APPLIED GENERAL TOPOLOGY

e_c-Filters and e_c-ideals in the functionally countable subalgebra of $C^*(X)$

Amir Veisi

Faculty of Petroleum and Gas, Yasouj University, Gachsaran, Iran (aveisi@yu.ac.ir)

Communicated by A. Tamariz-Mascarúa

Abstract

The purpose of this article is to study and investigate e_c-filters on X and e_c-ideals in $C_c^*(X)$ in which they are in fact the counterparts of z_c-filters on X and z_c-ideals in $C_c(X)$ respectively. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the e_c-ultrafilters on X. In addition, the sets of e_c-ultrafilters and z_c-ultrafilters are in one-to-one correspondence. It is also shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality. As another application of the new concepts, we characterized maximal ideals of $C_c^*(X)$. Finally, we show that whether the space X is compact, a proper ideal I of $C_c(X)$ is an e_c-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

1. INTRODUCTION

All topological spaces are completely regular Hausdorff spaces and we shall assume that the reader is familiar with the terminology and basic results of [6]. Given a topological space X, we let $C(X)$ denote the ring of all real-valued continuous functions defined on X. $C_c(X)$ is the subalgebra of $C(X)$ consisting of functions with countable image and $C_c^*(X)$ is its subalgebra consisting of bounded functions. In fact, $C_c^*(X) = C_c(X) \cap C^*(X)$, where elements of $C^*(X)$

are bounded functions of $C(X)$. Recall that for $f \in C(X), Z(f)$ denotes its zero-set:

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of a zero-set is known as a cozero-set and we denote this set by $\operatorname{coz}(f)$. Let us put $Z_c(X) = \{Z(f) : f \in C_c(X)\}$ and $Z_c^*(X) = \{Z(g) : g \in C_c^*(X)\}$. These two latter sets are in fact equal, since $Z(f) = Z(\frac{f}{1+|f|})$, where $f \in C_c(X)$. A nonempty subfamily $\mathcal{F}$ of $Z_c(X)$ is called a z_c-filter if it is a filter on X. If I is an ideal in $C_c(X)$ and $\mathcal{F}$ is a z_c-filter on X then, we denote $Z_c[I] = \{Z(f) : f \in I\}$, $\cap Z_c[I] = \cap\{Z(f) : f \in I\}$ and $Z_c^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. We see that $Z_c[I]$ is a z_c-filter and $Z_c^{-1}[Z_c[I]] \supseteq I$. If the equality holds, then I is called a z_c-ideal. Moreover, $Z_c^{-1}[\mathcal{F}]$ is a z_c-ideal and we always have $Z_c[Z_c^{-1}[\mathcal{F}]] = \mathcal{F}$. So maximal ideals in $C_c(X)$ are z_c-ideals. In [5], a Huasdorff space X is called countably completely regular (briefly, c-completely regular) if whenever F is a closed subset of X and $x \notin F$, there exists $f \in C_c(X)$ such that $f(x) = 0$ and $f(F) = 1$. In addition, two closed sets A and B of X are also called countably separated (in brief, c-separated) if there exists $f \in C_c(X)$ with $f(A) = 0$ and $f(B) = 1$. c-completely regular and zero-dimensional spaces are the same, see Theorem 1.1.

If we let $M_p^c = \{f \in C_c(X) : f(p) = 0\}$ $(p \in X)$, then the ring isomorphism $\frac{C_c(X)}{M_p^c} \cong \mathbb{R}$ gives that M_p^c is a maximal ideal, in fact, M_p^c is a fixed maximal ideal. Moreover, $\cap Z_c[M_p^c] = \{p\}$.

Our concentration is on the zero-dimensional spaces since in [5] the authors proved that for any space X there is a zero-dimensional Hausdorff space Y such that $C_c(X)$ and $C_c(Y)$ are isomorphic as rings, see Theorem 1.2.

In section 2, we study and investigate the e_c-filters on X and e_c-ideals in $C_c^*(X)$ which they are in fact the counterpart of [6, 2L]. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the e_c-ultrafilters on X. Moreover, the sets of e_c-ultrafilters and z_c-ultrafilters are in one-to-one correspondence. By using the latter facts, it is shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality. Finally, maximal ideals of $C_c^*(X)$ are characterized based on these concepts. In Section 3, our concentration is on the uniform norm topology on $C_c^*(X)$ which is the restriction of the uniform norm topology on $C^*(X)$. It is shown that whenever the space X is compact, a proper ideal I of $C_c(X)$ is an e_c-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

We recite the following results from [5].

Theorem 1.1 ([5, Proposition 4.4]). *Let X be a topological space. Then, X is a zero-dimensional space (i.e., a T_1-space with a base consisting of clopen sets) if and only if X is c-completely regular space.*

Theorem 1.2 ([5, Theorem 4.6]). *Let X be any topological space (not necessarily completely regular). Then, there is a zero-dimensional space Y which is a continuous image of X with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$.*

Remark 1.3 ([5, Remark 7.5]). There is a topological space X, such that there is no space Y with $C_c(X) \cong C(Y)$.

The following results are the known facts about $C_c(X)$ and we are seeking to get similar results for $C_c^*(X)$.

