**Definition 2.17** ([38]). Let  $Z$  be an arbitrary set,  $U$  a metric space. A mapping  $S$  is called intuitionistic fuzzy mapping if  $S$  is a mapping from  $Z$  into  $(IFS)^U$ .

**Definition 2.18.** A point  $u^* \in U$  is said to be an intuitionistic fuzzy fixed point of an intuitionistic fuzzy mapping  $S : U \rightarrow (IFS)^U$  if there exists  $\alpha \in [0, 1]$  such that  $u^* \in [S(u^*)]_{(\mathcal{T}, \mathcal{N}, \alpha)}$ .

Let  $(IFS)^U$  be the collection of all intuitionistic fuzzy subsets of  $U$  and define

$$\mathcal{F}_{IF}(U) = \left\{ E \in (IFS)^U : [E]_{(\mathcal{T}, \mathcal{N}, \alpha)} \in C(U) \text{ for all } \alpha \in [0, 1] \right\}.$$

For  $E_1, E_2 \in (IFS)^U$  and  $\alpha \in [0, 1]$  such that  $[E_1]_{(\mathcal{T}, \mathcal{N}, \alpha)}, [E_2]_{(\mathcal{T}, \mathcal{N}, \alpha)} \in C(U)$ , the following notations are defined by

$$D_\alpha(E_1, E_2) = \rho_H \left( [E_1]_{(\mathcal{T}, \mathcal{N}, \alpha)}, [E_2]_{(\mathcal{T}, \mathcal{N}, \alpha)} \right),$$

$$D_{IF}(E_1, E_2) = \sup_\alpha D_\alpha \left( [E_1]_{(\mathcal{T}, \mathcal{N}, \alpha)}, [E_2]_{(\mathcal{T}, \mathcal{N}, \alpha)} \right),$$

where  $D_{IF}$  is a metric on  $\mathcal{F}_{IF}(U)$ .

### 3. MAIN RESULTS

In what follows hereafter, we present our main results.

**Theorem 3.1.** Let  $(U, \rho)$  be a complete metric space and  $F, G$  be a pair of intuitionistic fuzzy mappings from  $U$  into  $(IFS)^U$ . For  $u \in U$ , there exist  $\alpha_F(\xi), \alpha_G(\xi) \in [0, 1]$  such that  $[F(\xi)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(\xi))}, [G(\xi)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(\xi))} \in C(U)$ . If for all  $u, v \in U$ ,

$$(3.1) \quad \rho_H \left( [F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(u))}, [G(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(v))} \right) \leq \rho(u, v) - \phi(\rho(u, v)),$$

where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous non-decreasing function with  $\phi(t) = 0$  if and only if  $t = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ .

Thus, there exists  $\omega \in U$  such that  $\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(\omega))} \cap [G(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(\omega))}$ .

*Proof.* Let  $u_0$  be an arbitrary but fixed element of  $U$ , then by assumptions, there exists  $\alpha_{F(u_0)} \in [0, 1]$  such that  $[F(u_0)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_0)})} \in C(U)$ . Choose  $u_1 \in [F(u_0)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_0)})}$ . It follows from Lemma 2.8, there exists  $u_2 \in [G(u_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_1))}$  such that

$$\begin{aligned} \rho(u_1, u_2) &\leq \rho_H \left( [F(u_0)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_0)})}, [G(u_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_1))} \right) \\ &\leq \rho(u_0, u_1) - \phi(\rho(u_0, u_1)). \end{aligned}$$

Again by Lemma 2.8, for  $u_2 \in [G(u_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_1))}$  there exists  $u_3 \in [F(u_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_2)})}$  such that

$$\begin{aligned} \rho(u_2, u_3) &\leq \rho_H \left( [G(u_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_1))}, [F(u_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_2)})} \right) \\ &\leq \rho(u_1, u_2) - \phi(\rho(u_1, u_2)). \end{aligned}$$

Continuing this process, for  $u_n \in U$  we obtain  $u_{n+1} \in U$  such that

$$\begin{aligned} u_{n+1} &\in [F(u_n)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_n)})}, \quad n = 0, 1, 2, \dots, \\ u_{n+2} &\in [G(u_{n+1})]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_{n+1}))}, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where,

$$\begin{aligned} \rho(u_{n+1}, u_{n+2}) &\leq \rho_H \left( [F(u_n)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u_n)})}, [G(u_{n+1})]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_{n+1}))} \right) \\ &\leq \rho(u_n, u_{n+1}) - \phi(\rho(u_n, u_{n+1})) \\ &\leq \rho(u_n, u_{n+1}), \quad n = 0, 1, 2, \dots. \end{aligned}$$

It follows that  $\{\rho(u_n, u_{n+1})\}$  is a non-increasing sequence of positive real numbers and hence tends to limit  $r \geq 0$ . If  $r > 0$ , then we obtain

$$(3.2) \quad \rho(u_{n+1}, u_{n+2}) \leq \rho(u_n, u_{n+1}) - \phi(r).$$

Therefore,

$$(3.3) \quad \rho(u_{n+N}, u_{n+N+1}) \leq \rho(u_n, u_{n+1}) - N\phi(r),$$

which is a contradiction for large enough  $N$ . Hence,  $\rho(u_n, u_{n+1}) \rightarrow 0$ .

Therefore, by a similar argument of [11], it follows  $\{u_n\}$  is a Cauchy sequence in  $U$ . As  $U$  is complete, therefore there exists  $\omega \in U$  such that  $u_n \rightarrow \omega$ .

Then by Lemma 2.9, we get

$$(3.4) \quad \begin{aligned} \rho \left( [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})}, u_{n+2} \right) &\leq \rho_H \left( [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})}, [G(u_{n+1})]_{(\mathcal{T}, \mathcal{N}, \alpha_G(u_{n+1}))} \right) \\ &\leq \rho(\omega, u_{n+1}) - \phi(\rho(\omega, u_{n+1})). \end{aligned}$$

Letting  $n \rightarrow \infty$  and using the fact that  $\phi(0) = 0$ , we obtain

$$\rho \left( [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})}, \omega \right) \leq 0.$$

This implies

$$\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})}.$$

Similarly,

$$\omega \in [G(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(\omega))}.$$

Hence, there exists  $\omega \in U$  such that  $\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(\omega))} \cap [G(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(\omega))}$ .  $\square$

**Corollary 3.2.** *Let  $(U, \rho)$  be a complete metric space and  $F : U \rightarrow (IFS)^U$  be a intuitionistic fuzzy mapping. For  $u \in U$ , there exists  $\alpha_F(u) \in [0, 1]$  such that  $[F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(u))} \in C(U)$ . If for all  $u, v \in U$ ,*

$$\rho_H \left( [F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(u))}, [F(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(v))} \right) \leq \rho(u, v) - \phi(\rho(u, v)),$$

where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous non-decreasing function with  $\phi(t) = 0$  if and only if  $t = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ .

Thus, there exists  $\omega \in U$  such that  $\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(\omega))}$ .

If we take  $\phi(t) = (1 - q)(t)$ , where  $0 < q < 1$ , then corollary 3.2 reduces to the following result.

**Corollary 3.3.** *Let  $(U, \rho)$  be a complete metric space and  $F : U \rightarrow (IFS)^U$  be intuitionistic fuzzy mapping. For  $u \in U$  there exists  $\alpha_F(u) \in [0, 1]$  such that  $[F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(u))} \in C(U)$ . If  $0 < q < 1$  and for all  $u, v \in U$ ,*

$$\rho_H \left( [F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(u))}, [F(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(v))} \right) \leq q\rho(u, v).$$

Thus, there exists  $\omega \in U$  such that  $\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(\omega))}$ .

**Corollary 3.4.** *Let  $(U, \rho)$  be a complete metric space and  $F, G : U \rightarrow I^U$  be a pair of fuzzy mappings. For  $u \in U$  there exists  $\alpha_F(u), \alpha_G(u) \in (0, 1]$  such that  $[F(u)]_{\alpha_F(u)}, [G(u)]_{\alpha_G(u)} \in C(U)$ . If for all  $u, v \in U$ ,*

$$\rho_H \left( [F(u)]_{\alpha_F(u)}, [G(v)]_{\alpha_G(v)} \right) \leq \rho(u, v) - \phi(\rho(u, v)),$$

where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous non-decreasing function with  $\phi(t) = 0$  if and only if  $t = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ .

Thus, there exists  $\omega \in U$  such that  $\omega \in [F(\omega)]_{\alpha_F(\omega)} \cap [G(\omega)]_{\alpha_G(\omega)}$ .

**Corollary 3.5.** *Let  $(W, \rho_W)$  be a complete metric linear space and  $F, G : W \rightarrow V(W)$  be a pair of fuzzy mappings satisfying the following condition*

$$D(F(u), G(v)) \leq \alpha\rho(u, v),$$

for each  $u, v \in W$ , where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a continuous non-decreasing function with  $\phi(t) = 0$  if and only if  $t = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ . Thus, there exists a  $\omega \in W$  such that  $\{\omega\} \subset F(\omega)$  and  $\{\omega\} \subset G(\omega)$ .

**Corollary 3.6.** *Let  $(W, \rho_W)$  be a complete metric linear space and  $F : W \rightarrow V(W)$  be a fuzzy mapping satisfying*

$$D(F(u), F(v)) \leq \rho(u, v) - \phi(\rho(u, v)),$$

for each  $u, v \in W$ . Thus, there exists  $\omega \in W$  such that  $\{\omega\} \subset F(\omega)$ .



**Example 3.7.** Let  $U = R^+$ ,  $\rho(u, v) = |u - v|$ , whenever  $u, v \in U$  and  $\gamma, \delta \in [0, 1]$ . Consider a pair of intuitionistic fuzzy mappings  $F = \langle \mu_F, \nu_F \rangle$ ,  $G = \langle \mu_G, \nu_G \rangle : U \rightarrow (IFS)^U$  as follow:

Case (i) : If  $u = v = 0$ , then we have

$$\mu_{F(0)}(t) = \mu_{G(0)}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{2}{5} & \text{if } 0 < t \leq 100^2 \\ 0 & \text{if } t > 100^2 \end{cases}$$

and

$$\nu_{F(0)}(t) = \nu_{G(0)}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{3}{5} & \text{if } 0 < t \leq 100^3 \\ 1 & \text{if } t > 100^3 . \end{cases}$$

If we take  $\alpha_{F(0)} = 1 = \alpha_{G(0)}$ , then we obtain

$$[F(0)]_{(\mathcal{T}, \mathcal{N}, 1)} = \{0\} = [G(0)]_{(\mathcal{T}, \mathcal{N}, 1)} .$$

Moreover,

$$\rho_H \left( [F(0)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(0)})}, [G(0)]_{(\mathcal{T}, \mathcal{N}, \alpha_{G(0)})} \right) = \rho(u, v) - \phi(\rho(u, v)) .$$

Case (ii) : If  $u \neq 0, v \neq 0$  then we have

$$\mu_{F(u)}(t) = \begin{cases} \gamma & \text{if } 0 \leq t \leq u - \frac{u^2}{2} \\ \frac{\gamma}{3} & \text{if } u - \frac{u^2}{2} < t \leq u - \frac{u^2}{4} \\ \frac{\gamma}{7} & \text{if } u - \frac{u^2}{4} < t < u \\ 0 & \text{if } u \leq t < \infty , \end{cases} \quad \nu_{F(u)}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq u - \frac{u^2}{2} \\ \frac{\gamma}{4} & \text{if } u - \frac{u^2}{4} < t \leq u - \frac{u^2}{6} \\ \frac{\gamma}{2} & \text{if } u - \frac{u^2}{6} < t < u \\ 1 & \text{if } u \leq t < \infty . \end{cases}$$

and

$$\mu_{G(u)}(t) = \begin{cases} \delta & \text{if } 0 \leq t \leq u - \frac{u^2}{2} \\ \frac{\delta}{2} & \text{if } u - \frac{u^2}{2} < t \leq u - \frac{u^2}{5} \\ \frac{\delta}{3} & \text{if } u - \frac{u^2}{5} < t < u \\ 0 & \text{if } u \leq t < \infty , \end{cases} \quad \nu_{G(u)}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq u - \frac{u^2}{2} \\ \frac{\delta}{8} & \text{if } u - \frac{u^2}{2} < t \leq u - \frac{u^2}{8} \\ \frac{\delta}{6} & \text{if } u - \frac{u^2}{8} < t < u \\ 1 & \text{if } u \leq t < \infty . \end{cases}$$

