Applied General Topology

Appl. Gen. Topol. 21, no. 1 (2020), 1-15
doi:10.4995/agt.2020.11488

Topological characterizations of amenability and congeniality of bases

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Communicated by J. Galindo

Abstract

We provide topological interpretations of the recently introduced notions of amenability and congeniality of bases of infinite dimensional algebras. In order not to restrict our attention only to the countable dimension case, the uniformity of the topologies involved is analyzed and therefore the pertinent ideas about uniform topological spaces are surveyed.

A basis B over an infinite dimensional F-algebra A is called amenable if F^B, the direct product indexed by B of copies of the field F, can be made into an A-module in a natural way. (Mutual) congeniality is a relation that serves to identify cases when different amenable bases yield isomorphic A-modules.

(Not necessarily mutual) congeniality between amenable bases yields an epimorphism of the modules they induce. We prove that this epimorphism is one-to-one only if the congeniality is mutual, thus establishing a precise distinction between the two notions.

2010 MSC: *16K40; 16D99; 15B99.*

Keywords: *uniform topologies; linear vector spaces; amenable bases; congeniality of bases; Schauder bases; infinite-dimensional modules and algebras.*

1. Introduction and Preliminaries

The notions of amenability and congeniality for bases of an infinite dimensional algebra were recently introduced in [1]. Since then, several papers have

Received 08 March 2019 – Accepted 28 November 2019

appeared exploring questions that arise naturally in that context (e.g. [2], [3], [5].) The purpose of our paper is to give topological insights on the naturality of those notions and to show some applications of these new perspectives. In addition, as is the case also in [5], we extend the notions studied in [1] to bases of any infinite dimensional module over an algebra of arbitrary dimension. We start with a brief summary of background definitions and results, which are straightforward adaptations of those in the literature. Unless otherwise stated, F denotes a field, $\mathcal{A}$ denotes an F-algebra and $\mathcal{T}$ denotes a (left) $\mathcal{A}$-module (therefore, itself an F-vector space.)

Given two sets I and J, an $I \times J$ matrix over the field F is a function $f : I \times J \to F$. As usual, I indexes rows of f and J indexes its columns. In this sense, the i-th row of f is $f|_{\{i\}\times J}$ and the j-th column of f is $f|_{I\times\{j\}}$.

We say that a matrix f is column finite if $|\{i \in I \mid f(i,j) \neq 0\}| < \infty$ for all $j \in J$. Similarly, we say that f is row finite if $|\{j \in J \mid f(i,j) \neq 0\}| < \infty$ for all $i \in I$. If we have two matrices, f and g, such that $f : I \times J \to F$ and $g : J \times K \to F$ then, when possible, we define the product fg of f and g, to be the matrix

$$fg(i,k) = \sum_{j\in J} f(i,j)g(j,k).$$

As infinite sums of non-zero elements are not defined, the product of two matrices is also not necessarily defined. When the product does exist, the result is a matrix with domain $I \times K$. Clearly, if either f is row finite or g is column finite then the product fg exists.

We follow the usual definitions of algebras over fields and of unitary modules over rings with unity. This paper contains results from the doctoral dissertation [7].

1.1. **Amenability and Congeniality.** The F-vector space $\mathcal{T}$ is isomorphic to $F^{(\mathcal{B})}$, the direct sum of copies of F indexed by $\mathcal{B}$. Consequently, $F^{(\mathcal{B})}$ inherits an $\mathcal{A}$-module structure from $\mathcal{T}$. A second vector space, $F^{\mathcal{B}}$, the collection of all infinite linear combinations of elements of $\mathcal{B}$ with coefficients in F, clearly contains $F^{(\mathcal{B})} \cong \mathcal{T}$ as a proper subspace. A natural question is for what bases does $F^{\mathcal{B}}$ have a (left) module structure over $\mathcal{A}$ that naturally extends that of $\mathcal{T}$. For the case $\mathcal{T} = \mathcal{A}$, the notion of an amenable basis was introduced in [1] to answer that question. A basis $\mathcal{B}$ of $\mathcal{T}$ is *amenable* if for every $r \in \mathcal{A}$, $c \in \mathcal{B}$ there exist only finitely many $d \in \mathcal{B}$ such that $(rd)_c \neq 0$. In other words, the set ${}_r\mathcal{B}_c = \{d \in \mathcal{B} \mid (rd)_c \neq 0\}$ is finite.

It is easy to see that $\mathcal{B}$ is amenable if for all $r \in \mathcal{A}$, the column-finite matrix $[\ell_r]_{\mathcal{B}}$, representing the F-linear map $\ell_r : \mathcal{T} \to \mathcal{T}$ *left multiplication by* r given by $\ell_r(a) = ra$, is also row-finite. Basically, the row-finiteness requirement allows us to multiply a vector v with infinite support on the left by $[\ell_r]_{\mathcal{B}}$; each entry of the product is the inner product of a vector with finite support and v, a finite sum. In this paper we aim to explain in what sense this module structure is *naturally induced by the knowledge of the products of elements from the basis* $\mathcal{B}$, as suggested in [1].

The expression $F^{\mathcal{B}} = {}^{\mathcal{B}}\mathcal{T}$ serves two purposes: it indicates that $\mathcal{B}$ is amenable and gives a name to the $\mathcal{A}$-module structure on $F^{\mathcal{B}}$. Given an infinite dimensional module $\mathcal{T}$ over an algebra $\mathcal{A}$ and an amenable basis $\mathcal{B}$ for $\mathcal{T}$, the resulting module ${}^{\mathcal{B}}\mathcal{T}$ is said to be a basic module (or the natural module extension of $\mathcal{T}$ by $\mathcal{B}$.)

The proof of Theorem 2.6 of [1] may easily be adapted to get the following result, which guarantees that amenable bases exist when the module considered is countable dimensional over an at most countable dimensional algebra; it is not know in general whether uncountable dimensional algebras must have amenable bases.

Theorem 1.1. *If* $\dim \mathcal{A}$ *is either finite or countable and* $\dim \mathcal{T}$ *is countable then* $\mathcal{T}$ *has an amenable basis.*

It is easy to see from [1] that some bases, but not necessarily all, are amenable. In fact, the following theorem is a direct consequence of a theorem in [5], which shows that, not only does such a module have a non amenable basis under the hypothesis, but it actually has a contrarian basis (as defined in [5].)

Theorem 1.2. *If a module* $\mathcal{T}$ *has an amenable basis* $\mathcal{B}$ *with an element* $b \in \mathcal{B}$ *that is not an eigenvector for any left multiplication map* l_r *with* $r \in \mathcal{A} \setminus F$, *then* $\mathcal{T}$ *also has a non-amenable basis.*

Following [1], given two bases $\mathcal{B}$ and $\mathcal{C}$ for a module $\mathcal{T}$, we say that $\mathcal{B}$ is congenial to $\mathcal{C}$ if $[I]^{\mathcal{C}}_{\mathcal{B}}$, the matrix representation of the identity map $I : \mathcal{T} \to \mathcal{T}$ with respect to the bases cB and $\mathcal{C}$, is row-finite. In general, this relation is not symmetric; when $\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is congenial to $\mathcal{B}$ then we say that $\mathcal{B}$ and $\mathcal{C}$ are mutually congenial. If $\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is not congenial to $\mathcal{B}$, we say that $\mathcal{B}$ is properly congenial to $\mathcal{C}$ and the matrix $[I]^{\mathcal{C}}_{\mathcal{B}}$ is a proper congeniality matrix. In other words, a proper congeniality matrix is a row and column finite matrix whose inverse is column but not row finite.