Proposition 1.4. *Let I be a proper ideal in $C_c(X)$ and $\mathcal{F}$ a z_c-filter on X. Then:*

(i) *$Z_c[I]$ is a z_c-filter and $Z_c^{-1}[\mathcal{F}]$ is a z_c-ideal of $C_c(X)$.*
(ii) *If I is maximal then $Z_c[I]$ is a z_c-ultrafilter, and the converse holds if I is a z_c-ideal.*
(iii) *$\mathcal{F}$ is a z_c-ultrafilter if and only if $Z_c^{-1}[\mathcal{F}]$ is a maximal ideal.*
(iv) *If $\mathcal{F}$ is a z_c-ultrafilter and $Z \in Z_c(X)$ meets each element of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

Corollary 1.5. *There is a one-to-one correspondence ψ between the sets of z_c-ideals of $C_c(X)$ and z_c-filters on X, defined by $\psi(I) = Z_c[I]$. In particular, the restriction of ψ to the set of maximal ideals is a one-to-one correspondence between the sets of maximal ideals of $C_c(X)$ and z_c-ultrafilters on X.*

2. e_c-FILTERS ON X AND e_c-IDEALS IN $C_c^*(X)$

For $f \in C_c^*(X)$ and $\epsilon > 0$, we define

$$E_\epsilon^c(f) = f^{-1}([-\epsilon, \epsilon]) = \{x \in X : |f(x)| \leq \epsilon\}.$$

Each such set is a zero set, since it is equal to $Z((|f| - \epsilon) \vee 0)$. Conversely, every zero set is also of this form, since for $g \in C_c^*(X)$ we have $Z(g) = E_\epsilon^c(|g| + \epsilon)$. For a nonempty subset I of $C_c^*(X)$ we denote $E_\epsilon^c[I] = \{E_\epsilon^c(f) : f \in I\}$, and $E_c(I) = \bigcup_\epsilon E_\epsilon^c[I]$. Moreover, if $\mathcal{F}$ is a nonempty subfamily of $Z_c^*(X)$, then we define $E_\epsilon^{c-1}[\mathcal{F}] = \{f \in C_c^*(X) : E_\epsilon^c(f) \in \mathcal{F}\}$ and $E_c^{-1}(\mathcal{F}) = \bigcap_\epsilon E_\epsilon^{c-1}[\mathcal{F}]$. So we have $E_c(I) = \{E_\epsilon^c(f) : f \in I \text{ and } \epsilon > 0\}$, and $E_c^{-1}(\mathcal{F}) = \{f \in C_c^*(X) : E_\epsilon^c(f) \in \mathcal{F}, \text{ for all } \epsilon\}$. Moreover, $E_c^{-1}(E_c(I)) = \{g \in C_c^*(X) : E_\delta^c(g) \in E_c(I), \text{for all } \delta > 0\}$ and $E_c(E_c^{-1}(\mathcal{F})) = \{E_\epsilon^c(f) : E_\delta^c(f) \in \mathcal{F}, \text{for all } \delta > 0\}$.

The next result is now immediate.

Corollary 2.1. *The following statements hold.*

(i) *$I \subseteq E_c^{-1}(E_c(I))$ and $E_c(E_c^{-1}(\mathcal{F})) \subseteq \mathcal{F}$.*
(ii) *The mappings E_c and E_c^{-1} preserve the inclusion.*
(iii) *If $f \in I$ then for each positive integer n, $E_\epsilon^c(f) = E_{\epsilon^n}^c(f^n)$.*
(iv) *If I is an ideal, then $E_c(I)$ is a z_c-filter.*

Proof. The proofs of (i), (ii) and (iii) are clear. (iv). This is presented in the proof of Proposition 2.5. $\square$

Examples 2.2 and 2.3 below show that the inclusions in (i) of the above corollary may be strict even when I is an ideal and $\mathcal{F}$ is a z_c-filter.

Example 2.2. Let X be the discrete space $\mathbb{N} \times \mathbb{N}$, $f(m,n) = \frac{1}{mn}$ and I the ideal in $C_c^*(X) (= C^*(X))$ generated by f^2. Obviously $f \notin I$. Since $\{x \in X : f(x) \leq \epsilon\} = \{x \in X : f^2(x) \leq \epsilon^2\}$, we have $E_\epsilon^c(f) \in E_c(I)$. So $I \subsetneqq E_c^{-1}(E_c(I))$.

Example 2.3. Let X be the zero-dimensional space $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathcal{F} = \{Z \in Z_c(X) : (0,0) \in Z\}$. Then $\mathcal{F}$ is a z_c-filter on X. Now, if we define $f(x,y) = \frac{|x|+|y|}{1+|x|+|y|}$ then $f \in C_c^*(X) (= C^*(X))$ and $Z(f) = \{(0,0)\}$. Given $\epsilon > 0$ and $g = f + \epsilon$, so we have $E_\epsilon^c(g) = \{(0,0)\}$. If we take $0 < \delta < \epsilon$ then $E_\delta^c(g) = \varnothing$. Hence $E_\epsilon^c(g)$ is not contained in $E_c(E_c^{-1}(\mathcal{F}))$. Therefore the latter set is contained in $\mathcal{F}$ properly, which gives the result.

