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Ideals in $B_1(X)$ and residue class rings of $B_1(X)$ modulo an ideal

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Abstract

This paper explores the duality between ideals of the ring $B_1(X)$ of all real valued Baire one functions on a topological space X and typical families of zero sets, called Z_B-filters, on X. As a natural outcome of this study, it is observed that $B_1(X)$ is a Gelfand ring but non-Noetherian in general. Introducing fixed and free maximal ideals in the context of $B_1(X)$, complete descriptions of the fixed maximal ideals of both $B_1(X)$ and $B_1^*(X)$ are obtained. Though free maximal ideals of $B_1(X)$ and those of $B_1^*(X)$ do not show any relationship in general, their counterparts, i.e., the fixed maximal ideals obey natural relations. It is proved here that for a perfectly normal T_1 space X, free maximal ideals of $B_1(X)$ are determined by a typical class of Baire one functions. In the concluding part of this paper, we study residue class ring of $B_1(X)$ modulo an ideal, with special emphasize on real and hyper real maximal ideals of $B_1(X)$.

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1. Introduction

In [1], we have introduced the ring of Baire one functions defined on any topological space X and have denoted it by $B_1(X)$. It has been observed that $B_1(X)$ is a commutative lattice ordered ring with unity containing the ring $C(X)$ of continuous functions as a subring. The collection of bounded Baire one functions, denoted by $B_1^*(X)$, is a commutative subring and sublattice of $B_1(X)$. Certainly, $B_1^*(X) \cap C(X) = C^*(X)$.
In this paper, we study the ideals, in particular, the maximal ideals of $B_1(X)$ (and also of $B_1^*(X)$). There is a nice interplay between the ideals of $B_1(X)$ and a typical family of zero sets (which we call a Z_B-filter) of the underlying space X. As a natural consequence of this duality of ideals of $B_1(X)$ and Z_B-filters on X, we obtain that $B_1(X)$ is Gelfand and in general, $B_1(X)$ is non-Noetherian.

Introducing the idea of fixed and free ideals in our context, we have characterized the fixed maximal ideals of $B_1(X)$ and also those of $B_1^*(X)$. We have shown that although fixed maximal ideals of the rings $B_1(X)$ and $B_1^*(X)$ obey a natural relationship, the free maximal ideals fail to do so. However, for a perfectly normal T_1 space X, free maximal ideals of $B_1(X)$ are determined by a typical class of Baire one functions.

In the last section of this paper, we have discussed residue class ring of $B_1(X)$ modulo an ideal and introduced real and hyper-real maximal ideals in $B_1(X)$.

2. Z_B-filters on X and Ideals in $B_1(X)$

Definition 2.1. A nonempty subcollection $\mathscr{F}$ of $Z(B_1(X))$ ([1]) is said to be a Z_B-filter on X, if it satisfies the following conditions:
(1) $\varnothing \notin \mathscr{F}$
(2) if $Z_1, Z_2 \in \mathscr{F}$, then $Z_1 \cap Z_2 \in \mathscr{F}$
(3) if $Z \in \mathscr{F}$ and $Z' \in Z(B_1(X))$ is such that $Z \subseteq Z'$, then $Z' \in \mathscr{F}$.

Clearly, a Z_B-filter $\mathscr{F}$ on X has finite intersection property. Conversely, if a subcollection $\mathscr{B} \subseteq Z(B_1(X))$ possesses finite intersection property, then $\mathscr{B}$ can be extended to a Z_B-filter $\mathscr{F}(\mathscr{B})$ on X, given by $\mathscr{F}(\mathscr{B}) = \{Z \in Z(B_1(X))$: there exists a finite subfamily $\{B_1, B_2, ..., B_n\}$ of $\mathscr{B}$ with $Z \supseteq \bigcap_{i=1}^{n} B_i\}$. Indeed this is the smallest Z_B-filter on X containing $\mathscr{B}$.

Definition 2.2. A Z_B-filter $\mathscr{U}$ on X is called a Z_B-ultrafilter on X, if there does not exist any Z_B-filter $\mathscr{F}$ on X, such that $\mathscr{U} \subsetneqq \mathscr{F}$.

Example 2.3. Let $A_0 = \{Z \in Z(B_1(\mathbb{R})) : 0 \in Z\}$. Then A_0 is a Z_B-ultrafilter on $\mathbb{R}$.

Applying Zorn's lemma one can show that, every Z_B-filter on X can be extended to a Z_B-ultrafilter. Therefore, a family $\mathscr{B}$ of $Z(B_1(X))$ with finite intersection property can be extended to a Z_B-ultrafilter on X.

Remark 2.4. A Z_B-ultrafilter $\mathscr{U}$ on X is a subfamily of $Z(B_1(X))$ which is maximal with respect to having finite intersection property. Conversely, if a family $\mathscr{B}$ of $Z(B_1(X))$ has finite intersection property and maximal with respect to having this property, then $\mathscr{B}$ is a Z_B-ultrafilter on X.

In what follow, by an ideal I of $B_1(X)$ we always mean a proper ideal.

Theorem 2.5. *If I is an ideal in $B_1(X)$, then $Z_B[I] = \{Z(f) : f \in I\}$ is a Z_B-filter on X.*

Proof. Since I is a proper ideal in $B_1(X)$, we claim $\varnothing \notin Z_B[I]$. If possible let $\varnothing \in Z_B[I]$. So, $\varnothing = Z(f)$, for some $f \in I$. As $f \in I \implies f^2 \in I$ and $Z(f^2) = Z(f) = \varnothing$, hence $\frac{1}{f^2} \in B_1(X)$ [1]. This is a contradiction to the fact that, I is a proper ideal and contains no unit.
Let $Z(f), Z(g) \in Z_B[I]$, for some $f, g \in I$. Our claim is $Z(f) \cap Z(g) \in Z_B[I]$. $Z(f) \cap Z(g) = Z(f^2 + g^2) \in Z_B[I]$, as I is an ideal and so, $f^2 + g^2 \in I$.
Now assume that $Z(f) \in Z_B[I]$ and $Z' \in Z(B_1(X))$ is such that $Z(f) \subseteq Z'$. Then we can write $Z' = Z(h)$, for some $h \in B_1(X)$. $Z(f) \subseteq Z' \implies Z(h) = Z(h) \cup Z(f)$. So, $Z(h) = Z(hf) \in Z_B[I]$, because $hf \in I$. Hence, $Z_B[I]$ is a Z_B-filter on X. □