It was proven in [1] that, given two mutually congenial bases $\mathcal{B}$ and $\mathcal{C}$, one of them is amenable if and only if the other one is and they induce isomorphic natural module extensions. On the other hand, if $\mathcal{B}$ is properly congenial to $\mathcal{C}$, the possible amenability of either basis is largely independent from that of the other one. It is know, however, that if they are both amenable then the map from ${}^{\mathcal{B}}\mathcal{T}$ into ${}^{\mathcal{C}}\mathcal{T}$ given by left multiplication by the proper congeniality matrix $[I]^{\mathcal{C}}_{\mathcal{B}}$ is a module homomorphism and, perhaps surprisingly, it is onto. This is one of the main results from [1]. The question about the significance of the potential injectivity of multiplication by $[I]^{\mathcal{C}}_{\mathcal{B}}$ was left open.

The final result in this paper, Theorem 3.0.2, shows that a proper congeniality matrix $[I]^{\mathcal{C}}_{\mathcal{B}}$ never induces a one-to-one left multiplication map and consequently draws a line separating mutual and proper congeniality. Our proof of this fact relies on the topological machinery built throughout the paper.

Proper congeniality has turned out to be one of the most interesting notions in this line of work; it has inspired the notions of simple and projective bases and these types of bases have, in turned, fueled many ongoing projects (see, for example, [1], [2] and [3].)

The unit vectors in $F^{(\mathcal{B})}$ are a basis $\mathcal{B}$ for $F^{(\mathcal{B})}$ which have the interesting property that they can be, in some sense, a surrogate basis for $F^{\mathcal{B}}$. Indeed, any element in $F^{\mathcal{B}}$ can be viewed as an *infinite linear combination* of the elements of $\mathcal{B}$ and the only way to represent $0 \in F^{\mathcal{B}}$ is using all zero coefficients. So, the elements of $\mathcal{B}$ satisfy a stronger sort of linear independence. This is reminiscent of the classical *Schauder bases* where a collection of elements in a topological vector space play the role of a basis because their infinite linear combinations converge to the element that they represent. Theorem 1.2.6 basically states that for any basis $\mathcal{B}$ there indeed exists a topology on $F^{\mathcal{B}}$ that makes $\mathcal{B} \subset F^{\mathcal{B}}$ into a Schauder basis.

1.2. **Nets and convergence.** For convenience, we will briefly introduce some topological concepts here. However, most necessary topology notions, such as subspace topologies and discrete topologies (when all subsets are open), may be found in any standard reference (c.g. [8]).

The product topology on the cartesian product of a family of topological spaces is the smallest topology making all projection maps continuous. In other words:

Definition 1.3 (Product topologies and projection maps)**.**

(1) For a collection of topological spaces $\{(X_i, \tau_i)\}_{i \in I}$, $\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i\}$ is a topological space under the topology generated by the base $B = \{U \subset \prod_{i \in I} X_i \mid U = \prod_{i \in I} U_i$ where each $U_i \in \tau_i$ and $U_i \neq X_i$ only finitely often$\}$.

(2) The map $\pi_j : \prod_{i \in I} X_i \to X_j$, given by $\pi_j(f) = f(j) \in X_j$, is called the j-th projection map.

Not all topologies may be described in terms of the behavior of sequences, a more powerful notion, called *nets*, is needed. Likewise, although the sequence of images under a continuous map of the elements of a convergent sequence converge to the image of the limit of that sequence, this property alone does not in general characterize continuous functions. In order to obtain a characterization of continuous functions, nets are once again needed. We will start by describing the prerequisite term of *directed sets*.

Definition 1.4 (Directed sets)**.** A *directed set* is a non-empty set A with a binary relation (a direction on A) $\leq$ that satisfies the following:

D.1 $a \leq a$ for all $a \in A$

D.2 if $a \leq b$ and $b \leq c$ then $a \leq c$

D.3 if $a, b \in A$ then there exists $c \in A$ such that $a \leq c$ and $b \leq c$.

Example 1.5. Let X be a topological space and let B_x be a local base (also called a neighborhood base) at a point $x \in X$. For $U, V \in B_x$ define $U \leq V$ if and only if $V \subset U$, then $\leq$ is a direction on B_x.

Definition 1.6 (Nets and their convergence)**.**

(1) A *net* is a function $f : A \to X$ where A is a directed set and X is a topological space. We write $\{x_\alpha\}_{\alpha \in A}$ to denote the net given by $f(\alpha) = x_\alpha$.
(2) For $U \subset X$ we say that a net $\{x_\alpha\}_{\alpha \in A}$ is residually (eventually) in U if there exists $\beta \in A$ such that $\alpha \geq \beta \implies x_\alpha \in U$.
(3) We say that $\{x_\alpha\}_{\alpha \in A}$ converges to $x \in X$ if and only if for all open neighborhoods U of x, $\{x_\alpha\}_{\alpha \in A}$ is eventually in U.

The significance of nets is highlighted by the following well-known result.

Proposition 1.7. *For arbitrary topologies,*

(1) a subset $U \subset X$ is open if and only if when a net $\{x_\alpha\}_{\alpha \in A}$ converges to some $x \in U$ then $\{x_\alpha\}_{\alpha \in A}$ is residually in U, and
(2) a subset $F \subset X$ is closed if and only if every convergent net $\{x_\alpha\}_{\alpha \in A} \subset F$ converges to some $x \in F$.
(3) a function $f : X \to Y$ is continuous if and only if for every net $x_\alpha \to x$ we have that $f(x_\alpha) \to f(x)$.

Theorem 1.8. *When F is equipped with the discrete topology and $F^{\mathcal{B}}$ with the corresponding product topology, then $F^{(\mathcal{B})}$ is dense in $F^{\mathcal{B}}$.*

Proof. Let $U(b_1, \dots, b_n | k_1, \dots, k_n) = \{f \in F^{\mathcal{B}} \mid f(b_i) = k_i, 1 \leq i \leq n\}$, note that sets of this form are open in the product topology as $\{k\}$ is open in F for all $k \in F$. Furthermore, the set $\{U(b_1, \dots, b_n \mid g(b_1), \dots, g(b_n)) \mid g \in F^{(\mathcal{B})},\ n \in \mathbb{N},\ b_1, \dots, b_n \in \mathcal{B}\}$ is a basis for the product topology on $F^{\mathcal{B}}$. Clearly $U(b_1, \dots, b_n \mid g(b_1), \dots, g(b_n))$ contains $g|_{\{b_1, \dots, b_n\}} \in F^{(\mathcal{B})}$. Thus, every open set in $F^{\mathcal{B}}$ has non-trivial intersection with $F^{(\mathcal{B})}$. Now, let $f \in F^{\mathcal{B}}$ and let U be open around it. We have seen that U intersects $F^{(\mathcal{B})}$ so that f is a cluster point of $F^{(\mathcal{B})}$ and so, $f \in cl(F^{(\mathcal{B})})$ so that $F^{(\mathcal{B})}$ is dense. □

1.3. **Uniform Topological Spaces.** The condition of Uniform continuity is strictly stronger than continuity. One could say that uniform continuity is the condition of being 'continuous the same way' everywhere. The main property of uniformly continuous functions that we will make use of here is that uniformly continuous functions preserve Cauchy sequences. That is, if $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy and $f : X \to Y$ is uniformly continuous then $\{f(x_n)\}_{n \in \mathbb{N}}$ is Cauchy in Y.