Definition 2.4. A z_c-filter $\mathcal{F}$ is called an e_c-filter if $\mathcal{F} = E_c(E_c^{-1}(\mathcal{F}))$, or equivalently, whenever $Z \in \mathcal{F}$ then there exist $f \in C_c^*(X)$ and $\epsilon > 0$ such that $Z = E_\epsilon^c(f)$ and $E_\delta^c(f) \in \mathcal{F}$, for each $\delta > 0$.

Proposition 2.5. *If I is a proper ideal in $C_c^*(X)$, then $E_c(I)$ is an e_c-filter.*

Proof. First, we show that $E_c(I)$ is a z_c-filter, i.e., it satisfies the following conditions.

    (i) $\varnothing \notin E_c(I)$.
    (ii) $E_\epsilon^c(f), E_\delta^c(g) \in E_c(I)$, then $E_\epsilon^c(f) \cap E_\delta^c(g) \in E_c(I)$.
    (iii) $E_\epsilon^c(f) \in E_c(I)$, $Z \in Z_c(X)$ with $Z \supseteq E_\epsilon^c(f)$, then $Z \in E_c(I)$.

(i). Suppose that for some $\epsilon > 0$ and $f \in I$, $E_\epsilon^c(f) = \varnothing$. So $\epsilon < |f|$, which yields f is a bounded away from zero. Hence I contains the unit f, which is impossible. (ii). This is equivalent to say that if $E_\epsilon^c(f), E_\delta^c(g) \in E_c(I)$, then $E_\epsilon^c(f) \cap E_\delta^c(g)$ contains a member of $E_c(I)$. Suppose that $f', g' \in I$ and $\epsilon', \delta' > 0$ such that $E_\epsilon^c(f) = E_{\epsilon'}^c(f')$ and $E_\delta^c(g) = E_{\delta'}^c(g')$. Without loss of generality, we may suppose that $\delta' < \epsilon' < 1$. Hence $f'^2 + g'^2 \in I$ and $E_{\delta'^2}^c(f'^2 + g'^2) \subseteq E_\epsilon^c(f) \cap E_\delta^c(g)$, which gives the result. (iii). Assume that $E_\epsilon^c(f) \subseteq Z(f')$, where $f \in I$ and $f' \in C_c^*(X)$. Since $E_\epsilon^c(f) = E_{\epsilon^2}^c(f^2)$ and $Z(f') = Z(|f'|)$, we can suppose that $f \geq 0$ and $f' \geq 0$. Now, define

$$g(x) = \begin{cases} 1, & \text{if } x \in E_\epsilon^c(f) \\ f'(x) + \frac{\epsilon}{f(x)}, & \text{if } x \in X \smallsetminus \text{int} E_\epsilon^c(f). \end{cases}$$

So g is continuous, since it is continuous on two closed sets whose union is X, in fact, $g \in C_c^*(X)$. Note that $fg \in I$ and

$$(fg)(x) = \begin{cases} f(x), & \text{if } x \in E_\epsilon^c(f) \\ (ff')(x) + \epsilon, & \text{if } x \in X \smallsetminus \text{int} E_\epsilon^c(f). \end{cases}$$

It is easily seen that $Z(f') = E_\epsilon^c(fg)$. So $Z(f') \in E_c(I)$. This shows that $E_c(I)$ is a z_c-filter. Now, apply (i) and (ii) of Corollary 2.1 for the ideal I and the z_c-filter $E_c(I)$, to get the inclusions $E_c(I) \subseteq E_c(E_c^{-1}(E_c(I)))$ and $E_c(E_c^{-1}(E_c(I))) \subseteq E_c(I)$, which yields $E_c(I)$ is an e_c-filter. $\qquad \square$

Definition 2.6. An ideal I in $C_c^*(X)$ is called e_c-ideal if $I = E_c^{-1}(E_c(I))$, or equivalently, if $f \in C_c^*(X)$ and $E_\epsilon^c(f) \in E_c(I)$ for all ϵ, then $f \in I$.

Proposition 2.7. *If $\mathcal{F}$ is a z_c-filter, then $E_c^{-1}(\mathcal{F})$ is an e_c-ideal in $C_c^*(X)$.*

Proof. Let $f, g \in E_c^{-1}(\mathcal{F})$, $h \in C_c^*(X)$ and let u be an upper bound for h and $\epsilon > 0$. Then $E_{\frac{\epsilon}{2}}^c(f)$, $E_{\frac{\epsilon}{2}}^c(g)$ and hence $E_{\frac{\epsilon}{2}}^c(f) \cap E_{\frac{\epsilon}{2}}^c(g)$ belong to $\mathcal{F}$. Hence $E_{\frac{\epsilon}{2}}^c(f) \cap E_{\frac{\epsilon}{2}}^c(g) \subseteq E_\epsilon^c(f+g)$ implies that $E_\epsilon^c(f+g) \in \mathcal{F}$, or equivalently, $f + g \in E_c^{-1}(\mathcal{F})$. Moreover, $E_{\frac{\epsilon}{u}}^c(f) \subseteq E_\epsilon^c(fh)$ implies $fh \in E_c^{-1}(\mathcal{F})$. Therefore $E_c^{-1}(\mathcal{F})$ is ideal. In view of Corollary 2.1, we have $E_c^{-1}(\mathcal{F}) \subseteq E_c^{-1}(E_c(E_c^{-1}(\mathcal{F}))) \subseteq E_c^{-1}(\mathcal{F})$ and so the equality holds, i.e., $E_c^{-1}(\mathcal{F})$ is an e_c-ideal. $\square$