Theorem 2.6. *Let $\mathscr{F}$ be a Z_B-filter on X. Then $Z_B^{-1}[\mathscr{F}] = \{f \in B_1(X) : Z(f) \in \mathscr{F}\}$ is an ideal in $B_1(X)$.*

Proof. We note that, $\varnothing \notin \mathscr{F}$. So the constant function $\mathbf{1} \notin Z_B^{-1}[\mathscr{F}]$. Hence $Z_B^{-1}[\mathscr{F}]$ is a proper subset of $B_1(X)$.
Choose $f, g \in Z_B^{-1}[\mathscr{F}]$. Then $Z(f), Z(g) \in \mathscr{F}$ and $\mathscr{F}$ being a Z_B-filter $Z(f) \cap Z(g) \in \mathscr{F}$. Now $Z(f) \cap Z(g) \subseteq Z(f - g)$. Hence $Z(f - g) \in \mathscr{F}$, $\mathscr{F}$ being a Z_B-filter on X. This implies $f - g \in Z_B^{-1}[\mathscr{F}]$.
For $f \in Z_B^{-1}[\mathscr{F}]$ and $h \in B_1(X)$, $Z(f.h) = Z(f) \cup Z(h)$. As $Z(f) \in \mathscr{F}$ and $\mathscr{F}$ is a Z_B-filter on X, it follows that $Z(f.h) \in \mathscr{F}$. Hence $f.h \in Z_B^{-1}[\mathscr{F}]$.
Thus $Z_B^{-1}[\mathscr{F}]$ is an ideal of $B_1(X)$. □

We may define a map $Z : B_1(X) \to Z(B_1(X))$ given by $f \mapsto Z(f)$. Certainly, Z is a surjection. In view of the above results, such Z induces a map Z_B between the collection of all ideals of $B_1(X)$, say $\mathscr{I}_B$ and the collection of all Z_B-filters on X, say $\mathscr{F}_B(X)$, i.e., $Z_B : \mathscr{I}_B \to \mathscr{F}_B(X)$ given by $Z_B(I) = Z_B[I]$, $\forall\, I \in \mathscr{I}_B$. The map Z_B is also a surjective map because for any $\mathscr{F} \in \mathscr{F}_B(X)$, $Z_B^{-1}[\mathscr{F}]$ is an ideal in $B_1(X)$. We also note that $Z_B[Z_B^{-1}[\mathscr{F}]] = \mathscr{F}$. So each Z_B-filter on X is the image of some ideal in $B_1(X)$ under the map $Z_B : \mathscr{I}_B \to \mathscr{F}_B(X)$.

Observation. The map $Z_B : \mathscr{I}_B \to \mathscr{F}_B(X)$ is not injective in general. Because, for any ideal I in $B_1(X)$, $Z_B^{-1}[Z_B[I]]$ is an ideal in $B_1(X)$, such that $I \subseteq Z_B^{-1}[Z_B[I]]$ and by our previous result $Z_B[Z_B^{-1}[Z_B[I]]] = Z_B[I]$. If one gets an ideal J in $B_1(X)$ such that $I \subseteq J \subseteq Z_B^{-1}[Z_B[I]]$, then we must have $Z_B[I] = Z_B[J]$. The following example shows that such an ideal is indeed possible to exist. In fact, in the following example, we get countably many ideals I_n in $B_1(\mathbb{R})$ such that the images of all the ideals are same under the map Z_B.

Example 2.7. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined as,

$$f_0(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and g.c.d. } (p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is well known that $f_0 \in B_1(\mathbb{R})$ (see [2]). Consider the ideal I in $B_1(X)$ generated by f_0, i.e., $I = \langle f_0 \rangle$. We claim that $f_0^{\frac{1}{3}} \notin I$. If possible, let $f_0^{\frac{1}{3}} \in I$. Then there exists $g \in B_1(\mathbb{R})$, such that $f_0^{\frac{1}{3}} = g f_0$. When $x = \frac{p}{q}$, where $p \in \mathbb{Z}, q \in \mathbb{N}$ and g.c.d $(p,q) = 1$, $g(x) = q^{\frac{2}{3}}$. We show that such g does not exist in $B_1(\mathbb{R})$. Let α be any irrational number in $\mathbb{R}$. We show that g is not continuous at α, no matter how we define $g(\alpha)$. Suppose $g(\alpha) = \beta$. There exists a sequence of rational numbers $\{\frac{p_m}{q_m}\}$, such that $\{\frac{p_m}{q_m}\}$ converges to α and $p_m \in \mathbb{Z}, q_m \in \mathbb{N}$ with g.c.d $(p_m, q_m) = 1$, $\forall\ m \in \mathbb{N}$. If g is continuous at α then $\{g(\frac{p_m}{q_m})\}$ converges to $g(\alpha)$, which implies that $q_m^{\frac{2}{3}}$ converges to β. But $q_m \in \mathbb{N}$, so $\{q_m^{\frac{2}{3}}\}$ must be eventually constant. Suppose there exists $n_0 \in \mathbb{N}$ such that $\forall\ m \geq n_0$, q_m is either c or $-c$ or q_m oscillates between c and $-c$, for some natural number c, i.e., $\{\frac{p_m}{c}\}$ converges to α or $-\alpha$ or oscillates. In any case, $\{\frac{p_m}{q_m}\}$ cannot converges to α. Hence we get a contradiction. So, g is not continuous at any irrational point. It is well known that, if, $f \in B_1(X,Y)$, where X is a Baire space, Y is a metric space and $B_1(X,Y)$ stands for the collection of all Baire one functions from X to Y then the set of points where f is continuous is dense in X [4]. Therefore, the set of points of $\mathbb{R}$ where g is continuous is dense in $\mathbb{R}$ and is a subset of $\mathbb{Q}$. Hence it is a countable dense subset of $\mathbb{R}$ (Since $\mathbb{R}$ is a Baire space). But using Baire's category theorem it can be shown that, there exists no function $f : \mathbb{R} \to \mathbb{R}$, which is continuous precisely on a countable dense subset of $\mathbb{R}$. So, we arrive at a contradiction and no such g exists. Hence $f_0^{\frac{1}{3}} \notin I$.

Observe that, $Z(f_0) = Z(f_0^{\frac{1}{3}})$ and $I \subseteq Z_B^{-1}[Z_B[I]]$. Again, $f_0^{\frac{1}{3}} \notin I$ but $f_0^{\frac{1}{3}} \in Z_B^{-1} Z_B[I]$, which implies $I \subsetneq Z_B^{-1}[Z_B[I]]$. By an earlier result $Z_B[I] = Z_B[Z_B^{-1}[Z_B[I]]]$, proving that the map $Z_B : \mathscr{I}_B \to \mathscr{F}_B(X)$ is not injective when $X = \mathbb{R}$.