If  $\alpha_{F(u)} = \gamma$  and  $\alpha_{G(u)} = \delta$ , then we have

$$[F(u)]_{(\mathcal{T}, \mathcal{N}, \gamma)} = \{t \in U : \mathcal{T}(\mu_{F(u)}(t), \mathcal{N}(\nu_{F(u)}(t))) = \gamma\} = \left[0, u - \frac{u^2}{2}\right]$$

and

$$[G(u)]_{(\mathcal{T}, \mathcal{N}, \delta)} = \{t \in U : \mathcal{T}(\mu_{G(u)}(t), \mathcal{N}(\nu_{G(u)}(t))) = \delta\} = \left[0, u - \frac{u^2}{2}\right] .$$

However,

$$\begin{aligned}
 \rho_H \left( [F(u)]_{(\mathcal{T}, \mathcal{N}, \gamma)}, [G(u)]_{(\mathcal{T}, \mathcal{N}, \delta)} \right) &= \left| u - \frac{u^2}{2} - v + \frac{v^2}{2} \right| \\
 &= \left| (u - v) \left( 1 - \frac{u + v}{2} \right) \right| \\
 &\leq |u - v| \left| 1 - \frac{|u - v|}{2} \right| \\
 &\leq |u - v| - \frac{|u - v|^2}{2} \\
 &= |u - v| - \varphi(|u - v|) \\
 &\leq \rho(u, v) - \varphi(\rho(u, v)).
 \end{aligned}$$

Thus, in both the cases, for  $\varphi(t) = \frac{1}{2}t^2$ , all the assumptions of theorem 3.1 are satisfied to obtain  $\omega \in [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})} \cap [G(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{G(\omega)})}$ .

**3.1. Application to Delay Differential Equations.** In this section, we will establish an existence result of delay differential equation with constant delay, where the only independent variable is the time variable.

Delay differential equations appear naturally in modelling the numerous biological systems. For instance, primary infection [14], drug therapy [29] and immune response [15]. They have also been used in the study of epidemiology [16], the respiratory system [36] and tumor growth [37]. Moreover, statistical analysis of ecological data [34], indicates the delay effects in many classes of population dynamics.

**General Form of Delay Differential Equation**

Consider the general form of equation with delay

$$u'(t) = g(t, u_t),$$

where  $u_t : [-\tau, 0] \rightarrow R^n$  is a function such that  $u_t(\lambda) = u(t + \lambda)$  for  $\lambda \in [-\tau, 0]$ , as shown in Fig. 2.

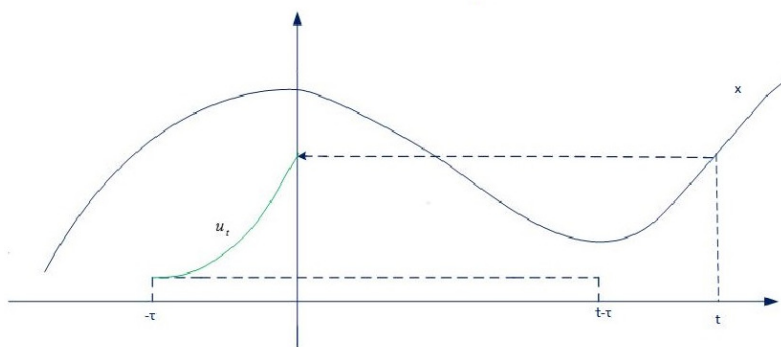


Fig. 2. General representation of delay differential equations

For an ordinary differential system, a unique solution is obtained using an initial point in Euclidean space at an initial time  $t_0$ . On the other hand, one needs information on the entire interval  $[t_0 - \tau, t_0]$ , for a delay differential equations.

Delay differential equations are solved by considering previous values of dependent variable  $u$  at every time step. For this, one requires initial function or initial history, the value of  $u(t)$  for the interval  $[-\tau, 0]$  is used to demonstrate the behavior of the system prior to the starting time.

**Theorem 3.8.** *Let  $U = C([a, b], R)$  be the space of all continuous real valued functions on  $[a, b]$  with a metric  $\rho : X \times X \rightarrow R$  defined by*

$$\rho(u, v) = \max_{t \in [a, b]} |u(t) - v(t)|,$$

for all  $u, v \in C[a, b]$ .

Assume that  $g : [t_0, b] \times R^2 \rightarrow R$  and  $\psi : [t_0 - \tau, b] \rightarrow R$  are continuous mappings, where  $t_0, b \in R$  and  $\tau > 0$ . If there exists  $\lambda_g > 0$  such that  $\lambda_g < \frac{1}{2(b-t_0)}$  and

$$(3.5) \quad |g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \lambda_g \sum_{i=1}^2 |u_i - v_i|,$$

for all  $u_i, v_i \in R, i = 1, 2, t \in [t_0, b]$ .

Thus, the delay differential equation

$$(3.6) \quad u'(t) = g(t, u(t), u(t - \tau)), \quad t \in [t_0, b]$$

with initial condition

$$(3.7) \quad u(t) = \psi(t), \quad t \in [t_0 - \tau, t_0]$$

has a solution  $u \in C([t_0 - \tau, b], R) \cap C^1([t_0, b], R)$ .

*Proof.* Let  $F : U \rightarrow (IFS)^U$  be intuitionistic fuzzy mapping and define an arbitrary mapping  $h$  from  $U$  into  $(0, 1]$ .

The integral reformulation of problem (3.5)-(3.7) is given by

$$(3.8) \quad u(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau, t_0] \\ \psi(t_0) + \int_{t_0}^t g(s, u(s), u(s - \tau)) ds, & t \in [t_0, b]. \end{cases}$$

Define an intuitionistic fuzzy mapping  $F = \langle \mu_F, \nu_F \rangle : U \rightarrow (IFS)^U$  as follows:

$$\mu_{F(u)}(e) = \begin{cases} h(u) & \text{if } e(t) = u(t) \text{ for all } t \in [t_0, b] \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{F(u)}(e) = \begin{cases} 0 & \text{if } e(t) = u(t) \text{ for all } t \in [t_0, b] \\ h(u) & \text{otherwise.} \end{cases}$$

If  $\alpha_{F(u)} = h(u)$ , then we have

$$\begin{aligned} [F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u)})} &= \{e \in U : \mathcal{T}(\mu_{F(u)}(e), \mathcal{N}(v_{F(u)}(e))) = h(u)\} \\ &= \{u\}. \end{aligned}$$