Corollary 2.8. *Maximal ideals of $C_c^*(X)$ and an arbitrary intersection of them are e_c-ideals.*

Proof. Let M be a maximal ideal of $C_c^*(X)$. If $E_c^{-1}(E_c(M))$ is not a proper e_c-ideal, then it contains the constant function 1 and $E_\epsilon^c(1) = \varnothing \in E_c(M)$ $(0 < \epsilon < 1)$ which is impossible, see Propositions 2.5 and 2.7. Hence $M = E_c^{-1}(E_c(M))$, i.e., M is an e_c-ideal. The second part is obtained by this fact, the fact that the intersection of a family of maximal ideals is an ideal contained in each of them and (ii) of Corollary 2.1. $\square$

The next corollary is an immediate result of Propositions 2.5 and 2.7.

Corollary 2.9. *The correspondence $I \mapsto E_c(I)$ is one-one from the set of e_c-ideals in $C_c^*(X)$ onto the set of e_c-filters on X.*

Lemma 2.10.   (i) *Let I and J be ideals in $C_c^*(X)$ and J an e_c-ideal. Then $I \subseteq J$ if and only if $E_c(I) \subseteq E_c(J)$.*
   (ii) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be z_c-filters on X and $\mathcal{F}_1$ an e_c-filter. Then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $E_c^{-1}(\mathcal{F}_1) \subseteq E_c^{-1}(\mathcal{F}_2)$.*

Proof. It is straightforward. $\square$

Proposition 2.11. *Let I be an ideal in $C_c^*(X)$ and $\mathcal{F}$ a z_c-filter on X. Then:*
   (i) *$E_c^{-1}(E_c(I))$ is the smallest e_c-ideal containing I.*
   (ii) *$E_c(E_c^{-1}(\mathcal{F}))$ is the largest e_c-filter contained in $\mathcal{F}$.*

Proof. (i). Propositions 2.5 and 2.7 respectively show that $E_c(I)$ is an e_c-filter and $E_c^{-1}(E_c(I))$ is an e_c-ideal. Now, suppose that $\mathcal{K}$ is an e_c-ideal containing I. So $E_c^{-1}(E_c(I)) \subseteq E_c^{-1}(E_c(\mathcal{K})) = \mathcal{K}$. Hence we are done. (ii). This is proved similarly. $\square$

The next theorem plays an important role in many of the following results.

Theorem 2.12. *Let $\mathcal{A}$ be a z_c-ultrafilter. Then a zero set Z meets every element of $E_c(E_c^{-1}(\mathcal{A}))$ if and only if $Z \in \mathcal{A}$.*

Proof. Since $\mathcal{A}$ is a filter and $E_c(E_c^{-1}(\mathcal{A})) \subseteq \mathcal{A}$, the sufficient condition is evident. For the necessary condition, it is recalled at first that if Z meets every element of $\mathcal{A}$ then $Z \in \mathcal{A}$, see (iv) of Proposition 1.4. Now, we claim that if Z meets every element of $E_c(E_c^{-1}(\mathcal{A}))$ as a particular subfamily of $\mathcal{A}$, then also $Z \in \mathcal{A}$. Otherwise, for some $Z' \in \mathcal{A}$, $Z \cap Z' = \varnothing$. Since the closed sets Z and

Z' are c-completely separated, there is $f \in C_c^*(X)$ (in fact $0 \le f \le 1$) such that $f(Z) = 1$ and $f(Z') = 0$. Notice that $Z' \subseteq Z(f) \subseteq E_\epsilon^c(f)$, for all ϵ, and; $E_\epsilon^c(f) \in \mathcal{A}$, since $Z' \in \mathcal{A}$. So $E_\epsilon^c(f) \in E_c(E_c^{-1}(\mathcal{A}))$. Now, if ϵ is taken less than 1, then $Z \cap E_\epsilon^c(f) = \varnothing$ which contradicts with our assumption of Z. So $Z \in \mathcal{A}$ and the proof is complete. $\qquad\square$

The following proposition shows that, as $Z_c^{-1}(\mathcal{A})$ is a maximal ideal in $C_c(X)$, $E_c^{-1}(\mathcal{A})$ is also a maximal ideal in $C_c^*(X)$, where $\mathcal{A}$ is a z_c-ultrafilter on X.

Proposition 2.13. *Let $\mathcal{A}$ be a z_c-ultrafilter on X. Then:*

   (i) *$E_c^{-1}(\mathcal{A})$ is a maximal ideal.*
   (ii) *$E_c^{-1}(\mathcal{A})$ is an e_c-ideal.*
   (iii) *$E_c^{-1}(\mathcal{A}) = E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$.*

Proof. (i). Let M be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{A})$. Hence $E_c(E_c^{-1}(\mathcal{A})) \subseteq E_c(M)$. Since every element of $E_c(M)$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.12 gives $E_c(M) \subseteq \mathcal{A}$. So $M = E_c^{-1}(E_c(M)) \subseteq E_c^{-1}(\mathcal{A})$ and hence $M = E_c^{-1}(\mathcal{A})$. (ii). It follows by (i). (iii). Since the maximal ideal $E_c^{-1}(\mathcal{A})$ is contained in the proper ideal $E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$, the result now holds. $\qquad\square$

An e_c-ultrafilter on X is meant a maximal e_c-filter, i.e., one not contained in any other e_c-filter. As usual, every e_c-filter $\mathcal{F}$ is contained in an e_c-ultrafilter. This is obtained by considering the collection of all e_c-filters containing $\mathcal{F}$ and the use of the Zorn's lemma, where the partially ordered relation on $\mathcal{F}$ is inclusion.