Observation: $\langle f_0 \rangle \subsetneq \langle f_0^{\frac{1}{3}} \rangle$. Analogously, it can be shown that $\langle f_0 \rangle \subsetneq \langle f_0^{\frac{1}{3}} \rangle \subsetneq \langle f_0^{\frac{1}{5}} \rangle \subsetneq ... \subsetneq \langle f_0^{\frac{1}{2m+1}} \rangle \subsetneq ...$ is a strictly increasing chain of proper ideals in $B_1(\mathbb{R})$. Hence $B_1(\mathbb{R})$ is not a Noetherian ring.

Theorem 2.8. *If M is a maximal ideal in $B_1(X)$ then $Z_B[M]$ is a Z_B-ultrafilter on X.*

Proof. By Theorem 2.5, $Z_B[M]$ is a Z_B-filter on X. Let $\mathscr{F}$ be a Z_B-filter on X such that, $Z_B[M] \subseteq \mathscr{F}$. Then $M \subseteq Z_B^{-1}[Z_B[M]] \subseteq Z_B^{-1}[\mathscr{F}]$. $Z_B^{-1}[\mathscr{F}]$ being a proper ideal and M being a maximal ideal, we have $Z_B^{-1}[\mathscr{F}] = M \implies$

$Z_B[M] = Z_B[Z_B^{-1}[\mathscr{F}]] = \mathscr{F}$. Hence every Z_B-filter that contains $Z_B[M]$ must be equal to $Z_B[M]$. This shows $Z_B[M]$ is a Z_B-ultrafilter on X. $\square$

Theorem 2.9. *If $\mathscr{U}$ is a Z_B-ultrafilter on X then $Z_B^{-1}[\mathscr{U}]$ is a maximal ideal in $B_1(X)$.*

Proof. By Theorem 2.6, we have $Z_B^{-1}[\mathscr{U}]$ is a proper ideal in $B_1(X)$. Let I be a proper ideal in $B_1(X)$ such that $Z_B^{-1}[\mathscr{U}] \subseteq I$. It is enough to show that $Z_B^{-1}[\mathscr{U}] = I$. Now $Z_B^{-1}[\mathscr{U}] \subseteq I \implies Z_B[Z_B^{-1}[\mathscr{U}]] \subseteq Z_B[I] \implies \mathscr{U} \subseteq Z_B[I]$. Since $\mathscr{U}$ is a Z_B-ultrafilter on X, we have $\mathscr{U} = Z_B[I] \implies Z_B^{-1}[\mathscr{U}] = Z_B^{-1}[Z_B[I]] \supseteq I$. Hence $Z_B^{-1}[\mathscr{U}] = I$ $\square$

Remark 2.10. Each Z_B-ultrafilter on X is the image of a maximal ideal in $B_1(X)$ under the map Z_B.

Let $\mathcal{M}(B_1(X))$ be the collection of all maximal ideals in $B_1(X)$ and $\Omega_B(X)$ be the collection of all Z_B-ultrafilters on X. If we restrict the map Z_B to the class $\mathcal{M}(B_1(X))$, then it is clear that the map $Z_B\Big|_{\mathcal{M}(B_1(X))} : \mathcal{M}(B_1(X)) \to \Omega_B(X)$ is a surjective map. Further, this restriction map is a bijection, as seen below.

Theorem 2.11. *The map $Z_B\Big|_{\mathcal{M}(B_1(X))} : \mathcal{M}(B_1(X)) \to \Omega_B(X)$ is a bijection.*

Proof. It is enough to check that $Z_B\Big|_{\mathcal{M}(B_1(X))} : \mathcal{M}(B_1(X)) \to \Omega_B(X)$ is injective. Let M_1 and M_2 be two members in $\mathcal{M}(B_1(X))$ such that $Z_B[M_1] = Z_B[M_2] \implies Z_B^{-1}[Z_B[M_1]] = Z_B^{-1}[Z_B[M_1]]$. But $M_1 \subseteq Z_B^{-1}[Z_B[M_1]]$ and $M_2 \subseteq Z_B^{-1}[Z_B[M_2]]$. By maximality of M_1 and M_2 we have, $M_1 = Z_B^{-1}[Z_B[M_1]]$ $= Z_B^{-1}[Z_B[M_2]] = M_2$. $\square$

Definition 2.12. An ideal I in $B_1(X)$ is called a Z_B-ideal if $Z_B^{-1}[Z_B[I]] = I$, i.e., $\forall\ f \in B_1(X)$, $f \in I \iff Z(f) \in Z_B[I]$.

Since $Z_B[Z_B^{-1}[\mathscr{F}_B]] = \mathscr{F}_B$, $Z_B^{-1}[\mathscr{F}_B]$ is a Z_B-ideal for any Z_B-filter $\mathscr{F}_B$ on X. If I is any ideal in $B_1(X)$, then, $Z_B^{-1}[Z_B[I]]$ is the smallest Z_B-ideal containing I. It is easy to observe

(1) Every maximal ideal in $B_1(X)$ is a Z_B ideal.
(2) The intersection of arbitrary family of Z_B-ideals in $B_1(X)$ is always a Z_B-ideal.
(3) The map $Z_B\Big|_{\mathscr{J}_B} : \mathscr{J}_B \to \mathscr{F}_B(X)$ is a bijection, where $\mathscr{J}_B$ denotes the collection of all Z_B-filters on X.

Example 2.13. Let $I = \{f \in B_1(\mathbb{R}) : f(1) = f(2) = 0\}$. Then I is a Z_B ideal in $B_1(\mathbb{R})$ which is not maximal, as $I \subsetneq \widehat{M_1} = \{f \in B_1(\mathbb{R}) : f(1) = 0\}$. The ideal I is not a prime ideal, as the functions $x - 1$ and $x - 2$ do not belong to I, but their product belongs to I. Also no proper ideal of I is prime. More generally, for any subset S of $\mathbb{R}, I_S = \{f \in B_1(\mathbb{R}) : f(S) = 0\}$ is a Z_B-ideal in $B_1(\mathbb{R})$.

It is well known that in a commutative ring R with unity, the intersection of all prime ideals of R containing an ideal I is called the **radical of** I and it is denoted by $\sqrt{I}$. For any ideal I, the radical of I is given by $\{a \in R : a^n \in I$, for some $n \in \mathbb{N}\}$ ([3]) and in general $I \subseteq \sqrt{I}$. For if $I = \sqrt{I}$, I is called a radical ideal.