However,

$$\rho_H\left([F(u)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(u)})}, [F(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(v)})}\right) = \max_{t \in [t_0 - \tau, b]} |u(t) - v(t)|.$$

Therefore, by assumptions, we obtain

$$\begin{aligned} \max_{t \in [t_0 - \tau, b]} |u(t) - v(t)| &= \max_{t \in [t_0 - \tau, b]} \left| \int_{t_0}^t g(s, u(s), u(s - \tau)) ds - \int_{t_0}^t g(s, v(s), v(s - \tau)) ds \right| \\ &\leq \max_{t \in [t_0 - \tau, b]} \int_{t_0}^t |g(s, u(s), u(s - \tau)) - g(s, v(s), v(s - \tau))| ds \\ &\leq \max_{t \in [t_0 - \tau, b]} \int_{t_0}^t \lambda_g (|u(s) - v(s)| + |u(s - \tau) - v(s - \tau)|) ds \\ &\leq \int_{t_0}^t \lambda_g \left( \max_{s \in [t_0 - \tau, b]} |u(s) - v(s)| + \max_{s \in [t_0 - \tau, b]} |u(s - \tau) - v(s - \tau)| \right) ds \\ &\leq \int_{t_0}^t \lambda_g (\rho(u, v) + \rho(u, v)) ds \\ &\leq 2\lambda_g d(u, v) \int_{t_0}^t ds \\ &\leq 2\lambda_g (b - t_0) \rho(u, v) \\ &\leq \rho(u, v) - (1 - q) \rho(u, v) \\ &\leq \rho(u, v) - \varphi(\rho(u, v)). \end{aligned}$$

Where,  $q = 2\lambda_g (b - t_0)$  and  $\varphi(u) = (1 - q)(u)$ . Thus, all the assumptions of Theorem 3.1 are satisfied for  $F = G$  to obtain  $\omega \in U$  such that

$$\omega \in \left[ [F(\omega)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(\omega)})} \right].$$

Hence,  $\omega$  is a solution of (3.5) and (3.6). □

**Example 3.9.** Consider the delay differential equation

$$(3.9) \quad u'(t) = t^3 + \frac{1}{10}u^5(t) + \frac{1}{10}u^5\left(t - \frac{1}{2}\right), \quad t \in [0, 1]$$

with initial condition

$$(3.10) \quad u(t) = t + 1, \quad t \in \left[-\frac{1}{2}, 0\right],$$

where  $\tau = \frac{1}{2}$  and  $\psi(t) = t + 1$ ,

$$g(t, u(t), u(t - \tau)) = t^3 + \frac{1}{10}u^5(t) + \frac{1}{10}u^5\left(t - \frac{1}{2}\right).$$

The associated integral equation of problem (3.9)-(3.10) is given by

$$u(t) = \begin{cases} t + 1, & t \in \left[-\frac{1}{2}, 0\right] \\ 1 + \int_0^t \left(s^3 + \frac{1}{10}u^5(s) + \frac{1}{10}u^5\left(s - \frac{1}{2}\right)\right) ds, & t \in [0, 1]. \end{cases}$$

If  $u_i, v_i \in R$ ,  $i = 1, 2$ , and  $t \in [0, 1]$  then we obtain

$$\begin{aligned} |g(t, u_1, u_2) - g(t, v_1, v_2)| &= \left| t^3 + \frac{1}{10}u_1^5 + \frac{1}{10}u_2^5 - t^3 - \frac{1}{10}v_1^5 - \frac{1}{10}v_2^5 \right| \\ &= \left| \frac{1}{10}(u_1^5 - v_1^5) + \frac{1}{10}(u_2^5 - v_2^5) \right| \\ &\leq \frac{1}{10} |u_1^5 - v_1^5| + \frac{1}{10} |u_2^5 - v_2^5| \\ &\leq \frac{1}{10} \sum_{i=1}^2 |u_i^5 - v_i^5|. \end{aligned}$$

Hence, for  $\lambda_g = \frac{1}{10}$ , all the conditions of Theorem 3.8 are satisfied to obtain a solution of the given delay differential equation.

#### 4. APPLICATION TO A SYSTEM OF RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

In recent time, fractional calculus has drawn the interests of researchers due to its wide range of applications in solving problems in diverse areas such as viscoelasticity, biological science, aerodynamics, statistical physics, etc. For some noted applications and developmental history of fractional calculus, the interested reader may see [24, 35]. Undoubtedly, the first problem of every fractional differential equation is the conditions for the existence of its solution. Thus, this section is devoted to providing existence conditions of solutions to Riemann-Liouville Cauchy type problem on a finite interval of the real line in a space of summable and continuous functions. Our investigations are based on reformulating the problem to Volterra integral equation of the second kind and using intuitionistic fuzzy maps. The nonlinear Riemann-Liouville fractional derivative  $(D_{a+}^{\xi}v)(u)$  of order  $\xi$ , defined for  $Re(\xi) > 0$  on a finite interval  $[a, b]$  is given by

$$(4.1) \quad (D_{a+}^{\xi}v)(u) = g(u, v(u)),$$

with initial conditions

$$(4.2) \quad (D_{a^+}^{\xi-i}v)(a^+) = d_i, \quad d_i \in \mathbb{C} \quad (i = 1, 2, 3, \dots, n),$$

where  $n = Re(\xi) + 1$  for  $\xi \notin \mathbb{N}$  and  $\xi = n$  for  $\xi \in \mathbb{N}$ . Notice that for  $\xi = n \in \mathbb{N}$ , problems (4.1)-(4.2) are reduced to classical Cauchy problem for the ordinary differential equation. The Cauchy type problem (4.1)-(4.2) with complex  $\xi \in \mathbb{C}$  was first studied by Kilbas [23] in the space of summable functions  $L(a, b)$ .