Proposition 2.14. *Let M be an ideal in $C_c^*(X)$ and $\mathcal{F}$ a z_c-filter on X. Then:*

   (i) *If M is a maximal ideal then $E_c(M)$ is an e_c-ultrafilter.*
   (ii) *If $\mathcal{F}$ is an e_c-ultrafilter then $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*
   (iii) *If M is an e_c-ideal, then M is maximal if and only if $E_c(M)$ is an e_c-ultrafilter.*
   (iv) *If $\mathcal{F}$ is an e_c-filter, then $\mathcal{F}$ is e_c-ultrafilter if and only if $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*

Proof. (i). Note that $M = E_c^{-1}(E_c(M))$. Let $\mathcal{F}'$ be an e_c-ultrafilter containing $E_c(M)$, then $M \subseteq E_c^{-1}(\mathcal{F}')$ and hence $M = E_c^{-1}(\mathcal{F}')$. Therefore $E_c(M) = E_c(E_c^{-1}(\mathcal{F}')) = \mathcal{F}'$, which yields the result. (ii). Let M be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{F})$. Then $\mathcal{F} \subseteq E_c(M)$. Hence $\mathcal{F} = E_c(M)$ and so $E_c^{-1}(\mathcal{F}) = M$. The proofs of (iii) and (iv) are similarly done and further details are omitted. $\qquad\square$

Corollary 2.15. *There is a one-to-one correspondence ψ between the sets of maximal ideals of $C_c^*(X)$ and e_c-ultrafilters on X, defined by $\psi(M) = E_c(M)$.*

Proposition 2.16. *Let $\mathcal{A}$ be a z_c-ultrafilter. Then it is the unique z_c-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$, and also $E_c(E_c^{-1}(\mathcal{A}))$ is the unique e_c-ultrafilter contained in $\mathcal{A}$. Hence every e_c-ultrafilter is contained in unique z_c-ultrafilter.*

Proof. Let $\mathcal{B}$ be a z_c-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$ and $Z \in \mathcal{B}$. Since Z meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.12 gives $\mathcal{B} \subseteq \mathcal{A}$ and hence $\mathcal{B} = \mathcal{A}$. So the first part of the proposition holds. Now, let $\mathcal{K}$ be an e_c-ultrafilter contained in $\mathcal{A}$. Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{K})) \subseteq E_c(E_c^{-1}(\mathcal{A}))$. Since the latter set is an e_c-filter, the inclusion cannot be proper, i.e., $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$. Hence the result is obtained. $\qquad\square$

Corollary 2.17. *The z_c-ultrafilters are in one-to-one correspondence with the e_c-ultrafilters.*

Proof. Consider the mapping ψ from the set of z_c-ultrafilters into the set of e_c-ultrafilters defined by $\psi(\mathcal{A}) = E_c(E_c^{-1}(\mathcal{A}))$. If $\psi(\mathcal{A}) = \psi(\mathcal{B})$, then we have that $E_c(E_c^{-1}(\mathcal{A})) = E_c(E_c^{-1}(\mathcal{B}))$ and it is contained in both $\mathcal{A}$ and $\mathcal{B}$. So each element of $\mathcal{B}$ meets each element of $E_c(E_c^{-1}(\mathcal{A}))$. Now, Theorem 2.12 gives $\mathcal{B} \subseteq \mathcal{A}$. Similarly, $\mathcal{A} \subseteq \mathcal{B}$. Therefore ψ is one-one. Let $\mathcal{K}$ be an e_c-ultrafilter and $\mathcal{A}$ the unique z_c-ultrafilter containing it (Proposition 2.16). Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$ and hence $\psi(\mathcal{A}) = \mathcal{K}$. Therefore ψ is onto. $\qquad\square$

Our next two theorems are applications that are based on the concepts of e_c-filters and e_c-ideals. In the first result (Theorem 2.18) we show that the maximal ideals of $C_c(X)$ are in one-to-one correspondence with those ones of $C_c^*(X)$ and the second result (Theorem 2.20) involves characterization of maximal ideals of $C_c^*(X)$.