Theorem 2.14. *A Z_B-ideal I in $B_1(X)$ is a radical ideal.*

Proof. $\sqrt{I} = \{f \in B_1(X) : \exists\ n \in \mathbb{N}$ such that $f^n \in I\} = \{f \in B_1(X) :$ such that $Z(f^n) \in Z_B[I]$ for some $n \in \mathbb{N}\}$ (As I is a Z_B-ideal in $B_1(X)$) $= \{f \in B_1(X) : Z(f) \in Z_B[I]\ \} = \{f \in B_1(X) : f \in I\} = I$. So I is a radical ideal in $B_1(X)$. □

Corollary 2.15. *Every Z_B-ideal I in $B_1(X)$ is the intersection of all prime ideals in $B_1(X)$ which contains I.*

Next theorem establishes some equivalent conditions on the relationship among Z_B-ideals and prime ideals of $B_1(X)$.

Theorem 2.16. *For a Z_B-ideal I in $B_1(X)$ the following conditions are equivalent:*

(1) *I is a prime ideal of $B_1(X)$.*
(2) *I contains a prime ideal of $B_1(X)$.*
(3) *if $fg = 0$, then either $f \in I$ or $g \in I$.*
(4) *Given $f \in B_1(X)$ there exists $Z \in Z_B[I]$, such that f does not change its sign on Z.*

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are immediate.
(3) $\implies$ (4): Let (3) be true. Choose $f \in B_1(X)$. Then $(f \vee 0).(f \wedge 0) = 0$. So by (3), $f \vee 0 \in I$ or $f \wedge 0 \in I$. Hence $Z(f \vee 0) \in Z_B[I]$ or $Z(f \wedge 0) \in Z_B[I]$. It is clear that $f \leq 0$ on $Z(f \wedge 0)$ and $f \geq 0$ on $Z(f \vee 0)$.

(4) $\implies$ (1): Let (4) be true. To show that I is prime. Let $g, h \in B_1(X)$ be such that $gh \in I$. By (4) there exists $Z \in Z_B[I]$, such that $|g| - |h| \geq 0$ on Z (say). It is clear that, $Z \cap Z(g) \subseteq Z(h)$. Consequently $Z \cap Z(gh) \subseteq Z(h)$. Since $Z_B[I]$ is a Z_B-filter on X, it follows that $Z(h) \in Z_B[I]$. So $h \in I$, since I is a Z_B-ideal. Hence, I is prime. □

Theorem 2.17. *In $B_1(X)$, every prime ideal P can be extended to a unique maximal ideal.*

Proof. If possible let P be contained in two distinct maximal ideals M_1 and M_2. So, $P \subseteq M_1 \cap M_2$. Since maximal ideals in $B_1(X)$ are Z_B-ideals and intersection of any number of Z_B-ideals is Z_B-ideal, $M_1 \cap M_2$ is a Z_B-ideal containing the prime ideal P. By Theorem 2.16, $M_1 \cap M_2$ is a prime ideal. But in a commutative ring with unity, for two ideals I and J, if, $I \nsubseteq J$ and $J \nsubseteq I$, then $I \cap J$ is not a prime ideal. Thus $M_1 \cap M_2$ is not prime ideal and we get a contradiction. So, every prime ideal can be extended to a unique maximal ideal. □

Corollary 2.18. *$B_1(X)$ is a Gelfand ring for any topological space X.*

Definition 2.19. A Z_B-filter $\mathscr{F}_B$ on X is called a prime Z_B-filter on X, if, for any $Z_1, Z_2 \in Z(B_1(X))$ with $Z_1 \cup Z_2 \in \mathscr{F}_B$ either $Z_1 \in \mathscr{F}_B$ or $Z_2 \in \mathscr{F}_B$.

The next two theorems are analogous to Theorem 2.12 in [3] and therefore, we state them without proof.

Theorem 2.20. *If I is a prime ideal in $B_1(X)$, then $Z_B[I] = \{Z(f) : f \in I\}$ is a prime Z_B-filter on X.*

Theorem 2.21. *If $\mathscr{F}_B$ is a prime Z_B-filter on X then $Z_B^{-1}[\mathscr{F}_B] = \{f \in B_1(X) : Z(f) \in \mathscr{F}_B\}$ is a prime ideal in $B_1(X)$.*

Corollary 2.22. *Every prime Z_B-filter can be extended to a unique Z_B-ultrafilter on X.*

Corollary 2.23. *A Z_B-ultrafilter $\mathscr{U}$ on X is a prime Z_B-filter on X, as $\mathscr{U} = Z_B[M]$, for some maximal ideal M in $B_1(X)$.*

3. Fixed ideals and free ideals in $B_1(X)$

In this section, we introduce fixed and free ideals of $B_1(X)$ and $B_1^*(X)$ and completely characterize the fixed maximal ideals of $B_1(X)$ as well as those of $B_1^*(X)$. It is observed here that a natural relationship exists between fixed maximal ideals of $B_1^*(X)$ and the fixed maximal ideals of $B_1(X)$. However, free maximal ideals do not behave the same. In the last part of this section, we find a class of Baire one functions defined on a perfectly normal T_1 space X which precisely determines the fixed and free maximal ideals of the corresponding ring.

Definition 3.1. A proper ideal I of $B_1(X)$ (respectively, $B_1^*(X)$) is called **fixed** if $\bigcap Z[I] \neq \varnothing$. If I is not fixed then it is called **free**.

For any Tychonoff space X, the fixed maximal ideals of the ring $B_1(X)$ and those of $B_1^*(X)$ are characterized.

Theorem 3.2. *$\{\widehat{M}_p : p \in X\}$ is a complete list of fixed maximal ideals in $B_1(X)$, where $\widehat{M}_p = \{f \in B_1(X) : f(p) = 0\}$. Moreover, $p \neq q \implies \widehat{M}_p \neq \widehat{M}_q$.*

Proof. Choose $p \in X$. The map $\Psi_p : B_1(X) \to \mathbb{R}$, defined by $\Psi_p(f) = f(p)$ is clearly a ring homomorphism. Since the constant functions are in $B_1(X)$, Ψ_p is surjective and $\ker \Psi_p = \{f \in B_1(X) : \Psi_p(f) = 0\} = \{f \in B_1(X) : f(p) = 0\} = \widehat{M}_p$ (say).