Al-Bassam [12] studied problem (4.1)-(4.2) for a real  $0 < \xi \leq 1$  in the space of continuous functions  $C[a, b]$ , provided that  $g(u, v)$  is a real-valued continuous function in a domain  $H \subset \mathbb{R}^2$ . Most likely, he was the first to show that the method of contraction mapping could be employed to prove the existence of solution to (4.1)-(4.2). It was however observed by Kilbas [23] that the condition given by Al-Bassam [12] was not suitable for solving the problem. Afterwards, Delbosco and Rodino [19] studied the nonlinear Riemann-Liouville Cauchy problem:

$$(4.3) \quad (D_{a^+}^\xi v)(u) = g(u, v(u)), \quad v^{(i)}(0) = v_i \in \mathbb{R} \quad (i = 1, 2, 3, \dots, n)$$

with  $0 \leq u \leq 1$ ,  $\lambda > 0$  and  $g(u, v)$  is a continuous function on  $[0, 1] \times \mathbb{R}$ . They showed the equivalence to the corresponding Volterra integral equation and applied Schauder's fixed point theorem to prove that problem (4.3) has at least one solution  $v(u)$  defined on  $[0, \tau]$  provided that  $u^\kappa g(u, v)$  is continuous on  $[0, 1] \times \mathbb{R}$  for some  $\kappa \in [0, 1)$ . Later on, problems (4.1)-(4.2) and (4.3) were studied by several authors (see, [2, 4, 32]). But the above investigations were not complete due to the missing of some techniques of nonlinear functional analysis [23]. For details in this observation, the interested readers may go through the survey paper by Kilbas and Trujillo [33].

As far as we know, no contribution exists in the literature concerning with the study of existence conditions of the Riemann-Liouville Cauchy type problem (4.1)-(4.2) in the setting of intuitionistic fuzzy mappings and even then for fuzzy and multivalued mappings. Thus, in this section, we establish existence conditions for the solution of problem (4.1)-(4.2) in the space  $L_1(a, b) = L(a, b)$  of summable functions on a finite interval  $[a, b]$  of  $\mathbb{R}$  by appealing to intuitionistic fuzzy mappings defined on a complete metric space.

For our convenience, we recall the definitions of Riemann-Liouville fractional integrals and fractional derivatives on a finite interval of the real line and present specific results. For these basic concepts and notations, we follow the books of Kilbas et al. [23] and Samko et al. [31].

The Riemann-Liouville fractional integrals  $I_{a^+}^\xi g$  and  $I_{b^-}^\xi g$  of order  $\xi \in \mathbb{C}$  where  $Re(\xi) > 0$  are defined by

$$(4.4) \quad (I_{a^+}^\xi g)(u) = \frac{1}{\Gamma(\xi)} \int_a^u \frac{g(t)}{(u-t)^{1-\xi}} dt$$

and

$$(4.5) \quad (I_{b^-}^\xi g)(u) = \frac{1}{\Gamma(\xi)} \int_u^b \frac{g(t)}{(u-t)^{1-\xi}} dt,$$

where  $\Gamma(\cdot)$  is the Gamma function. The integrals (4.4) and (4.5) are called the left-sided and right-sided fractional integrals, respectively.

The Riemann-Liouville fractional derivatives  $D_{a^+}^\xi v$ ,  $D_{b^-}^\xi v$  of order  $\xi \in \mathbb{C}$  are defined by

$$\begin{aligned} (D_{a^+}^\xi v)(u) &= \left(\frac{d}{du}\right)^n \left(I_{a^+}^{n-\xi}\right)(u) \\ &= \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{du}\right)^n \int_a^u \frac{v(t)}{(u-t)^{\xi-n+1}} dt \quad (n = [Re(\xi)] + 1) \end{aligned}$$

and

$$\begin{aligned} (D_{b^-}^\xi v)(u) &= \left(-\frac{d}{du}\right)^n \left(I_{b^-}^{n-\xi}\right)(u) \\ &= \frac{1}{\Gamma(n-\xi)} \left(-\frac{d}{du}\right)^n \int_u^b \frac{v(t)}{(u-t)^{\xi-n+1}} dt \quad (n = [Re(\xi)] + 1), \end{aligned}$$

respectively, where  $[Re(\xi)]$  means the integral part of  $Re(\xi)$ .

we denote by  $L_p(a, b)$ , where  $1 \leq p \leq \infty$ , the set of all Lebesgue complex-valued measurable functions  $g$  on  $\Omega$  for which  $\|g\|_p < \infty$  with

$$\|g\|_p = \left(\int_\Omega |f(t)|^p dt\right)^{\frac{1}{p}}$$

and

$$\|g\|_\infty = \text{ess sup}_{a \leq u \leq b} |g(u)|.$$

The following result shows that fractional differentiation is an operator inverse to the fractional integral operator from the left.

**Lemma 4.1** ([31]). *If  $Re(\xi) > 0$  and  $g(u) \in L_p(a, b)$ , where  $1 \leq p \leq \infty$ , then the following equalities*

$$\left(D_{a^+}^\xi I_{a^+}^\xi g\right)(u) = g(u) \quad \text{and} \quad \left(D_{b^-}^\xi I_{b^-}^\xi g\right)(u) = g(u),$$

hold almost everywhere on  $[a, b]$ .

**Lemma 4.2** ([31]). *The fractional integral operator  $I_{a^+}^\xi$  with  $\xi > 0$  is bounded in  $L(a, b)$  satisfying*

$$\|I_{a^+}^\xi z\|_1 \leq \frac{(b-a)^\xi}{\Gamma(\xi+1)} \|z\|_1.$$

**Lemma 4.3** ([23]). *Let  $\xi \in \mathbb{C}$  and  $n - 1 < Re(\xi) < n$  ( $n \in \mathbb{N}$ ). Let  $H$  be an open set in  $\mathbb{C}$  and  $g : [a, b] \times H \rightarrow \mathbb{C}$  be a function such that  $g(u, v) \in L(a, b)$  for any  $v \in H$ . If  $v(u) \in L(a, b)$ , then  $v(u)$  satisfies almost everywhere the Riemann-Liouville Cauchy type problem (4.1)-(4.2) if and only if  $v(u)$  satisfies the integral equation*

$$(4.6) \quad v(u) = \sum_{i=1}^n \frac{d_i}{\Gamma(\xi-i+1)} (u-a)^{\xi-i} + \frac{1}{\Gamma(\xi)} \int_a^u \frac{g(t, v(t))}{(u-t)^{1-\xi}} dt$$

Our main result of this section runs as follows:

**Theorem 4.4.** *Let  $H$  be an open set in  $\mathbb{C}$  and  $g_1, g_2 : [a, b] \times H \rightarrow \mathbb{C}$  be functions such that  $g_1(u, v), g_2(u, v) \in L(a, b) = U$ , where  $L(a, b)$  is the set of all Lebesgue complex-valued measurable functions on  $[a, b]$  endowed with the metric  $\rho : U \times U \rightarrow \mathbb{R}$  defined as*

$$\rho(v_1, v_2) = \|v_1 - v_2\| = \int_a^u |v_1(u) - v_2(u)| du,$$

for all  $v_1, v_2 \in U$  and  $a < u < b$ . Assume that for all  $g_1, g_2 \in L(a, b)$ , there exists  $\varrho > 0$  such that

$$\|g_1(u, v_1) - g_2(u, v_2)\| \leq \varrho \|v_1 - v_2\|.$$

Thus, the system of Riemann-Liouville Cauchy type problems (SRLCTPs) given by

$$(4.7) \quad (D_{a^+}^\xi v)(u) = g_1(u, v(u)),$$

with initial conditions

$$(4.8) \quad (D_{a^+}^{\xi-i} v)(a^+) = d_i, \quad d_i \in \mathbb{C} \quad (i = 1, 2, 3, \dots, n)$$

and

$$(4.9) \quad (D_{a^+}^\xi v)(u) = g_2(u, v(u)),$$

with initial conditions

$$(4.10) \quad (D_{a^+}^{\xi-i} v)(a^+) = d_i, \quad d_i \in \mathbb{C} \quad (i = 1, 2, 3, \dots, n),$$

have a common solution in  $L(a, b)$ .

*Proof.* By Lemma 4.3, the common solution of (4.7)-(4.10) is also the common solution of their integral reformulation, respectively given as:

$$(4.11) \quad v(u) = \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\lambda-i} + \frac{1}{\Gamma(\xi)} \int_a^u \frac{g_1(t, v(t))}{(u - t)^{1-\xi}} dt$$

$$(4.12) \quad v(u) = \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\lambda-i} + \frac{1}{\Gamma(\xi)} \int_a^u \frac{g_2(t, v(t))}{(u - t)^{1-\xi}} dt$$

Clearly, the set  $U$  equipped with the given metric  $\rho$  is a complete metric space. Let  $r, s : U \rightarrow (0, 1]$  be any two arbitrary mappings and  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a continuous non-decreasing function. Choose  $u_1 \in (a, b)$  such that

$$\varrho \frac{(u_1 - a)^\xi}{\Gamma(\xi + 1)} \leq \frac{\rho(v_1, v_2) - \varphi(\rho(v_1, v_2))}{1 + \rho(v_1, v_2)}.$$

For  $v \in U$ , we have

$$\omega_v(t) = v_0(t) + \frac{1}{\Gamma(\xi)} \int_a^u \frac{g_1(t, v(t))}{(u - t)^{1-\xi}} dt$$



and

$$\tau_v(t) = v_0(t) + \frac{1}{\Gamma(\xi)} \int_a^u \frac{g_2(t, v(t))}{(u-t)^{1-\xi}} dt,$$

where

$$v_0(t) = \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\xi - i}.$$

Consider a pair of intuitionistic fuzzy mappings  $F, G : U \rightarrow (IFS)^U$  defined as follows:

$$\mu_{F(v)}(q) = \begin{cases} r(v), & q(t) = \omega_v(t), \quad t \in [a, b] \\ 0, & q(t) \neq \omega_v(t), \end{cases}$$

$$\nu_{F(v)}(q) = \begin{cases} 0, & q(t) = \omega_v(t), \quad t \in [a, b] \\ r(v), & q(t) \neq \omega_v(t) \end{cases}$$

and

$$\mu_{G(v)}(q) = \begin{cases} s(v), & q(t) = \tau_v(t), \quad t \in [a, b] \\ 0, & q(t) \neq \tau_v(t), \end{cases}$$

$$\nu_{G(v)}(q) = \begin{cases} 0, & q(t) = \tau_v(t), \quad t \in [a, b] \\ s(v), & q(t) \neq \tau_v(t). \end{cases}$$

If we take  $\alpha_{F(v)} = r(v)$  and  $\alpha_{G(v)} = s(v)$ , then we have

$$\begin{aligned} [F(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(v)})} &= \{q \in U : \mathcal{T}(\mu_{F(v)}(q), \mathcal{N}(\nu_{F(v)}(q))) = r(v)\} \\ &= \{\omega_v\} \end{aligned}$$

and

$$\begin{aligned} [G(v)]_{(\mathcal{T}, \mathcal{N}, \alpha_{G(v)})} &= \{q \in U : \mathcal{T}(\mu_{G(v)}(q), \mathcal{N}(\nu_{G(v)}(q))) = s(v)\} \\ &= \{\tau_v\}. \end{aligned}$$

Therefore, for  $v_1, v_2 \in U$ , we obtain

$$[F(v_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(v_1)})} = \{\omega_{v_1}\}$$

and

$$[G(v_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_{G(v_2)})} = \{\tau_{v_2}\}.$$

Consequently,

$$\rho_H \left( [F(v_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_{F(v_1)})}, [G(v_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_{G(v_2)})} \right) = \|\omega_{v_1} - \tau_{v_2}\|_1.$$

For the remaining steps, we employ a standard method for nonlinear Volterra integral equations of the proof of result on a subinterval of  $[a, b]$ , (see, [25, 23]). Notice that equations (4.11)-(4.12) are valid in any interval  $[a, u_1] \subset [a, b]$  for  $a < u_1 < b$ . Thus, for an interval  $[a, u_1]$ , a metric  $\rho : L(a, u_1) \times L(a, u_1) \rightarrow \mathbb{R}$  is defined by

$$\rho(v_1, v_2) = \|v_1 - v_2\|_1 = \int_a^{u_1} |v_1(u) - v_2(u)| du.$$

Since  $f, g \in L(a, b)$ , therefore, by Lemma 4.2,  $u_1 = b$  and  $z = g_1(u, v_1) - g_2(u, v_2)$ , we have

$$\begin{aligned}
 \rho_H \left( [F(v_1)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(v_1))}, [G(v_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(v_2))} \right) &= \|\omega_{v_1} - \tau_{v_2}\|_1 \\
 &= \left\| \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{g_1(t, v_1(t))}{(u-t)^{1-\xi}} dt - \int_a^{u_1} \frac{g_2(t, v_2(t))}{(u-t)^{1-\xi}} dt \right\|_1 \\
 &\leq \left\| \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{[g_1(t, v_1) - g_2(t, v_2)]}{(u-t)^{1-\xi}} dt \right\|_1 \\
 &\leq \left\| I_{a^+}^\xi [g_1(t, v_1) - g_2(t, v_2)] \right\|_1 \\
 &\leq \frac{(u_1 - a)^\xi}{\Gamma(\xi + 1)} \|g_1(t, v_1) - g_2(t, v_2)\|_1 \\
 &\leq \varrho \frac{(u_1 - a)^\xi}{\Gamma(\xi + 1)} \rho(v_1, v_2) \\
 &\leq \varrho \frac{(u_1 - a)^\xi}{\Gamma(\xi + 1)} (1 + \rho(v_1, v_2)) \\
 &\leq \rho(v_1, v_2) - \varphi(\rho(v_1, v_2)).
 \end{aligned}$$