We start by recalling the familiar definition, in the context of metric spaces, of a uniformly continuous function.

Definition 1.9. Let X, Y be metric spaces, whose metrics are d_X, d_Y respectively. We say a function $f : X \to Y$ is uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that when $d_X(x, x') < \delta$ we have that $d_Y(f(x), f(x')) < \epsilon$.

In this paper we deal with potentially uncountable products, which may fail to be metrizable. Thus, we introduce Uniform Spaces, which will allow us to talk about uniformly continuous functions in the absence of a metric. We first develop some notation; the reader may find a more thorough treatment in [8].

Definition 1.10. If X is a set, we denote by Δ the diagonal $\{(x,x) \mid x \in X\}$ in $X \times X$. If $U, V \subset X \times X$, then $U \circ V$ is the set $\{(x,y) \mid$ for some $z \in X, (x,z) \in V$ and $(z,y) \in U\}$. Notice that U and V are just relations on X and $\circ$ is a natural extension of the notion of composition of functions.

The definition we will use comes from the notion that x and y are close together in a metric space if and only if (x,y) is close to the diagonal in $X \times X$.

Definition 1.11. A *diagonal uniformity* on a set X is a collection $\mathcal{D}(X)$, or just $\mathcal{D}$, of subsets of $X \times X$, called *entourages*, which satisfy:

(D.a) $D \in \mathcal{D} \implies \Delta \subset D$
(D.b) $D_1, D_2 \in \mathcal{D} \implies D_1 \cap D_2 \in \mathcal{D}$
(D.c) $D \in \mathcal{D} \implies E \circ E \subset D$ for some $E \in \mathcal{D}$
(D.d) $D \in \mathcal{D} \implies E^{-1} \subset D$ for some $E \in \mathcal{D}$ where $E^{-1} = \{(y,x) \mid (x,y) \in E\}$
(D.e) $D \in \mathcal{D}, D \subset E \implies E \in \mathcal{D}$

Proposition 1.12. *If $\mathcal{D}$ is a diagonal uniformity and $D \in \mathcal{D}$ then $D^{-1} \in \mathcal{D}$.*

Proof. Let $D \in \mathcal{D}$ then, by $(D.d)$ we have that there is some $E \in \mathcal{D}$ such that $E^{-1} \subset D$. Therefore, $E \subset D^{-1}$ so by $(D.e)$ we have that $D^{-1} \in \mathcal{D}$. □

Each uniformity gives rise to a topology. To see this, we define neighborhoods of points $x \in X$ for a uniform space.

Definition 1.13. For $x \in X$ and $D \in \mathcal{D}$ let $D[x] = \{y \in X \mid (x,y) \in D\}$. This is extended to subsets $A \subset X$ as follows: $D[A] = \bigcup_{x \in A} D[x]$.

Definition 1.14. A collection $\mathcal{E} \subset \mathcal{D}$ is a base for the uniformity on $\mathcal{D}$ if for all $D \in \mathcal{D}$ there is some $E \in \mathcal{E}$ such that $E \subset D$. That is, $\mathcal{D}$ can be recovered from $\mathcal{E}$ by applying $(D.e)$.

That these sets, $D[x]$, are neighborhoods is justified by the following theorem which can be found [8].

Theorem 1.15. *For each $x \in X$, the collection $\mathcal{U}_x = \{D[x] \mid D \in \mathcal{D}\}$ forms a neighborhood base at x, making X a topological space.*

The way that X is made a topological space is as follows. We say that O is $\mathcal{D}$-open if for all $x \in O$ there is some $U \in \mathcal{D}$ such that $U[x] \subset O$. That is, O is open if and only if is a neighborhood of all its points. This topology will be called the uniform topology on X generated by $\mathcal{D}$.

Example 1.16. Let $D_\epsilon = \{(x,y) \mid d(x,y) < \epsilon\} \subset X \times X$ where X is a metric space with metric $d : X \times X \to \mathbb{R}$. The set $\mathcal{D} = \{D_\epsilon \mid \epsilon > 0\}$ is a uniformity on X. In fact, the uniform topology generated by $\mathcal{D}$ coincides with the metric topology on X.

Example 1.17. Let X be a set and let $\mathcal{D} = \{D \subset X \times X \mid \Delta \subset D\}$. The uniform topology generated by $\mathcal{D}$ is the discrete topology on X. Indeed, let $A \subset X$ then $\Delta[x] = \{x\} \subset A$ for all $x \in A$. This uniformity is called the discrete uniformity on X.

Now we generalize the notion of a uniformly continuous function to all uniform spaces.

Definition 1.18. Let X and Y be sets with diagonal uniformities $\mathcal{D}$ and $\mathcal{E}$ respectively. A function $f : X \to Y$ is *uniformly continuous* if and only if for each $E \in \mathcal{E}$, there is some $D \in \mathcal{D}$ such that $(x, y) \in D \implies (f(x), f(y)) \in E$.

As we primarily deal with product spaces in this document it will be useful to have a way to define a uniformity on a product of uniform spaces. In particular, we will do so in such a way that the uniform topology on the product coincides with the product of the uniform topologies.

Definition 1.19. If X_α is a set for each $\alpha \in A$ and $X = \prod X_\alpha$, the αth *biprojection* is the map $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ defined by $P_\alpha(x, y) = (\pi_\alpha(x), \pi_\alpha(y))$ where π_α is the αth projection. That is, $\pi_\alpha(x) = x(\alpha) \in X_\alpha$ for each $x \in X$.

Theorem 1.20. *If $\mathcal{D}_\alpha$ is a diagonal uniformity on X_α, for each $\alpha \in A$, then the sets $P_{\alpha_1}^{-1}(D_{\alpha_1}) \cap \cdots \cap P_{\alpha_n}^{-1}(D_{\alpha_n})$, where $D_{\alpha_n} \in \mathcal{D}_{\alpha_n}$ for $i = 1, \dots, n$, form a base for a uniformity $\mathcal{D}$ on $X = \prod X_\alpha$ whose associated topology is the product topology on X.*

1.4. **Complete Uniform Spaces.** Let us deal next with completeness for uniform spaces; a notion which, when dealing with metric spaces, relies on Cauchy sequences. Consequently, we we will lean on the more general concept of Cauchy nets.