By First isomorphism theorem of rings we get $B_1(X)/\widehat{M}_p$ is isomorphic to the field $\mathbb{R}$. $B_1(X)/\widehat{M}_p$ being a field we conclude that $\widehat{M}_p$ is a maximal ideal in $B_1(X)$. Since $p \in \bigcap Z_B[M]$, the ideal $\widehat{M}_p$ is a fixed ideal.

For any Tychonoff space X, we know that $p \neq q \implies M_p \neq M_q$, where $M_p = \{f \in C(X) : f(p) = 0\}$ is the fixed maximal ideal in $C(X)$. Since $\widehat{M_p} \cap C(X) = M_p$, it follows that for any Tychonoff space X, $p \neq q \implies \widehat{M_p} \neq \widehat{M_q}$.
Let M be any fixed maximal ideal in $B_1(X)$. There exists $p \in X$ such that for all $f \in M$, $f(p) = 0$. Therefore, $M \subseteq \widehat{M_p}$. Since M is a maximal ideal and $\widehat{M_p}$ is a proper ideal, we get $M = \widehat{M_p}$. □

Theorem 3.3. *$\{\widehat{M_p^*} : p \in X\}$ is a complete list of fixed maximal ideals in $B_1^*(X)$, where $\widehat{M_p^*} = \{f \in B_1^*(X) : f(p) = 0\}$. Moreover, $p \neq q \implies \widehat{M_p^*} \neq \widehat{M_q^*}$.*

Proof. Similar to the proof of Theorem 3.2. □

The following two theorems show the interrelations between fixed ideals of $B_1(X)$ and $B_1^*(X)$.

Theorem 3.4. *If I is any fixed ideal of $B_1(X)$ then $I \cap B_1^*(X)$ is a fixed ideal of $B_1^*(X)$.*

Proof. Straightforward. □

Lemma 3.5. *Given any $f \in B_1(X)$, there exists a positive unit u of $B_1(X)$ such that $uf \in B_1^*(X)$.*

Proof. Consider $u = \frac{1}{|f|+1}$. Clearly u is a positive unit in $B_1(X)$ [1] and $uf \in B_1^*(X)$ as $|uf| \leq 1$. □

Theorem 3.6. *Let an ideal I in $B_1(X)$ be such that $I \cap B_1^*(X)$ is a fixed ideal of $B_1^*(X)$. Then I is a fixed ideal of $B_1(X)$.*

Proof. For each $f \in I$, there exists a positive unit u_f of $B_1(X)$ such that $u_f f \in I \cap B_1^*(X)$. Therefore, $\bigcap\limits_{f \in I} Z(f) = \bigcap\limits_{f \in I} Z(u_f f) \supseteq \bigcap\limits_{g \in B_1^*(X) \cap I} Z(g) \neq \emptyset$. Hence I is fixed in $B_1(X)$. □

Since for any discrete space X, $C(X) = B_1(X)$ and $C^*(X) = B_1^*(X)$, considering the example 4.7 of [3], we can conclude the following:

(1) For any maximal ideal M of $B_1(X)$, $M \cap B_1^*(X)$ need not be a maximal ideal in $B_1^*(X)$.
(2) All free maximal ideals in $B_1^*(X)$ need not be of the form $M \cap B_1^*(X)$, where M is a maximal ideal in $B_1(X)$.

Theorem 3.7. *If X is a perfectly normal T_1 space then for each $p \in X$, $\chi_p : X \to \mathbb{R}$ given by*

$$\chi_p(x) = \begin{cases} 1 & \text{if } x = p \\ 0 & \text{otherwise.} \end{cases}$$

is a Baire one function.

Proof. For any open set U of $\mathbb{R}$,

$$\chi_p^{-1}(U) = \begin{cases} X & \text{if } 0,1 \in U \\ X \setminus \{p\} & \text{if } 0 \in U \text{ but } 1 \notin U \\ \{p\} & \text{if } 0 \notin U \text{ but } 1 \in U \\ \varnothing & \text{if } 0 \notin U \text{ but } 1 \notin U. \end{cases}$$

Since X is a perfectly normal space, the open set $X \setminus \{p\}$ is a F_σ set. Hence in any case χ_p pulls back an open set to a F_σ set. So χ_p is a Baire one function [5]. □

In view of Theorem 3.7 we obtain the following facts about any perfectly normal T_1 space.

Observation 3.8. If M is a maximal ideal of $B_1(X)$ where X is a perfectly normal T_1 space then

(1) For each $p \in X$ either $\chi_p \in M$ or $\chi_p - 1 \in M$.
This follows from $\chi_p(\chi_p - 1) = 0 \in M$ and M is prime.
(2) If $\chi_p - 1 \in M$ then $\chi_q \in M$ for all $q \neq p$.
For if $\chi_q - 1 \in M$ for some $q \neq p$ then $Z(\chi_p - 1), Z(\chi_q - 1) \in Z_B[M]$. This implies $\varnothing = Z(\chi_p - 1) \cap Z(\chi_q - 1) \in Z_B[M]$ which contradicts that $Z_B[M]$ is a Z_B-ultrafilter.
(3) M is fixed if and only if $\chi_p - 1 \in M$ for some $p \in X$.
If M is fixed then $M = \widehat{M_p}$ for some $p \in X$ and therefore, $\chi_p - 1 \in M$. Conversely let $\chi_p - 1 \in M$ for some $p \in X$. Then $\{p\} = Z(\chi_p - 1) \in Z_B[M]$ shows that M is fixed.
(4) M is free if and only if M contains $\{\chi_p : p \in X\}$.
Follows from Observation (3).

The following theorem ensures the existence of free maximal ideals in $B_1(X)$ where X is any infinite perfectly normal T_1 space.