Hence, by Theorem 3.1, there exists a common solution  $v^* \in L(a, u_1)$  to the Volterra integral equations (4.11)-(4.12) in the interval  $[a, u_1]$ .

Next, consider the interval  $[u_1, u_2]$ , where  $u_2 = u_1 + \zeta_1$  and  $\zeta_1 > 0$  are such that  $u_2 < b$ . Rewrite equations (4.11)-(4.12) as follows:

$$\begin{aligned}
 v(u) &= \frac{1}{\Gamma(\xi)} \int_{u_1}^u \frac{g_1(t, v(t))}{(u-t)^{1-\xi}} dt + \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\xi - i} \\
 (4.13) \quad &+ \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{g_1(t, v(t))}{(u-t)^{1-\xi}} dt.
 \end{aligned}$$

$$\begin{aligned}
 v(u) &= \frac{1}{\Gamma(\xi)} \int_{u_1}^u \frac{g_2(t, v(t))}{(u-t)^{1-\xi}} dt + \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\xi - i} \\
 (4.14) \quad &+ \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{g_2(t, v(t))}{(u-t)^{1-\xi}} dt.
 \end{aligned}$$

Again, equations (4.13)-(4.14) can be rewritten as

$$v(u) = v_0(u) + \frac{1}{\Gamma(\xi)} \int_{u_1}^u \frac{g_1(t, v(t))}{(u-t)^{1-\xi}} dt,$$

$$v(u) = v_0(u) + \frac{1}{\Gamma(\xi)} \int_{u_1}^u \frac{g_2(t, v(t))}{(u-t)^{1-\xi}} dt,$$

where

$$\begin{aligned}
 v_0(u) &= \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\xi - i} + \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{g_1(t, v(t))}{(u - t)^{1 - \xi}} \\
 (4.15) \quad &= \sum_{i=1}^n \frac{d_i}{\Gamma(\xi - i + 1)} (u - a)^{\xi - i} + \frac{1}{\Gamma(\xi)} \int_a^{u_1} \frac{g_2(t, v(t))}{(u - t)^{1 - \xi}}
 \end{aligned}$$

are the known functions. The idea of (4.15) is to ignore the previous interval  $[a, u_1]$  for which a solution is known. Next, by re-considering any two arbitrary mappings  $r, s : U \rightarrow (0, 1]$ , a pair of intuitionistic fuzzy mappings  $F, G : U \rightarrow (IFS)^U$  and a nondecreasing continuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that

$$\rho \frac{(u_2 - a)^\xi}{\Gamma(\xi + 1)} \leq \frac{\rho(v_2, v_3) - \phi(\rho(v_2, v_3))}{1 + \rho(v_2, v_3)} \quad (v_2, v_3 \in L(a, b), \quad a < u_2 < b).$$

So, one can obtain

$$\rho_H \left( [F(v_2)]_{(\mathcal{T}, \mathcal{N}, \alpha_F(v_2))}, [G(v_3)]_{(\mathcal{T}, \mathcal{N}, \alpha_G(v_3))} \right) \leq \rho(v_2, v_3) - \phi(\rho(v_2, v_3)).$$

Again, Theorem 3.1 can be applied to find a solution  $v^*(u) \in L(u_1, u_2)$  to the integral equations (4.11)-(4.12) on the interval  $[u_1, u_2]$ . By repeating this procedure inductively on the intervals  $[u_2, u_3], \dots, [u_n, u_{n+1}]$ , where  $u_{n+1} = u_n + \zeta_n$  and  $\zeta_n > 0$  are such that  $u_{n+1} < b$ , therefore, we can conclude according to Theorem 3.1 that there exists a common solution  $v(u) = v^*(u) \in L(a, b)$  to the Riemann-Liouville Cauchy type problems (4.7)-(4.10) on the interval  $[a, b]$ .  $\square$

*Remark 4.5.* The result of Theorem 4.4 only gives existence conditions for the Riemann-Liouville Cauchy type problem (4.7)-(4.8) and its equivalent integral equation (4.11) in the space  $L(a, b)$  for  $\xi \in \mathbb{C}$  and  $n - 1 < Re(\xi) < n$  ( $n \in \mathbb{N}$ ). The case of the problem (4.7)-(4.8) for order  $\xi = n + im$ , ( $n \in \mathbb{N}, m \in \mathbb{R}, m \neq 0$ ) may be considered in due course.

### CONCLUSION

In the framework of IF-sets, we have established a common fixed point theorem using weakly contractive condition for a pair of intuitionistic fuzzy mappings in the context of  $(\mathcal{T}, \mathcal{N}, \alpha)$ -cut set of an IF-set in a complete metric space. Moreover, in our research work, we have constructed the iterations to establish the fixed point of intuitionistic fuzzy mappings. By building on the constructive approach, one will be able to define a procedure for obtaining the solution of certain functional equations arising in dynamical systems. On other hand, there is a rich variety of dynamics with multifaceted mathematical structures such as industrial control devices and systems handling imprecise information. Therefore, the knowledge of cut sets of an IF-set is beneficial to handle such uncertain and imprecise informations and processes, because these sets can transform an IF-set into a crisp set.

As an application, we have investigated the existence of solution of time dependent delay differential equations with constant delay and Riemann-Liouville Cauchy type fractional differential equations, which involve completeness property of function spaces. Moreover, an example has been given to support the validity of existence theorem of the considered delay differential equation.