Theorem 2.18. *Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the set of maximal ideals of $C_c(X)$ (resp. $C_c^*(X)$). Then $\mathcal{M}$ and $\mathcal{M}^*$ have the same cardinality.*

Proof. If $M \in \mathcal{M}$ then $Z_c[M]$ is a z_c-ultrafilter and hence $E_c^{-1}(Z_c[M]) \in \mathcal{M}^*$, see Propositions 1.4 and 2.13. Define

$$\varphi : \mathcal{M} \to \mathcal{M}^* \text{ which } M \mapsto E_c^{-1}(Z_c[M]).$$

If $\varphi(M) = \varphi(M')$ then $E_c(E_c^{-1}(Z_c[M])) = E_c(E_c^{-1}(Z_c[M']))$ and it is contained in both $Z_c[M]$ and $Z_c[M']$. Since each element of $Z_c[M']$ meets each element of $E_c(E_c^{-1}(Z_c[M]))$, Theorem 2.12 yields $Z_c[M'] \subseteq Z_c[M]$. Similarly, $Z_c[M] \subseteq Z_c[M']$. Therefore $M = M'$. This verifies φ is one-one. To show that φ is onto, suppose that $M^* \in \mathcal{M}^*$. Hence $E_c(M^*)$ is an e_c-ultrafilter (Proposition 2.14). Now, let $\mathcal{A}$ be the unique z_c-ultrafilter containing $E_c(M^*)$ (Proposition 2.16), then $Z_c^{-1}[\mathcal{A}]$ is a maximal ideal in $C_c(X)$ (Proposition 1.4) and $M^* = E_c^{-1}(\mathcal{A})$. Recall that if $\mathcal{F}$ is a z_c-filter, then we always have $Z_c[Z_c^{-1}[\mathcal{F}]] = \mathcal{F}$ and $E_c(E_c^{-1}(\mathcal{F})) \subseteq \mathcal{F}$, but the equality occurs if $\mathcal{F}$ is an e_c-filter. Now, if we let $M = Z_c^{-1}[\mathcal{A}]$ then $\varphi(M) = E_c^{-1}(Z_c[M]) = E_c^{-1}(\mathcal{A}) = M^*$. Hence φ is onto, which it completes the proof. $\qquad\square$

Remark 2.19. Combining Corollaries 1.5, 2.15 and 2.17 gives another proof of the above theorem.

Theorem 2.20. *Let M be an ideal in $C_c^*(X)$. Then M is maximal if and only if whenever $f \in C_c^*(X)$ and each $E_\epsilon^c(f)$ meets every element of $E_c(M)$, then $f \in M$.*

Proof. Necessity: Suppose that $f \notin M$. So $(M, f) = C_c^*(X)$. Hence $h + fg = 1$, for some $h \in M$ and $g \in C_c^*(X)$. Let u be an upper bound for g and $0 < \epsilon < 1$. Then

$$\varnothing = E_\epsilon^c(1) = E_\epsilon^c(h + fg) \supseteq E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2}}^c(fg) \supseteq E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2u}}^c(f),$$

which contradicts with our assumption, since $E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2u}}^c(f) \neq \varnothing$. So we are done.

Sufficiency: Let M' be a maximal ideal of $C_c^*(X)$ containing M and $f \in M'$. Then $E_c(M) \subseteq E_c(M')$ and $E_\epsilon^c(f) \in E_c(M')$, for all ϵ. Since $E_c(M')$ is an e_c-filter, $E_\epsilon^c(f)$ meets every element of $E_c(M')$. Hence it also meets each element of $E_c(M)$. Now, by hypothesis $f \in M$. Therefore $M = M'$, which gives the result. $\square$

3. Uniform norm topology on $C_c^*(X)$ and related closed ideals

Consider the supremum-norm on $C^*(X)$, i.e., $\|f\| = \sup_{x \in X} |f(x)|$, where $f \in C^*(X)$. So its restriction on $C_c^*(X)$ is also the supremum-norm. This defines a metric d as usual, $d(f, g) = \|f - g\|$. The resulting metric topology is called the uniform norm topology on $C_c^*(X)$. Convergence in this topology is uniform convergence of the functions. A base for the neighborhood system at g consists of all sets of the form

$$\{f : \|f - g\| \leq \epsilon\} \qquad (\epsilon > 0).$$

Equivalently, a base at g is given by all sets

$$\{f : |f(x) - g(x)| \leq u(x) \text{ for every } x \in X\},$$

where u is a positive unit of $C_c^*(X)$.

If I is an ideal in $C_c^*(X)$ then its closure in $C_c^*(X)$ is denoted by cl$I$.

Proposition 3.1. *Let I be an ideal in $C_c^*(X)$. Then:*

(i) *cl$I$ is ideal.*
(ii) *If I is a proper ideal then cl$I$ is also a proper ideal.*
(iii) *If I is an e_c-ideal then it is a closed ideal.*

Proof. (i). Let $f, g \in$ cl$I$, $h \in C_c^*(X)$ and let u be an upper bound for h and $\epsilon > 0$ is fixed. Then for some $f' \in N_{\frac{\epsilon}{2}}(f) \cap I$ and $g' \in N_{\frac{\epsilon}{2}}(g) \cap I$ we have $f' + g' \in N_\epsilon(f + g) \cap I$. Moreover, there exists $f_1 \in N_{\frac{\epsilon}{u}}(f) \cap I$ and hence $f_1 h \in N_\epsilon(fh) \cap I$. So cl$I$ contains $f + g$ and fh. Hence cl$I$ is ideal. (ii). If cl$I$ is not a proper ideal then $1 \in$ cl$I$ and hence $N\epsilon(1) \cap I$ contains a unit element of $C_c^*(X)$ such as f, since $1 - \epsilon < f < 1 + \epsilon$ gives f is bounded away from zero (of course, when $0 < \epsilon < 1$). But this is impossible since $f \in I$. Thus cl$I$ is a proper ideal.