Definition 1.21. Let X be a uniform space with diagonal uniformity $\mathcal{D}$ and let $\{x_\lambda\}_{\lambda \in \Lambda} \subset X$ be a net. We say that $\{x_\lambda\}_{\lambda \in \Lambda}$ is a *Cauchy Net* if for all $D \in \mathcal{D}$ there exists $\lambda_D \in \Lambda$ such that $\gamma, \delta > \lambda_D \implies (x_\gamma, x_\delta) \in D$.

Much like the usual case with uniformly continuous functions and Cauchy sequences we have that uniformly continuous functions preserve Cauchy Nets. That is, if $f : X \to Y$ where X, Y are uniform spaces and $\{x_\lambda\} \subset X$ is a Cauchy net then $\{f(x_\lambda)\} \subset Y$ is a Cauchy net.

Definition 1.22. A uniform space X is *complete* if every Cauchy net converges to some element of X.

Theorem 1.23. *If $\{x_\lambda\}_{\lambda \in \Lambda} \subset X$ is a Cauchy net and $f : X \to Y$ is uniformly continuous then $\{f(x_\lambda)\}_{\lambda \in \Lambda}$ is also a Cauchy net.*

Proof. By definition, for each entourage V in X we have that there is some $\lambda_V \in \Lambda$ such that $\lambda, \lambda' > \lambda_V$ implies that $(x_\lambda, x_{\lambda'}) \in V$.
Let V be an entourage in Y. As f is uniformly continuous there is some entourage U in X such that $(x, y) \in U$ implies that $(f(x), f(y)) \in V$. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a Cauchy net. Let $\lambda_{X,U}$ be such that $\lambda, \lambda' \geq \lambda_{X,U} \Rightarrow (x_\lambda, x_{\lambda'}) \in U$. Then, $(f(x_\lambda), f(x_{\lambda'})) \in V$ so that $\{f(x_\lambda)\}_{\lambda \in \Lambda}$ is a Cauchy net. □

Uniformly continuous functions between complete uniform spaces can be continuously extended to the whole space; furthermore, that extension is also uniformly continuous.

Theorem 1.24. *If X, Y are complete uniform spaces and $A \subset X$ is dense then every $f : A \to Y$ that is uniformly continuous has a unique uniformly continuous extension $\hat{f} : X \to Y$.*

The above result, an easy consequence of Theorem 8.3.10 in [4], will help us determine when natural module extensions exist.

1.5. **Topological Vector Spaces.** It turns out that $F^{(\mathcal{B})}$ and $F^{\mathcal{B}}$ are topological vector spaces so we write down some useful properties of these spaces.

Definition 1.25. A *topological $F-$vector space* E is an F-vector space E that is endowed with a topology such that the mappings $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ from $E \times E$ to E and $F \times E$ to E respectively are continuous. ($E \times E$ and $F \times E$ having the product topologies.)

Example 1.26. Let F be a field and endow F with the discrete topology. Then, for non-empty A, $X = \prod_{\alpha \in A} F$ is a topological vector space with the product topology.

Proof. We proceed by showing that addition and scalar multiplication are continuous. Let $+ : X \times X \to X$ be such that $+(x, y) = x + y$. We aim to show that this map is continuous by showing that $\pi_j \circ + : X \times X \to F_j$ is continuous for each $j \in A$. Note, of course, that each F_j is just a copy of F. Note that $\pi_j \circ +(x, y) = \pi_j(x + y) = x(j) + y(j)$. As singletons make up a base for the topology on F we look at the inverse image of a singleton under this map. $(\pi_j \circ +)^{-1}(z) = \cup_{x \in F} \pi_j^{-1}(x) \times \pi_j^{-1}(z - x)$. As $\pi_j^{-1}(x) \times \pi_j^{-1}(z - x)$ is open in $X \times X$ for each $x \in F$ we have that $(\pi_j \circ +)^{-1}(z)$ is the arbitrary union of open sets and so is open itself. Thus $\pi_j \circ +$ is continuous for each $j \in A$ so that $+ : X \times X \to X$ is continuous.

Now we show that $\cdot : F \times X \to X$ defined by $\cdot(\lambda, f) = \lambda f$ is continuous. Let $j \in A$ and note that $\pi_j \circ \cdot(\lambda, f) = \lambda f(j)$. Again, we examine the inverse image of singletons, $(\pi_j \circ \cdot)^{-1}(z) = \cup_{\lambda \in F} \{\lambda\} \times \pi_j^{-1}\left(\frac{z}{\lambda}\right)$ each of which is open in $F \times X$ so that the inverse image is an arbitrary union of open sets. Thus, $\cdot$ is continuous as well. □

As we saw in definition 1.19 we have that the space defined in example 1.26 is indeed a uniform space. With that in mind, we show that, for linear operators, continuity and uniform continuity are the same. To do this, we rely on some more elementary facts.

Proposition 1.27. *Let $\mathcal{U}$ be all the basic open sets around 0 as defined by the product topology. Let $D_U = \{(x, y) \mid x - y \in U\}$ and $\mathcal{D} = \{D_U \mid U \in \mathcal{U}\}$. Then, $\mathcal{D}$ is a base for the product uniformity, $\mathcal{E}$, on $X = \prod_{\alpha \in A} F$. That is $\mathcal{D} \subset \mathcal{E}$ and for every entourage $E \in \mathcal{E}$ there is some entourage $D \in \mathcal{D}$ with $D \subset E$.*

Proof. Let E be a base entourage from $\mathcal{E}$, that is, $E = P_{\alpha_1}^{-1}(E_{\alpha_1}) \cap \cdots \cap P_{\alpha_n}^{-1}(E_{\alpha_n})$, where $E_{\alpha_n} \in \mathcal{E}_{\alpha_n}$ for $i = 1, \ldots, n$ where $\mathcal{E}_\alpha$ is the discrete uniformity on F_α. Take $U = \{x \in X \mid x(\alpha_i) = 0 \text{ for all } i = 1, \ldots, n\}$. Then, $D_U =$

$\{(x, y) \mid x(\alpha_i) = y(\alpha_i) \text{ for } i = 1, \ldots, n\}$ thus $D_U \subset E$ as $(x(\alpha), y(\alpha)) \in E_\alpha$ for all $E_\alpha \in \mathcal{E}_\alpha$. Finally, note that $D_U = P_{\alpha_1}^{-1}(\Delta_{\alpha_1}) \cap \cdots \cap P_{\alpha_n}^{-1}(\Delta_{\alpha_n})$. □

Proposition 1.28. *Let X be as in example 1.26 and let $f : X \to X$. Then, f is uniformly continuous if and only if for all open sets U around 0 there exists an open set V around 0 such that $(x, y) \in D_V \implies (f(x), f(y)) \in D_U$ where $D_U = \{(x, y) \mid x - y \in U\}$.*

Proof. This follows from the definition of uniformly continuous and the fact that sets of the form D_U give a base for the uniformity on X. □

Corollary 1.29. *Let X be as in example 1.26 and let $f : X \to X$. Then, f is uniformly continuous if and only if for all open sets U around 0 there exists an open set V around 0 such that $x - y \in V \implies f(x) - f(y) \in U$.*

Proposition 1.30. *Let X be as in example 1.26 and let $f : X \to X$ be a linear map. Then, f is continuous on all of X if f is continuous at 0.*