Theorem 3.9. *For a perfectly normal T_1 space X, the following statements are equivalent:*

(1) *X is finite.*
(2) *Every maximal ideal in $B_1(X)$ is fixed.*
(3) *Every ideal in $B_1(X)$ is fixed.*

Proof. (1) $\implies$ (2): Since a finite T_1 space is discrete, $C(X) = B_1(X) = X^{\mathbb{R}}$. X being finite, it is compact and therefore all the maximal ideals of $C(X)$ $\big(= B_1(X)\big)$ are fixed.
(2) $\implies$ (3): Proof obvious.
(3) $\implies$ (1): Suppose X is infinite. We shall show that there exists a free (proper) ideal in $B_1(X)$.
Consider $I = \{f \in B_1(X) : \overline{X \setminus Z(f)} \text{ is finite}\}$ (Here finite includes $\varnothing$).
Of course $I \neq \varnothing$, as $\mathbf{0} \in I$. Since X is infinite, $\mathbf{1} \notin I$ and so, I is proper. We show that, I is an ideal in $B_1(X)$. Let $f, g \in I$. Then $\overline{X \setminus Z(f)}$ and

$\overline{X \setminus Z(g)}$ are both finite. Now $\overline{X \setminus Z(f-g)} \subseteq \overline{X \setminus Z(f)} \cup \overline{X \setminus Z(g)}$ implies that $\overline{X \setminus Z(f-g)}$ is finite. Hence $f - g \in I$. Similarly, $\overline{X \setminus Z(f.g)} \subseteq \overline{X \setminus Z(f)}$ for any $f \in I$ and $g \in B_1(X)$. So, $\overline{X \setminus Z(f.g)}$ is finite and hence $f.g \in I$. Therefore, I is an ideal in $B_1(X)$. We claim that I is free.
For any $p \in X$, consider $\chi_p : X \to \mathbb{R}$ given by

$$\chi_p(x) = \begin{cases} 1 & \text{if } x = p \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 3.7, χ_p is a Baire one function. Also, $\overline{X \setminus Z(\chi_p)} = \overline{X \setminus (X \setminus \{p\})} = \overline{\{p\}} = \{p\} =$ finite and $\chi_p(p) \neq 0$. Hence, I is free. □

4. Residue class ring of $B_1(X)$ modulo ideals

An ideal I in a partially ordered ring A is called **convex** if for all $a, b, c \in A$ with $a \leq b \leq c$ and $c, a \in I \implies b \in I$. Equivalently, for all $a, b \in A, 0 \leq a \leq b$ and $b \in I \implies a \in I$.
If A is a lattice ordered ring then an ideal I of A is called **absolutely convex** if for all $a, b \in A$, $|a| \leq |b|$ and $b \in I \implies a \in I$.

Example 4.1. If $t : B_1(X) \to B_1(Y)$ is a ring homomorphism, then $\ker t$ is an absolutely convex ideal.

Proof. Let $g \in \ker t$ and $|f| \leq |g|$, where $f \in B_1(X)$. $g \in \ker t \implies t(g) = 0 \implies t(|g|) = |t(g)| = 0$. Since any ring homomorphism $t : B_1(X) \to B_1(Y)$ preserves the order, $t(|f|) = 0 \implies |t(f)| = 0 \implies t(f) = 0 \implies f \in \ker t$. □

Let I be an ideal in $B_1(X)$. In what follows we shall denote any member of the quotient ring $B_1(X)/I$ by $I(f)$ for $f \in B_1(X)$. i.e., $I(f) = f + I$. Now we begin with two well known theorems.

Theorem 4.2 ([3])**.** *Let I be an ideal in a partially ordered ring A. The corresponding quotient ring A/I is a partially ordered ring if and only if I is convex, where the partial order is given by $I(a) \geq 0$ iff $\exists\, x \in A$ such that $x \geq 0$ and $a \equiv x(\mod I)$.*

Theorem 4.3 ([3])**.** *On a convex ideal I in a lattice-ordered ring A the following conditions are equivalent.*

1. *I is absolutely convex.*
2. *$x \in I$ implies $|x| \in I$.*
3. *$x, y \in I$ implies $x \vee y \in I$.*
4. *$I(a \vee b) = I(a) \vee I(b)$, whence A/I is a lattice ordered ring.*
5. *$\forall\, a \in A$, $I(a) \geq 0$ iff $I(a) = I(|a|)$.*

Remark 4.4. For an absolutely convex ideal I of A, $I(|a|) = I(a \vee -a) = I(a) \vee I(-a) = |I(a)|$, $\forall\, a \in A$.

Theorem 4.5. *Every Z_B-ideal in $B_1(X)$ is absolutely convex.*

Proof. Suppose I is any Z_B-ideal and $|f| \leq |g|$, where $g \in I$ and $f \in B_1(X)$. Then $Z(g) \subseteq Z(f)$. Since $g \in I$, it follows that $Z(g) \in Z_B[I]$, hence $Z(f) \in Z_B[I]$. Now I being a Z_B-ideal, $f \in I$. □

Corollary 4.6. *In particular every maximal ideal in $B_1(X)$ is absolutely convex.*

Theorem 4.7. *For every maximal ideal M in $B_1(X)$, the quotient ring $B_1(X)/M$ is a lattice ordered field.*

Proof. Proof is immediate. □

The following theorem is a characterization of the non-negative elements in the lattice ordered ring $B_1(X)/I$, where I is a Z_B-ideal.

Theorem 4.8. *Let I be a Z_B-ideal in $B_1(X)$. For $f \in B_1(X)$, $I(f) \geq 0$ if and only if there exists $Z \in Z_B[I]$ such that $f \geq 0$ on Z.*

Proof. Let $I(f) \geq 0$. By condition (5) of Theorem 4.3, we write $I(f) = I(|f|)$. So, $f - |f| \in I \implies Z(f - |f|) \in Z_B[I]$ and $f \geq 0$ on $Z(f - |f|)$.
Conversely, let $f \geq 0$ on some $Z \in Z_B[I]$. Then $f = |f|$ on $Z \implies Z \subseteq Z(f - |f|) \implies Z(f - |f|) \in Z_B[I]$. I being a Z_B-ideal we get $f - |f| \in I$, which means $I(f) = I(|f|)$. But $|f| \geq 0$ on Z gives $I(|f|) \geq 0$. Hence, $I(f) \geq 0$. □

Theorem 4.9. *Let I be any Z_B-ideal and $f \in B_1(X)$. If there exists $Z \in Z_B[I]$ such that $f(x) > 0$, for all $x \in Z$, then $I(f) > 0$.*

Proof. By Theorem 4.8, $I(f) \geq 0$. But $Z \cap Z(f) = \varnothing$ and $Z \in Z_B[I] \implies Z(f) \notin Z_B[I] \implies f \notin I \implies I(f) \neq 0 \implies I(f) > 0$. □

The next theorem shows that the converse of the above theorem holds if the ideal is a maximal ideal in $B_1(X)$.