(iii). Let $g \in$ cl$I$ and $\epsilon > 0$ arbitrary. Then for some $f \in N_{\frac{\epsilon}{2}}(g) \cap I$ and all $x \in E_{\frac{\epsilon}{2}}^c(f)$, we have

$$|g(x)| = |g(x) - f(x) + f(x)| \leq |g(x) - f(x)| + |f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $E_{\frac{\epsilon}{2}}^c(f) \subseteq E_\epsilon^c(g)$. Since the z_c-filter $E_c(I)$ contains $E_{\frac{\epsilon}{2}}^c(f)$, it also contains $E_\epsilon^c(g)$, for all ϵ. So $g \in E_c^{-1}(E_c(I)) = I$ and therefore $\mathrm{cl}I \subseteq I$. This proves that I is closed and hence the proof is complete. $\qquad\square$

Immediately, we find there is no proper dense ideal in $C_c^*(X)$, and further maximal ideals of $C_c^*(X)$ and hence every intersection of them are closed, see Corollary 2.8 and (iii) of the above proposition.

We recall that [6, 1D(1)] plays a useful role in the context of $C(X)$. The following is the counterpart for $C_c(X)$.

Proposition 3.2. *If $f, g \in C_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in C_c(X)$.*

In the remainder of this section, the zero-dimensional topological space X will be assumed to be compact. Hence it is c-pseudo-compact, i.e, $C_c(X) = C_c^*(X)$.

Lemma 3.3. *Let X be a compact space, I an ideal in $C_c(X)$, $f \in C_c(X)$ and $Z(f)$ a neighborhood of $\cap Z_c[I]$. Then $f \in I$.*

Proof. First, we recall that X is compact if and only if the intersection of members of any collection consisting of nonempty closed subsets of X with the finite intersection property (i.e., the intersection of each of a finite number of them is nonempty) is nonempty. The lemma is obvious when $I = C_c(X)$. Now, if I is a proper ideal in $C_c(X)$ then $Z_c[I]$ satisfies the finite intersection property and hence $\cap Z_c[I] \neq \varnothing$. By assumption $\cap Z_c[I] \subseteq \mathrm{int}Z(f)$. Hence $X \smallsetminus \mathrm{int}Z(f) \subseteq \bigcup_{g \in I} \mathrm{coz}(g)$ and so $X = \bigcup_{g \in I} \mathrm{coz}(g) \cup \mathrm{int}Z(f)$. By compactness of X, there are a finite number of elements of I, say $g_1, g_2, \ldots, g_n$, such that

$$X = \bigcup_{i=1}^n \mathrm{coz}(g_i) \cup \mathrm{int}Z(f).$$

Now, if we let $g = \sum_{i=1}^n g_i^2$ then $g \in I$ and $\varnothing \neq Z(g) = \bigcap_{i=1}^n Z(g_i) \subseteq \mathrm{int}Z(f)$. In view of Proposition 3.2, f is a multiple of g and hence it is contained in I. So the proof is complete. $\qquad\square$

Proposition 3.4. *If $g \in C_c(X)$ and $\epsilon > 0$ is fixed, then there exists $f \in C_c(X)$ such that $\|g - f\| \leq \epsilon$ and $Z(f)$ is a neighborhood of $Z(g)$.*

Proof. The trivial solution is $f = g$, of course when $Z(g)$ is open. In general, it suffices to define

$$f(x) = \begin{cases} g(x) - \epsilon, & \text{if } x \in g^{-1}([\epsilon, +\infty)) \\ 0, & \text{if } x \in E_\epsilon^c(g) \\ g(x) + \epsilon, & \text{if } x \in g^{-1}((-\infty, -\epsilon]). \end{cases}$$

We note that X is the union of three closed sets $g^{-1}([\epsilon, +\infty))$, $E_\epsilon^c(g)$ and $g^{-1}((-\infty, -\epsilon])$ and further f is continuous on each of them. Therefore f is continuous on X, i.e., $f \in C(X)$. Notice that the definition of f makes the

cardinality of the range of f the same cardinality of the range of g. Hence this leads us $f \in C_c(X)$. Moreover, $\|g - f\| \leq \epsilon$. Evidently, $Z(g) \subseteq g^{-1}((-\epsilon, \epsilon)) \subseteq \text{int} Z(f)$ which yields $Z(f)$ is a neighborhood of $Z(g)$. $\qquad \square$

Theorem 3.5. *Let I be a proper ideal in $C_c(X)$, $\overline{I} = \cap \{M_p^c : M_p^c \supseteq I\}$ and $J = \{g \in C_c(X) : Z(g) \supseteq \cap Z_c[I]\}$. Then:*

(i) $\overline{I} = J$.

(ii) $\cap Z_c[I] = \cap Z_c[\overline{I}]$.

Proof. (i). Let $g \in J$ and M_p^c be a fixed maximal ideal of $C_c(X)$ containing I. Then $Z(g) \supseteq \cap Z_c[I] \supseteq \cap Z_c[M_p^c] = \{p\}$. So $g(p) = 0$ and hence $g \in M_p^c$. Therefore $g \in \overline{I}$. For the reverse inclusion, we show that if $g \notin J$ then $g \notin \overline{I}$. If $g \notin J$ then there exists $x \in \cap Z_c[I] \setminus Z(g)$. So $I \subseteq M_x^c$ but $g \notin M_x^c$. This means that $g \notin \overline{I}$. The proof of (i) is now complete.