Proof. Let $x \in X$ and let U be open around $f(x)$. Let $U' = U - f(x)$ and note that U' is open around 0. By continuity at 0 there exists $V' \subset X$ around 0 such that $f(V') \subset U'$. Let $V = V' + x$. V is open around x and $f(V) = f(V') + f(x) \subset U' + f(x) = U$. Thus, for linear maps, continuity at 0 is sufficient to show continuity in general. □

Theorem 1.31. *Let X be as in example 1.26 and let $f : X \to X$ be a linear map. Then, f is continuous if and only if f is uniformly continuous.*

Proof. That uniform continuity implies continuity is trivial. Let $U \subset X$ be an open neighborhood of 0. Let V be such that $f(V) \subset U$ which is guaranteed by continuity. Now, suppose $x - y \in V$ so that $f(x - y) \in U$. However, by linearity of f, $f(x - y) = f(x) - f(y)$ so that $x - y \in V \implies f(x) - f(y) \in U$ so that f is uniformly continuous. □

While some of the above results hold for topological vector spaces in general, we opted for leaving our discussion self-contained and focused only on the specific case considered here.

2. Topological characterization of amenable bases and the natural module extensions they induce.

It would certainly be nice if, in general, we could define rt for $r \in \mathcal{A}$ and $t \in F^{\mathcal{B}}$ to be a limit as can be done when the power series are viewed as a module over polynomials with the standard basis. However, if $\mathcal{T}$ has a higher dimension than ω this will not work. In particular, elements of $F^{\mathcal{B}}$ with uncountable support can not even be found as a limit of elements of $\mathcal{T}$ when $\mathcal{B}$ is uncountable. That is, the limit of any sequence of finite linear combinations can be at most a countable linear combination, or in the language of our function spaces, a function $f : \mathcal{B} \to F$ with at most countable support.

2.1. **Module Multiples Defined By Approximations.** For $r \in \mathcal{A}$, $\ell_r : \mathcal{T} \to \mathcal{T}$ denotes the map given by $\ell_r(t) = rt$, $\gamma_{\mathcal{B}} : F^{(\mathcal{B})} \to \mathcal{T}$ denotes the bijection such that $\gamma_{\mathcal{B}}(f) = \sum_{b \in \mathcal{B}} f(b)b$, and $f_t = [t]_{\mathcal{B}} = \gamma_{\mathcal{B}}^{-1}(t) \in F^{\mathcal{B}}$ for $t \in \mathcal{T}$ (the coordinates of t with respect to $\mathcal{B}$.) We think of $[t]_{\mathcal{B}}$ as a matrix with one column.

Also, for $r \in \mathcal{A}$, the $[\ell_r]_{\mathcal{B}}$ represent the map ℓ_r with respect to $\mathcal{B}$ in the usual way, namely, the rows and columns are labeled by $\mathcal{B}$ and, for $b \in \mathcal{B}$, the b-th column of $[\ell_r]_{\mathcal{B}}$ is given by the equation $[\ell_r]_{\mathcal{B}}|_{\mathcal{B} \times \{b\}} = rf_b := [rb]_{\mathcal{B}}$. With this notation, one gets, as is customary in the theory of matrix representations of linear transformations, the identity granting that, for $t \in \mathcal{T}$ and $r \in \mathcal{A}$, the product $[\ell_r]_{\mathcal{B}}[t]_{\mathcal{B}}$ is equal to $[rt]_{\mathcal{B}} = \gamma_{\mathcal{B}}^{-1}(rt) = r\gamma_{\mathcal{B}}^{-1}(t) = rf_t$.

In other words, multiplication on the left of $f \in F^{(\mathcal{B})}$ by $[\ell_r]_{\mathcal{B}}$ is a matrix representation of left multiplication by r with respect to the basis $\mathcal{B}$.

Consequently, for $t \in \mathcal{T}$ the product $[\ell_r]_{\mathcal{B}}[t]_{\mathcal{B}}$ is equal to $(\gamma_{\mathcal{B}}^{-1} \circ \ell_r \circ \gamma_{\mathcal{B}})(f_t)$.

We may then think of left multiplication by $[\ell_r]_{\mathcal{B}}$ as a function from $F^{(\mathcal{B})}$ to $F^{(\mathcal{B})}$. As $[\ell_r]_{\mathcal{B}}$ already has a function description we avoid using parenthesis when we mean the matrix product and we will write $[\ell_r]_{\mathcal{B}} \cdot : F^{(\mathcal{B})} \to F^{(\mathcal{B})}$. Clearly, this means that $[\ell_r]_{\mathcal{B}} \cdot = \gamma_{\mathcal{B}}^{-1} \circ \ell_r \circ \gamma_{\mathcal{B}}$

Handling the uncountable dimension situation requires a little finesse beyond the standard approach with sequences. If $f \in F^{\mathcal{B}}$ has uncountable support, there is no sequence in $F^{(\mathcal{B})}$ that converges to f. We must therefore rely on nets. Indeed, recall that a local base at $f \in F^{\mathcal{B}}$, B_f is given by sets of the form $U(b_1, \ldots, b_n \mid f(b_1), \ldots, f(b_n))$ and that B_f is a directed set under the direction in Example 1.5. That is, $U \leq V$ if and only if $V \subseteq U$.

Now we construct a net whose directed set is $(B_f, \leq)$. For $U = U(b_1, \ldots, b_n \mid f(b_1), \ldots, f(b_n)) \in B_f$ let $f_U = f|_{\{b_1, \ldots, b_n\}} \cup \mathbf{0}|_{\mathcal{B} \setminus \{b_1, \ldots, b_n\}}$.

Definition 2.1. The net constructed above will be called the *canonical net* for f.

Claim 2.2. *The canonical net for f, $\{f_U \mid U \in B_f\}$, converges to f.*

Proof. Let $U \in B_f$ be a basic open neighborhood of f. Let $V \in B_f$ be such that $U \leq V$ so that $V \subset U$. Therefore, $U = U(b_1, \ldots, b_n \mid f(b_1), \ldots, f(b_n))$ and $V = U(b_1, \ldots, b_n, b_{n+1}, \ldots, b_{n+k} \mid f(b_1), \ldots, f(b_n), f(b_{n+1}), \ldots, f(b_{n+k}))$. Thus, $f_V|_{\{b_1, \ldots, b_n\}} \equiv f_U|_{\{b_1, \ldots, b_n\}}$ so that $f_V \in U$. That is, the net is eventually in U for any basic neighborhood of f. □

These canonical nets are integral in our description of naturality. Note, the canonical net for f consists of better and better approximations of f. Therefore, if we can define scalar multiplication naturally, we would like $[\ell_r]_{\mathcal{B}} f_U$, $U \in B_f$ to consist of better and better approximations to $[\ell_r]_{\mathcal{B}} f$.

Definition 2.3. Let $r \in \mathcal{A}$ and $f \in F^{\mathcal{B}}$. We say that $[L_r]_{\mathcal{B}} \cdot : F^{\mathcal{B}} \to F^{\mathcal{B}}$ is *the natural extension* of $[\ell_r] \cdot$ if $\{[\ell_r]_{\mathcal{B}} f_U\} \to [L_r]_{\mathcal{B}} f$ for all $f \in F^{\mathcal{B}}$ where $\{f_U\}_{U \in B_f}$ is the canonical net described above.