In future, the presented results will be useful to handle several realistic uncertain situations. On one hand, as an application, one can implement these results for the existence of delay differential equations with variable delays and  $n$ -systems of Cauchy problems of Riemann Liouville type.

**ACKNOWLEDGEMENTS.** *The authors are grateful to the editors and the anonymous referee(s) for careful checking of the details and for their helpful comments to improve this paper.*

#### REFERENCES

- [1] H. M. Abu-Donia, Common fixed points theorems for fuzzy mappings in metric space under  $\varphi$ -contraction condition, *Chaos Solitons & Fractals* 34 (2007), 538–543.
- [2] A. Z. Al-Abedein, Existence theorem on differential equation of generalized order, *Al-Rafidain J. Sci. Mosul University, Iraq*, 1 (1976), 95–104.
- [3] Y. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, in: *New Results in Operator Theory and Its Applications*, Birkhäuser, Basel (1997), 7–22.
- [4] H. L. Arora and J. G. Alshamani, Stability of differential equations of noninteger order through fixed point in the large, *Indian J. Pure Appl. Math.* 11, no. 3 (1980), 307–313.
- [5] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy sets and Systems* 20, no. 1 (1986), 87–96.
- [6] A. Azam, M. Arshad and P. Vetro, On a pair of fuzzy  $\varphi$ -contractive mappings, *Mathematical and Computer Modelling* 52, no. 1 (2010), 207–214.
- [7] A. Azam and M. Rashid, A fuzzy coincidence theorem with applications in a function space, *Journal of Intelligent and Fuzzy Systems* 27, no. 4 (2014), 1775–1781.
- [8] A. Azam, R. Tabassum and M. Rashid, Coincidence and fixed point theorems of intuitionistic fuzzy mappings with applications, *Journal of Mathematical Analysis* 8, no. 4 (2017), 56–77.
- [9] A. Azam and R. Tabassum, Existence of common coincidence point of intuitionistic fuzzy maps, *Journal of Intelligent and Fuzzy Systems* 35 (2018), 4795–4805.
- [10] J. S. Bae, Fixed point theorems for weakly contractive multivalued maps, *Journal of Mathematical Analysis and Applications* 284, no. 2 (2003), 690–697.
- [11] I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, *Fixed Point Theory and Applications* 2006 (2006), 1–7.
- [12] M. A. Al-Bassam, Some existence theorems on differential equations of generalized order, *J. Reine Angew. Math.* 218, no. 1 (1965), 70–78.
- [13] V. Berinde, Approximating fixed points of weak contractions, *Fixed Point Theory* 4 (2003), 131–142.
- [14] S. M. Ciupe, B. L. de Bivort, D. M. Bortz and P. W. Nelson, Estimates of kinetic parameters from HIV patient data during primary infection through the eyes of three different models, *Math. Biosci.*, to appear.

- [15] K. Cooke, Y. Kuang and B. Li, Analyses of an antiviral immune response model with time delays, *Canad. Appl. Math. Quart.* 6, no. 4 (1998), 321–354.
- [16] K. L. Cooke, P. van den Driessche and X. Zou, Interaction of maturation delay and nonlinear birth in population and epidemic models, *J. Math. Biol.* 39 (1999), 332–352.
- [17] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive multi-valued mappings, *Journal of Mathematical Analysis and Applications* 192, no. 2 (1995), 655–666.
- [18] S. K. De, R. Biswas and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Sets and Systems* 117, no. 2 (2001), 209–213.
- [19] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Appl.* 204, no. 2 (1996), 609–625.
- [20] S. Heilpern, Fuzzy mappings and fixed point theorems, *Journal of Mathematical Analysis and Applications* 83, no. 2 (1981), 566–569.
- [21] Z. Jia, L. Amselang and P. Gros, Content-based image retrieval from a large image database, *Pattern Recognition* 11, no. 5 (2008), 1479–1495.
- [22] A. Kharal, Homeopathic drug selection using intuitionistic fuzzy sets, *Homeopathy* 98, no. 1 (2009), 35–39.
- [23] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science Limited, 2006.
- [24] S. Konjik, L. Oparnica and D. Zorica, Waves in viscoelastic media described by a linear fractional model, *Integral Transforms Spec. Funct.* 22 (2011), 283–291.
- [25] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis*, Nauka, Moscow, 1968.
- [26] D. F. Li, Multiattribute decision making models and methods using intuitionistic fuzzy sets, *J. Comput. Syst. Sci.* 70 (2005), 73–85.
- [27] D. Martinetti, V. Janiš and S. Montes, Cuts of intuitionistic fuzzy sets respecting fuzzy connectives, *Information Sciences* 232 (2013), 267–275.
- [28] S. B. Nadler Jr, Multi-valued contraction mappings, *Pacific Journal of Mathematics* 30, no. 2 (1969), 475–488.
- [29] P. W. Nelson, J. D. Murray and A. S. Perelson, A model of HIV-1 pathogenesis that includes an intracellular delay. *Math. Biosci.* 163 (2000), 201–215.
- [30] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Analysis* 4, no. 47 (2001), 2683–2693.
- [31] A. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives : Theory and Applications*, Gordon and Breach Science Publishers, Switzerland, 1993.
- [32] A. M. A. El-Sayed and A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comp.* 68, no. 1 (1995), 15–25.
- [33] A. A. Kilbas and J. J. Trujillo, Differential equations of fractional order: methods, results and problems, I. *Appl. Anal.* 78, no. 1-2 (2001), 153–192.
- [34] P. Turchin and A. D. Taylor, Complex dynamics in ecological time series, *Ecology* 73 (1992), 289–305.
- [35] D. Valerjo, D. Machadoa and J. T. Kryakova, Some pioneers of the applications of fractional calculus, *Fract. Calc. Appl. Anal.* 17 (2014), 552–578.
- [36] B. Vielle and G. Chauvet, Delay equation analysis of human respiratory stability, *Math. Biosci.* 152, no. 2 (1998), 105–122.
- [37] M. Villasana and A. Radunskaya, A delay differential equation model for tumor growth, *J. Math. Biol.* 47, no. 3 (2003), 270–294.
- [38] Y. H. Shen, F. X. Wang and W. Chen, A note on intuitionistic fuzzy mappings, *Iranian Journal of Fuzzy Systems* 9, no. 5 (2012), 63–76.
- [39] L. A. Zadeh, Fuzzy sets, *Information and Control* 8, no. 3 (1965), 338–353.