(ii). By (i), we have $\cap Z_c[\overline{I}] = \cap Z_c[J] \supseteq \cap Z_c[I]$. On the other hand, $I \subseteq \overline{I}$ implies $Z_c[I] \subseteq Z_c[\overline{I}]$ and therefore $\cap Z_c[I] \supseteq \cap Z_c[\overline{I}]$. So it gives the result. $\qquad \square$

Corollary 3.6. *Let I be a proper ideal in $C_c(X)$ and $\overline{I}$ as defined in Theorem 3.5. Then $\overline{I} = \text{cl} I$.*

Proof. Since maximal ideals are closed, $\bigcap_{I \subseteq M} M$ is also closed, where M is a maximal ideal in $C_c(X)$. Therefore $\text{cl} I \subseteq \bigcap_{I \subseteq M} M \subseteq \overline{I}$. Let $g \in \overline{I}$ and $N_\epsilon(g)$ is a neighborhood of g. By Proposition 3.4, there is f such that $Z(f)$ is a neighborhood of $Z(g)$ and $\|g - f\| \leq \epsilon$. Hence, by Theorem 3.5, $\cap Z_c[I] \subseteq Z(g) \subseteq \text{int} Z(f)$ and therefore Lemma 3.3 implies $f \in I$. Now, since $f \in N_\epsilon(g) \cap I$, it gives $g \in \text{cl} I$. So $\overline{I} \subseteq \text{cl} I$ and we are done. $\qquad \square$

We conclude the article with the following results for the proper ideals of $C_c(X)$. Corollary 3.7 is a consequence of Corollary 2.8, Proposition 3.1 (iii) and Corollary 3.6; by the same results, plus Corollary 3.7 we obtain Corollary 3.8; finally Corollary 3.9 is the combination of Corollaries 3.7 and 3.8.

Corollary 3.7. *An ideal I of $C_c(X)$ is closed in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.*

Corollary 3.8. *An ideal I of $C_c(X)$ is an e_c-ideal if and only if it is closed in $C_c(X)$.*

Corollary 3.9. *An ideal I of $C_c(X)$ is an e_c-ideal if and only if it is an intersection of maximal ideals of $C_c(X)$.*

References

[1] F. Azarpanah, Intersection of essential ideals in $C(X)$, Proc. Amer. Math. Soc. 125 (1997), 2149–2154.

[2] R. Engelking, General Topology, Heldermann Verlag Berlin, 1989.

[3] A. A. Estaji, A. Karimi Feizabadi and M. Abedi, Zero-sets in point-free topology and strongly z-ideals, Bull. Iranian Math. Soc. 41, no. 5 (2015), 1071–1084.

[4] N. J. Fine, L. Gillman and J. Lambek, Rings of quotients of rings of functions, Lecture Notes Series Mc-Gill University Press, Montreal, 1966.

[5] M. Ghadermazi, O. A. S. Karamzadeh and M. Namdari, On functionally countable subalgebra of $C(X)$, Rend. Sem. Mat. Univ. Padova 129 (2013), 47–69.

[6] L. Gillman and M. Jerison, Rings of continuous functions, Springer-Verlag, 1976.

[7] M. Henriksen, R. Raphael and R. G. Woods, SP-scattered spaces; a new generalization of scattered spaces, Comment. Math. Univ. Carolin. 48, no. 3 (2007), 487–505.

[8] O. A. S. Karamzadeh, M. Namdari and S. Soltanpour, On the locally functionally countable subalgebra of $C(X)$, Appl. Gen. Topol. 16, no. 2 (2015), 183–207.

[9] O. A. S. Karamzadeh and M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, Proc. Amer. Math. Soc. 93 (1985), 179–184.

[10] M. R. Koushesh, The Banach algebra of continuous bounded functions with separable support, Studia Mathematica 210, no. 3 (2012), 227-237.

[11] R. Levy and M. D. Rice, Normal P-spaces and the G_δ-topology, Colloq. Math. 47 (1981), 227–240.

[12] M. A. Mulero, Algebraic properties of rings of continuous functions, Fund. Math. 149 (1996), 55–66.

[13] M. Namdari and A. Veisi, Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image, Inter. Math. Forum 7 (2012), 561–571.

[14] D. Rudd, On two sum theorems for ideals of $C(X)$, Michigan Math. J. 17 (1970), 139–141.

[15] W. Rudin, Continuous functions on compact spaces without perfect subsets, Proc. Amer. Math. Soc. 8 (1957), 39–42.

[16] A. Veisi, The subalgebras of the functionally countable subalgebra of $C(X)$, Far East J. Math. Sci. (FJMS) 101, no. 10 (2017), 2285–2297.

[17] A. Veisi, Invariant norms on the functionally countable subalgebra of $C(X)$ consisting of bounded functions with countable image, JP Journal of Geometry and Topology 21, no. 3 (2018), 167–179.

[18] S. Willard, General Topology, Addison-Wesley, 1970.