Theorem 4.10. *Let M be any maximal ideal in $B_1(X)$ and $M(f) > 0$ for some $f \in B_1(X)$ then there exists $Z \in Z_B[M]$ such that $f > 0$ on Z.*

Proof. By Theorem 4.8, there exists $Z_1 \in Z_B[M]$ such that $f \geq 0$ on Z_1. Now $M(f) > 0 \implies f \notin M$ which implies that there exists $g \in M$, such that $Z(f) \cap Z(g) = \varnothing$. Choosing $Z = Z_1 \cap Z(g)$, we observe $Z \in Z_B[M]$ and $f(x) > 0$, for all $x \in Z$. □

Corollary 4.11. *For a maximal ideal M of $B_1(X)$ and for some $f \in B_1(X)$, $M(f) > 0$ if and only if there exists $Z \in Z_B[M]$ such that $f(x) > 0$ on Z.*

Now we show Theorem 4.10 doesn't hold for every non-maximal ideal I.

Theorem 4.12. *Suppose I is any non-maximal Z_B-ideal in $B_1(X)$. There exists $f \in B_1(X)$ such that $I(f) > 0$ but f is not strictly positive on any $Z \in Z_B[I]$.*

Proof. Since I is non-maximal, there exists a proper ideal J of $B_1(X)$ such that $I \subsetneq J$. Choose $f \in J \setminus I$. $f^2 \notin I \implies I(f^2) > 0$. Choose any $Z \in Z_B[I]$. Certainly, $Z \in Z_B[J]$ and so, $Z \cap Z(f^2) \in Z_B[J] \implies Z \cap Z(f^2) \neq \varnothing$. So f is not strictly positive on the whole of Z. □

In what follows, we characterize the ideals I in $B_1(X)$ for which $B_1(X)/I$ is a totally ordered ring.

Theorem 4.13. *Let I be a Z_B-ideal in $B_1(X)$, then the lattice ordered ring $B_1(X)/I$ is totally ordered ring if and only if I is a prime ideal.*

Proof. $B_1(X)/I$ is a totally ordered ring if and only if for any $f \in B_1(X)$, $I(f) \geq 0$ or $I(-f) \leq 0$ if and only if for all $f \in B_1(X)$, there exists $Z \in Z_B[I]$ such that f does not change its sign on Z if and only if I is a prime ideal (by Theorem 2.16). □

Corollary 4.14. *For every maximal ideal M in $B_1(X)$, $B_1(X)/M$ is a totally ordered field.*

Theorem 4.15. *Let M be a maximal ideal in $B_1(X)$. The function $\Phi : \mathbb{R} \to B_1(X)/M$ (respectively, $\Phi : \mathbb{R} \to B_1^*(X)/M$) defined by $\Phi(r) = M(\mathbf{r})$, for all $r \in \mathbb{R}$, where $\mathbf{r}$ denotes the constant function with value r, is an order preserving monomorphism.*

Proof. It is clear from the definitions of addition and multiplication of the residue class ring $B_1(X)/M$ that the function is a homomorphism.
To show ϕ is injective. Let $M(\mathbf{r}) = M(\mathbf{s})$ for some $r, s \in \mathbb{R}$ with $r \neq s$. Then $\mathbf{r} - \mathbf{s} \in M$. This contradicts to the fact that M is a proper ideal. Hence $M(\mathbf{r}) \neq M(\mathbf{s})$, when $r \neq s$.
Let $r, s \in \mathbb{R}$ with $r > s$. Then $r - s > 0$. The function $\mathbf{r} - \mathbf{s}$ is strictly positive on X. Since $X \in Z(B_1(X))$, by Theorem 4.9, $M(\mathbf{r} - \mathbf{s}) > 0 \implies M(\mathbf{r}) > M(\mathbf{s}) \implies \Phi(r) > \Phi(s)$. Thus Φ is an order preserving monomorphism. □

For a maximal ideal M in $B_1(X)$, the residue class field $B_1(X)/M$ (respectively $B_1^*(X)/M$) can be considered as an extension of the field $\mathbb{R}$.

Definition 4.16. The maximal ideal M of $B_1(X)$ (respectively, $B_1^*(X)$) is called real if $\Phi(\mathbb{R}) = B_1(X)/M$ (respectively, $\Phi(\mathbb{R}) = B_1^*(X)/M$) and in such case $B_1(X)/M$ is called **real** residue class field. If M is not real then it is called **hyper-real** and $B_1(X)/M$ is called hyper-real residue class field.

Definition 4.17 ([3])**.** A totally ordered field F is called **archimedean** if given $\alpha \in F$, there exists $n \in \mathbb{N}$ such that $n > \alpha$. If F is not archimedean then it is called **non-archimedean**.

If F is a non-archimedean ordered field then there exists some $\alpha \in F$ such that $\alpha > n$, for all $n \in \mathbb{N}$. Such an α is called an infinitely large element of F and $\frac{1}{\alpha}$ is called infinitely small element of F which is characterized by the relation $0 < \frac{1}{\alpha} < \frac{1}{n}$, $\forall\, n \in \mathbb{N}$. The existence of an infinitely large (equivalently, infinitely small) element in F assures that F is non-archimedean.

In the context of archimedean field, the following is an important theorem available in the literature.

Theorem 4.18 ([3]). *A totally ordered field is archimedean iff it is isomorphic to a subfield of the ordered field* $\mathbb{R}$ *.*

We thus get that the real residue class field $B_1(X)/M$ is archimedean if M is a real maximal ideal of $B_1(X)$.

Theorem 4.19. *Every hyper-real residue class field* $B_1(X)/M$ *is non-archimedean.*

Proof. Proof follows from the fact that the identity is the only non-zero homomorphism on the ring $\mathbb{R}$ into itself. □

Corollary 4.20. *A maximal ideal* M *of* $B_1(X)$ *is hyper-real if and only if there exists* $f \in B_1(X)$ *such that* $M(f)$ *is an infinitely large member of* $B_1(X)/M$*.*

Theorem 4.21. *Each maximal ideal* M *in* $B_1^*(X)$ *is real.*

Proof. It is equivalent to show that $B_1^*(X)/M$ is archimedean. Choose $f \in B_1^*(X)$. Then $|f(x)| \leq n$, for all $x \in X$ and for some $n \in \mathbb{N}$. i.e., $|M(f)| = M(|f|) \leq M(\mathbf{n})$. So there does not exist any infinitely large member in $B_1^*(X)/M$ and hence $B_1^*(X)/M$ is archimedean. □

Corollary 4.22. *If* X *is a topological space such that* $B_1(X) = B_1^*(X)$ *then each maximal ideal in* $B_1(X)$ *is real.*

The following theorem shows how an unbounded Baire one function f on X is related to an infinitely large member of the residue class field $B_1(X)/M$.