Here we must justify the use of the word 'the' in our above definition. That is, we must verify that this extension is indeed unique. This is handled by a well-known topological fact. The arbitrary product of Hausdorff spaces is again a Hausdorff space. As limits are unique in Hausdorff spaces we know that $\{[\ell_r]_{\mathcal{B}} f_U\}_{U \in B_f}$ can converge to at most one element in $F^{\mathcal{B}}$, thus, if one natural extension exists, it is the only one.

With it in mind that $[\ell_r]_{\mathcal{B}} f = rf$ for $r \in \mathcal{A}$ and $f \in F^{(\mathcal{B})}$ we see that $F^{(\mathcal{B})}$ is a module and $\{[\ell_r]_{\mathcal{B}} \cdot \mid r \in \mathcal{A}\}$ is a collection of matrices that describe scalar multiplication in $F^{(\mathcal{B})}$. In fact, the definition of $\gamma_{\mathcal{B}}$ shows us that this module coincides with $\mathcal{T}$.

Definition 2.4. We say that ${}_{\mathcal{A}}F^{\mathcal{B}}$ is the *natural extension* of the left $\mathcal{A}$-module structure on $F^{(\mathcal{B})}$ if $\{[L_r]_{\mathcal{B}} \cdot \mid r \in \mathcal{A}\}$ is a set of maps that describe scalar multiplication by r and each $[L_r]_{\mathcal{B}} \cdot$ is the natural extension of $[\ell_r]_{\mathcal{B}} \cdot$.

Note, if any $[\ell_r]_{\mathcal{B}} \cdot$ fails to have a natural extension then ${}_{\mathcal{A}}F^{\mathcal{B}}$ can not be a natural extension of $F^{(\mathcal{B})}$ with the aforementioned module structure. Note, for any $f \in F^{\mathcal{B}}$ the scaled canonical net, $\{rf_U\}_{U \in B_f}$ is a net in $F^{(\mathcal{B})}$. Whether it converges for every f is the determining factor for scaling by r being a valid operation on $F^{\mathcal{B}}$. To help us understand when this happens, we introduce the notion of a uniformity on our topoligical spaces as well as what it means for a net to be Cauchy. Much like in the case with Cauchy sequences in metric spaces, a Cauchy net is one such that if we desire two elements of the net to be close to each other, we need only traverse far enough into the net. The notion of closeness will be handled by the uniformity as we lack a metric in general. Under this new structure we will utilize the notion of a uniformly continuous function. In particular, uniformly continuous functions map Cauchy nets to Cauchy nets. Therefore, if multiplication by r is found to be uniformly continuous, then we shall be able to define rf as the $\lim_{U \in B_f} rf_U$ as this limit will therefore exist. The details are the subject of the following subsection.

2.2. **Uniform Continuity and Amenability.** We will see that if $[\ell_r]_{\mathcal{B}}$ is a row finite matrix, then it is uniformly continuous and, therefore, has a uniformly continuous extension as guaranteed by theorem 1.24.

Theorem 2.5. *If $[\ell_r]_{\mathcal{B}} \cdot : F^{(\mathcal{B})} \to F^{(\mathcal{B})}$ is row finite then it is uniformly continuous.*

Proof. As $[\ell_r]_{\mathcal{B}}$ is row finite we know that for each $b \in \mathcal{B}$ there are only finitely many $b' \in \mathcal{B}$ such that $[\ell_r]_{\mathcal{B}}(b, b') \neq 0$. As it is F-linear, we show that it is continuous at 0 to establish that it is uniformly continuous. Let $\mathcal{U}(b_1, \ldots, b_n \mid 0, \ldots, 0)$ be an open set around 0. That is, $\mathcal{U} = \{f \in F^{\mathcal{B}} \mid f(b_i) = 0, i = 1, \ldots, n\}$. Let $\mathcal{B}_{\mathcal{U}} = \{b \in \mathcal{B} \mid [\ell_r]_{\mathcal{B}}(b_i, b) \neq 0 \text{ for any } i = 1, \ldots, n\}$. Note, $\mathcal{B}_{\mathcal{U}}$ is finite as each row of $[\ell_r]_{\mathcal{B}}$ is finite. Let $\mathcal{V} = \mathcal{U}(\mathcal{B}_{\mathcal{U}} \mid 0) = \{f \in F^{\mathcal{B}} \mid f(b) = 0$ for all $b \in \mathcal{B}_{\mathcal{U}}\}$.

Now suppose that $f_1, f_2 \in F^{(\mathcal{B})}$ such that $f := f_1 - f_2 \in \mathcal{V}$. Then, for $i \in \{1, \ldots, n\}$, $[\ell_r]_{\mathcal{B}} f(b_i) = \sum_{b' \in \mathcal{B}} [\ell_r]_{\mathcal{B}}(b_i, b') f(b')$, yet, $[\ell_r]_{\mathcal{B}}(b_i, b') = 0$ for all

$b' \in \mathcal{B} \setminus \mathcal{B}_{\mathcal{U}}$ so that $[\ell_r]_{\mathcal{B}} f(b_i) = \sum_{b' \in \mathcal{B}_{\mathcal{U}}} [\ell_r]_{\mathcal{B}}(b_i, b') f(b') = 0$ as $f(b') = 0$ for all $b' \in \mathcal{B}_{\mathcal{U}}$ so that $[\ell_r]_{\mathcal{B}} f \in \mathcal{U}$. Thus, $[\ell_r]_{\mathcal{B}} \cdot$ is uniformly continuous. □

Theorem 2.6. *If $[\ell_r]_{\mathcal{B}}$ is row finite then, $[L_r]_{\mathcal{B}} \cdot : F^{\mathcal{B}} \to F^{\mathcal{B}}$ defined by*

$$[L_r]_{\mathcal{B}} f(b) = \sum_{b' \in \mathcal{B}} [\ell_r]_{\mathcal{B}}(b, b') f(b')$$

is the uniformly continuous extension of $[\ell_r]_{\mathcal{B}} \cdot$.

Proof. As $[\ell_r]_{\mathcal{B}}$ is row finite we know that for each $b \in \mathcal{B}$ there are only finitely many $b' \in \mathcal{B}$ such that $[\ell_r]_{\mathcal{B}}(b, b') \neq 0$ so that $[L_r]_{\mathcal{B}} f(b)$ is defined. Therefore, $[L_r]_{\mathcal{B}}$ is indeed a function from $F^{\mathcal{B}}$ to $F^{\mathcal{B}}$. The proof from here is identical to the above proof, except that f_1, f_2 are taken from $F^{\mathcal{B}}$. □

Therefore, we have established that if $\mathcal{B}$ is amenable, then $[\ell_r]_{\mathcal{B}}$ is uniformly continuous, that it has a uniformly continuous extension, and that this extension is the natural extension of $[\ell_r]_{\mathcal{B}}$. Therefore, with the collection $\{[L_r]_{\mathcal{B}} \cdot \mid r \in \mathcal{A}\}$ $F^{\mathcal{B}}$ has a left $\mathcal{A}$-module structure. Furthermore, this structure is the natural extension of the $\mathcal{A}$-module structure on $F^{(\mathcal{B})}$.