Theorem 4.23. *Given a maximal ideal* M *of* $B_1(X)$ *and* $f \in B_1(X)$*, the following statements are equivalent:*

(1) $|M(f)|$ *is infinitely large member in* $B_1(X)/M$*.*
(2) f *is unbounded on each zero set in* $Z_B[M]$*.*
(3) *for all* $n \in \mathbb{N}$*,* $Z_n = \{x \in X : |f(x)| \geq n\} \in Z_B[M]$*.*

Proof. (1) $\iff$ (2): $|M(f)|$ is not infinitely large in $B_1(X)/M$ if and only if $\exists\ n \in \mathbb{N}$ such that, $|M(f)| = M(|f|) \leq M(\mathbf{n})$ if and only if $|f| \leq \mathbf{n}$ on some $Z \in Z_B[M]$ if and only if f is bounded on some $Z \in Z_B[M]$.
(2) $\implies$ (3): Choose $n \in \mathbb{N}$, we shall show that $Z_n \in Z_B[M]$. By (2), Z_n intersects each member in $Z_B[M]$. Now $Z_B[M]$ being a Z_B-ultrafilter, $Z_n \in Z_B[M]$.
(3) $\implies$ (2): Let each $Z_n \in Z_B[M]$, for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $|f| \geq n$ on some zero set in $Z_B[M]$. Hence $|M(f)| = M(|f|) \geq M(\mathbf{n})$, for all $n \in \mathbb{N}$. That means $|M(f)|$ is infinitely large member in $B_1(X)/M$. □

Theorem 4.24. $f \in B_1(X)$ *is unbounded on* X *if and only if there exists a maximal ideal* M *in* $B_1(X)$ *such that* $M(f)$ *is infinitely large in* $B_1(X)/M$*.*

Proof. Let f be unbounded on X. So, each Z_n in Theorem 4.23 is non-empty. We observe that $\{Z_n : n \in \mathbb{N}\}$ is a subcollection of $Z(B_1(X))$ having finite

intersection property. So there exists a Z_B-ultrafilter $\mathscr{U}$ on X such that $\{Z_n : n \in \mathbb{N}\} \subseteq \mathscr{U}$. Therefore, there is a maximal ideal M in $B_1(X)$ for which $\mathscr{U} = Z_B[M]$ and so, $Z_n \in Z_B[M]$, for all $n \in \mathbb{N}$. By Theorem 4.23 $M(f)$ is infinitely large.
Converse part is a consequence of (1) $\implies$ (2) of Theorem 4.23. □

Corollary 4.25. *If a completely Hausdorff space X is not totally disconnected then there exists a hyper-real maximal ideal M in $B_1(X)$.*

Proof. It is enough to prove that there exists an unbounded Baire one function in $B_1(X)$. We know that if a completely Hausdorff space is not totally disconnected, then there always exists an unbounded Baire one function [1]. □

In the next theorem we characterize the real maximal ideals of $B_1(X)$.

Theorem 4.26. *For the maximal ideal M of $B_1(X)$ the following statements are equivalent:*

(1) *M is a real maximal ideal.*
(2) *$Z_B[M]$ is closed under countable intersection.*
(3) *$Z_B[M]$ has countable intersection property.*

Proof. (1) $\implies$ (2): Assume that (2) is false, i.e., there exists a sequence of functions $\{f_n\}$ in M for which $\bigcap\limits_{n=1}^{\infty} Z(f_n) \notin Z_B[M]$. Set $f(x) = \sum\limits_{n=1}^{\infty} \big(|f_n(x)| \wedge \frac{1}{4^n}\big)$, $\forall x \in X$. It is clear that, the function f defined on X is actually a Baire one function ([1]) and $Z(f) = \bigcap\limits_{n=1}^{\infty} Z(f_n)$. Thus, $Z(f) \notin Z_B[M]$. Hence $f \notin M \implies M(f) > 0$ in $B_1(X)/M$.
Fix a natural number m. Then $Z(f_1) \bigcap Z(f_2) \bigcap Z(f_3) ... \bigcap Z(f_m) = Z(\text{say}) \in Z_B[M]$. Now for any point $x \in Z$, $f(x) = \sum\limits_{n=m+1}^{\infty} \big(|f_n(x)| \wedge \frac{1}{4^n}\big) \leq \sum\limits_{n=m+1}^{\infty} \frac{1}{4^n} = 3^{-1}4^{-m}$. This shows that, $0 < M(f) \leq M(3^{-1}4^{-m})$, $\forall\, m \in \mathbb{N}$. Hence $M(f)$ is an infinitely small member in $B_1(X)/M$. So, M becomes a hyper-real maximal ideal and then (1) is false.
(2) $\implies$ (3): Trivial, as $\varnothing \notin Z_B[M]$.
(3) $\implies$ (1): Assume that (1) is false, i.e. M is hyper-real. So, there exists $f \in B_1(X)$ so that $|M(f)|$ is infinitely large in $B_1(X)/M$. Therefore for each $n \in \mathbb{N}$, Z_n defined in Theorem 4.23, belongs to $Z_B[M]$. Since $\mathbb{R}$ is archimedean, we have $\bigcap\limits_{n=1}^{\infty} Z_n = \varnothing$. Thus (3) is false. □

So far we have seen that for any topological space X, all fixed maximal ideals of $B_1(X)$ are real. Though the converse is not assured in general, we show in the next example that in $B_1(\mathbb{R})$ a maximal ideal is real if and only if it is fixed.

Example 4.27. Suppose M is any real maximal ideal in $B_1(\mathbb{R})$. We claim that M is fixed. The identity $i : \mathbb{R} \to \mathbb{R}$ belongs to $B_1(\mathbb{R})$. Since M is a real maximal ideal, there exists a real number r such that $M(i) = M(\mathbf{r})$. This implies $i - \mathbf{r} \in M$. Hence $Z(i - \mathbf{r}) \in Z_B[M]$. But $Z(i - \mathbf{r})$ is a singleton. So, $Z_B[M]$ is fixed, i.e., M is fixed.

In view of Observation 3.8(3), we conclude that a maximal ideal M in $B_1(\mathbb{R})$ is real if and only if there exists a unique $p \in \mathbb{R}$ such that $\chi_p - 1 \in M$.

If X is a P-space then $C(X)$ possesses real free maximal ideals. In such case however, $B_1(X) = C(X)$. Consequently, $B_1(X)$ possesses real free maximal ideals, when X is a P-space. It is still a natural question, what are the topological spaces X for which $B_1(X)$ ($\supseteq C(X)$) contains a free real maximal ideal?

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