Lemma 2.7. *Any injective net $\{b_\lambda\}_{\lambda \in \Lambda}$ converges to 0.*

Proof. Let U be open around 0 and let $b_1, \ldots, b_n \in \mathcal{B}$ be such that if $f(b_i) = 0$ for all $1 \leq i \leq n$ then $f \in U$. Choose $\lambda \in \Lambda$ such that $\gamma > \lambda$ implies that $b_\gamma \notin \{b_1, \ldots, b_n\}$. Then, $\gamma > \lambda$ implies $b_\gamma \in U$. □

Theorem 2.8. *If $[\ell_r]_{\mathcal{B}} \cdot : F^{(\mathcal{B})} \to F^{(\mathcal{B})}$ is continuous, then $[\ell_r]_{\mathcal{B}}$ is row finite.*

Proof. Suppose that $[\ell_r]_{\mathcal{B}}$ fails to be row finite so that there is some infinite row. That is, there is some $b \in \mathcal{B}$ such that $[\ell_r]_{\mathcal{B}}(b, b') \neq 0$ for infinitely many b'. Let $f : \mathbb{N} \to \mathcal{B}$ be an injective map such that for all $b' \in f(\mathbb{N})$ we have that $[\ell_r]_{\mathcal{B}}(b, b') \neq 0$. Now, f describes an injective net so that lemma 2.7 applies. By continuity, we see that $[\ell_r]_{\mathcal{B}} f(n) \to 0$. Therefore, there is some $N \in \mathbb{N}$ such that for $n > N$ we have that $[[\ell_r]_{\mathcal{B}} f(n)](b) = 0$. This is a contradiction as the b-th coordinate of $[\ell_r]_{\mathcal{B}} f(n)$ is non-zero for all $n \in \mathbb{N}$. Therefore we must conclude that $[\ell_r]_{\mathcal{B}}$ is indeed row finite. □

Corollary 2.9. *The basis $\mathcal{B}$ is amenable if and only if $[\ell_r]_{\mathcal{B}} : F^{(\mathcal{B})} \to F^{(\mathcal{B})}$ is uniformly continuous for all $r \in \mathcal{A}$.*

Theorem 2.10. *The left $\mathcal{A}$-module $F^{(\mathcal{B})}$ has a natural extension if and only if $\mathcal{B}$ is amenable.*

3. Characterizations of Mutual and Proper Congeniality

It is known that if $\mathcal{B}$ is mutually congenial to $\mathcal{C}$ then ${}^{\mathcal{B}}\mathcal{T}$ and ${}^{\mathcal{C}}\mathcal{T}$ are isomorphic. In fact, they are 'naturally 'isomorphic as the congeniality map is an isomorphism between them. Furthermore, for $\mathcal{T}$ of countable dimension, it is known that congeniality maps are onto. Here we show that if a congeniality map is one-to-one as well as being onto and, as always, continuous, then it is, in fact, a mutual congeniality map. We do so by establishing continuity of the inverse map.

Theorem 3.1. *Let $\mathcal{C}$ and $\mathcal{B}$ be countable and let $g : \mathcal{C} \times \mathcal{B} \to F$ be a row and column finite matrix such that $g\cdot$ maps $F^{\mathcal{B}}$ bijectively onto $F^{\mathcal{C}}$, then g^{-1} is continuous.*

The proof of this lemma uses a lot of the same tools that are used in the proof that congeniality maps are onto, which is provided in [1].

Proof. Enumerate $\mathcal{B} = \{b_i\}_{i \in Z^+}$. Let V be the row-span of g. For $n \in \mathbb{Z}^+$, define $V_n = \{v \in V : v(b_j) = 0 \ if \ j > n\}$; one could informally write '$V_n = F^{\{b_1,\ldots,b_n\}} \cap V$ '. Then $V_1 \subseteq V_2 \subseteq \cdots \subseteq \cup_{n=1}^{\infty} V_i = V$. Also, for any infinite matrix h and for $n \in N$, $h_{>n}$ or $h_{\geq n+1}$ denote the infinite matrix made up of all rows of h but beginning with the $(n+1)$-th.

Note that V_n is a subspace of V and $\dim V_n \leq n$. Recursively, choose, for all $n \in \mathbb{Z}^+$, $\mathcal{C}_n$, a basis for V_n, in such a way that $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. Then $\mathcal{C} = \bigcup_{n \in \mathbb{Z}^+} \mathcal{C}_n$ is a basis for V. Let f be a matrix such that $fg = h$ is a matrix having, as rows, the elements of $\mathcal{C}$ in such a way that the elements of $\mathcal{C}_i$ appear no later than those of $\mathcal{C}_j$ when $i < j$. Clearly, f is a row-finite invertible matrix with an inverse f^{-1} that is also row-finite.

Let $\{n_k \mid k \geq 1\}$ and be strictly increasing sequences of positive integers and let $\{m_k \mid k \geq 1\}$ be a sequence of positive integers so that the rows of h may be viewed as a sequence of $m_k \times n_k$ rectangular matrices C_k with $m_k \leq n_k$. Let $\ell_k = \sum_{j=1}^{k} m_j$. in the following manner:

Now we have

$$\text{fg} = \text{h} = \left(\begin{array}{c|c|c} C_1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{C_2} & 0 \\ \hline \multicolumn{3}{c}{\cdots} \end{array} \right),$$

and $h\cdot : F^{\mathcal{B}} \to F^{\mathcal{C}} \to F^{\mathcal{E}}$ where $\mathcal{E} = \{e_i\}_{i \in Z^+}$ and $e_i = \gamma_{\mathcal{C}} \left(f^{-1}_{\mathcal{C} \times \{e_i\}} \right)$. We begin by examining the image of sets of the form $U_j^p := U(b_j \mid p) = \{s \in F^{\mathcal{B}} \mid s(b_j) = p\}$ for some $b_j \in \mathcal{B}$ and $p \in F$. Let $b_j \in \mathcal{B}$ and $p \in F$ be arbitrary. Let $k = \min\{k \in Z^+ \mid j \leq n_k\}$. Let $F_p^{\mathcal{B}_n} = \{t \in F^{\{b_1, \ldots b_{n_k}\}} \mid t(b_j) = p\}$ and for $t \in F_p^{\mathcal{B}_n}$ let $U(t) = \{s \in F^{\mathcal{B}} \mid s(b_j) = t(b_j)$ for all $1 \leq j \leq n_k\}$ and let $\hat{t} = t \cup 0|_{\{b_j \ \mid \ j > n_k\}} \in F^{\mathcal{B}}$. Note that $hs(b_i) = h\hat{t}(b_i)$ for $1 \leq i \leq \ell_k$ for all $s \in U(t)$. Thus, $hU(t) \subset \hat{U}(t) = \{v \in F^{\mathcal{E}} \mid v(e_j) = h\hat{t}(e_j)$ for all $1 \leq j \leq \ell_k\}$. Now, for $v \in \hat{U}(t)$ we show that $hx = v$ has a solution $x \in U(t)$.

The existence of a solution for the system can be obtained as the limit of a convergent sequence. We build the sequence as follows. Let $d_k = v|_{\{e_j \ \mid \ 1 \ \leq \ j \ \leq \ \ell_k\}}$ and for every $y > k$, $d_y = v|_{\{e_j \ \mid \ \ell_{y-1}+1 \ \leq \ j \ \leq \ \ell_y\}}$ and, for all $y \geq k$, $D_y = v|_{\{e_j \ \mid \ 1 \ \leq \ j \ \leq \ \ell_y\}}$. Clearly, t is such that such that $h_k t = d_k$ where

$$h_{y-1} = \left(\begin{array}{c|c|c} C_1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{C_2} & 0 \\ \hline \multicolumn{3}{c}{\cdots} \\ \hline \multicolumn{3}{c}{C_{y-1}} \end{array} \right)$$

for $y > k$. For $y > k+1$ Suppose a $y-1$-th approximation X_{y-1} has been obtained; in other words, $h_{y-1}X_{y-1} = D_{y-1}$. We construct X_y, but first we partition C_y as $\left(C_{y,1} \mid C_{y,2}\right)$ where $C_{y,1}$ has n_{y-1} columns. We note that the rows of $C_{y,2}$ are linearly independent. To see this we assume that they aren't linearly independent and that $a_1 r'_1 + \dots a_{m_y} r'_{m_y} = 0$ where r'_i is a row of $C_{y,2}$ and not all a_i are zero. Now consider the same linear combination, except of the rows of C_y. By the linear independence of the rows of C_y we know $a_1 r_1 + a_{m_y} r_{m_y} \neq 0$, however, its non-zero entries must exist in the first n_{y-1} entries. Thus, $a_1 r_1 + a_{m_y} r_{m_y} \in V_{y-1} = span\{\bigcup_{j=1}^{y-1} \mathcal{C}_j\}$ so that $\mathcal{C}_y$ is a linearly dependent collection. This is a contradiction, so the rows of $C_{y,2}$ are indeed linearly independent.

Now we obtain a solution x_y to the equation $C_{y,2}x_y = D_{y-1} - C_{y,1}X_{y-1}$ and setting $X_y = X_{y-1} \cup x_y$. The existence of x_y can be assured because m_y, the number of rows of $C_{y,2}$, is less than or equal to $n_y - n_{y-1}$, the number of its columns, due to linear independence. We note that X_y is a vector of length n_y, let $S_y = X_y \cup 0|_{\{b_j \mid j > n_y\}} \in F^{(\mathcal{B})}$. Now, clearly $\{S_y\}_{y=1}^{\infty}$ is a Cauchy sequence. That is, if we want $S_n - S_m$ to have its first non-zero entry after ℓ then we choose j such that $n_j > \ell$ and let $n, m > n_j$. Let $S = \lim_{y \to \infty} S_k$ which is guaranteed to exist by completeness.

Furthermore, $(hS_y)(e_j) = (h_y X_y)(e_j) = D_y(e_j)$ for $1 \leq j \leq \ell_y$ and $hS_y \to hS$ by the continuity of h. Let $\hat{D}_y = D_y \cup 0|_{\{e_j \mid j > \ell_y\}}$ and note that $\hat{D}_y$ is a Cauchy sequence and clearly $\hat{D}_y \to \cup D_y = v$. Additionally, $hS_y - D_y \to 0$ as $(hS_y)(e_j) = (h_y X_y)(e_j) = D_y(e_j)$ for $1 \leq j \leq \ell_y$. Therefore, if we want $hS_y(e_z) - D_y(e_z) = 0$ we need only choose $y > Y$ where $1 \leq z \leq \ell_Y$. Thus, $hS_y \to v$ so $v = hS$. Thus, $\hat{U}(t) \subset hU(t)$ so that $\hat{U}(t) = hU(t)$ which is open in the product topology on $F^{\mathcal{E}}$.

Then, as $U_j^p = \cup_{t \in F_p^{\mathcal{B}_n}} U(t)$ we have that $hU_j = h\left(\cup_{t \in F_p^{\mathcal{B}_n}} U(t)\right) = \cup_{t \in F_p^{\mathcal{B}_n}} hU(t)$ so that hU_j is open in $F^{\mathcal{E}}$. Now, let $U = U(b_{j_1}, \dots, b_{j_z} \mid p_1, \dots, p_z)$ be a basic open set in $F^{\mathcal{B}}$. Then, $U = \bigcap_{1=1}^{z} U_{j_i}^{p_i}$ so that $hU = h(\bigcap_{1=1}^{z} U_{j_i}^{p_i}) \subset \bigcap_{1=1}^{z} hU_{j_i}^{p_i}$. However, as $g\cdot$ and $f\cdot$ are bijections we know that $fg\cdot = h\cdot$ is a bijection as well so that $hU = h(\bigcap_{1=1}^{z} U_{j_i}^{p_i}) = \bigcap_{1=1}^{z} hU_{j_i}^{p_i}$ so that hU is open in $F^{\mathcal{E}}$.

As f is row finite we know that $f\cdot : F^{\mathcal{C}} \to F^{\mathcal{E}}$ is continuous. Therefore, $f^{-1}hU$ is open in $F^{\mathcal{C}}$. However, $f^{-1}hU = gU$ so that $g\cdot$ maps open sets to open sets. Therefore, $g\cdot$ is an open map. Now, by bijectivity we know that $(g\cdot)^{-1} : F^{\mathcal{C}} \to F^{\mathcal{B}}$ is well-defined. Furthermore, $[(g\cdot)^{-1}]^{-1}U$ is open for all open U. Thus, $(g\cdot)^{-1}$ is continuous and therefore is uniformly continuous by virtue of linearity. We also have that $(g\cdot)^{-1}|_{F^{(\mathcal{C})}} = g^{-1} \cdot |_{F^{(\mathcal{C})}}$ so that $g^{-1}\cdot$ is continuous and $g^{-1}\cdot = (g\cdot)^{-1}$. □

Theorem 3.2. *Let $\mathcal{B}$ and $\mathcal{C}$ be amenable. Then $[I]_{\mathcal{B}}^{\mathcal{C}}\cdot$ bijectively maps ${}^{\mathcal{B}}\mathcal{T}$ onto ${}^{\mathcal{C}}\mathcal{T}$ if and only if $\mathcal{B}$ is mutually congenial to $\mathcal{C}$.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be amenable such that $[I]^{\mathcal{C}}_{\mathcal{B}}\cdot$ bijectively maps ${}^{\mathcal{B}}\mathcal{T}$ onto ${}^{\mathcal{C}}\mathcal{T}$. Then, $[I]^{\mathcal{C}}_{\mathcal{B}}$ satisfies the conditions of theorem 3.1 so that $([I]^{\mathcal{C}}_{\mathcal{B}}\cdot)^{-1} = [I]^{\mathcal{B}}_{\mathcal{C}}\cdot$ is continuous.

The other direction is trivial. □

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