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# On ideal sequence covering maps

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Abstract

*In this paper we introduce the concept of ideal sequence covering map which is a generalization of sequence covering map, and investigate some of its properties. The present article contributes to the problem of characterization to the certain images of metric spaces which was posed by Y. Tanaka [22], in more general form. The entire investigation is performed in the setting of ideal convergence extending the recent results in [11, 15, 16].*

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## 1. Introduction

In the past 50 years, papers and surveys on discussion for theory of images of metric spaces were published in large amounts. This issue has become a typical research direction in developments of general topology, which has made outstanding contributions for progress and prosperity of this subject. Taking spaces as images of metric spaces, we deal with structures of metric spaces. In 1971, Swiec [20, 21] introduced the concept of sequence covering maps which is closely related to the question about compact covering and $s$-images of metric spaces (see also [4]). In [11] Lin discuss about sequence covering maps and its properties also solved many open problem related to this concept.

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The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [3] and Schoenberg [19] and its topological consequences were studied first by Fridy [5] and Salat [6]. It seems more appropriate to follow the more general approach of [6] where the notion of $\mathcal{I}$-convergence of sequence was introduced by using the ideas of ideal of the set of positive integers. In [15, 16] Renukadevi studied some of the results of Lin in statistical format.
As a natural consequences, in the present paper we continue the investigation proposed in [11] and study similar problems in more general form. In Section 3.1 we introduce the concepts of ideal sequence covering map and ideal sequence covering compact map and study some of its properties. Also in Section 3.2 we propose the concept of ideal sequentially quotient map and examine related properties. The entire investigation is performed in the setting of ideal convergence extending the recent results in [11, 14, 15, 16].

## 2. Preliminaries

We start by recalling the basic definition of ideals and filters. A family $\mathcal{I} \subset 2^Y$ of subsets of a non- empty set $Y$ is said to be an ideal in $Y$ if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}$ and $B \subset A$ imply $B \in \mathcal{I}$. Further an admissible ideal $\mathcal{I}$ of $Y$ satisfies $\{x\} \in Y$ for each $x \in Y$. Such ideals are called free ideals. If $\mathcal{I}$ is a proper non-trivial ideal in $Y$ (i.e. $Y \notin \mathcal{I}$ , $\mathcal{I} \neq \varnothing$ ), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \setminus A\}$ is called the filter associated with the ideal $\mathcal{I}$. $\mathcal{I}_{fin} = \{A \subset \mathbb{N} : A \; is \; finite\}$. It is an ideal and $\mathcal{I}_{fin}$-convergence implies original convergence. Now density of a subset $A$ of $\mathbb{N}$ is $d(A) = \lim_{n\to\infty} \frac{1}{n}|\{k : k \in A, k \leq n\}|$, provided the limit exists. $\mathcal{I}_d = \{A : A \subset \mathbb{N}, d(A) = 0\}$ is an ideal and $\mathcal{I}_d-$convergence implies the notion of statistical convergence (see [7, 12, 17, 26]). Throughout all topological spaces are assumed to be Hausdorff, all maps are onto and continuous and $\mathbb{N}$ stands for the set of all natural number. Let $X$ be a topological space and $P \subset X$. A sequence $\{x_n\}$ converging to $x$ in $X$ is eventually in $P$ if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in $P$ if $\{x_{n_k}\}$ is eventually in $P$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Throughout by a space we will mean a topological space, unless otherwise mentioned. Let us recall some definitions.

**Definition 2.1** ([11]). Let $X$ be a space and $P \subset X$.

(a) Let $x \in P$. Then $P$ is called a *sequential neighbourhood* of $x$ in $X$ if whenever $\{x_n\}$ is a sequence converging to $x$, then $\{x_n\}$ is eventually in $P$.
(b) $P$ is called a *sequentially open* subset in $X$ if $P$ is a sequential neighbourhood of $x$ in $X$ for each $x \in P$.

**Definition 2.2** ([9]). Let $X$ be a space, and let $\mathcal{P}$ be a cover of $X$.

(1) $\mathcal{P}$ is a cs-cover of $X$, if for any convergent sequence $S$ in $X$, there exists $P \in \mathcal{P}$ such that S is eventually in P.

(2) $\mathcal{P}$ is an sn-cover of $X$, if each element of $\mathcal{P}$ is a sequential neighbourhood of some point of $x$ and for each $x \in X$, there exists $P \in \mathcal{P}$ such that P is the sequential neighbourhood of $x$.

**Definition 2.3** ([20]). A space $X$ is strongly Fréchet if whenever $\{A_n | n \in \mathbb{N}\}$ is a decreasing sequence of sets in $X$ and $x$ is a point which is in the closure of each $A_n$, then for each $n \in \mathbb{N}$ there exists an element $x_n \in A_n$ such that the sequence $x_n \to x$.

**Definition 2.4** ([23]). A space $X$ is said to have property $\omega$D if every countably infinite discrete subset has an infinite subset A such that there exists a discrete open family $\{U_x | x \in A\}$ with $U_x \cap A = \{x\}$ for each $x \in A$.

**Definition 2.5** ([1]). A class of mappings is said to be hereditary if whenever $f : X \to Y$ is in the class, then for each subspace H of $Y$, the restriction of f to $f^{-1}(H)$ is in the class.

**Definition 2.6** ([20]). Let $f : X \to Y$ be a mapping.

(a) $f$ is a sequence covering map if for every convergent sequence $\{y_n\}$ in $Y$, there is a convergent sequence $\{x_n\}$ in $X$ with each $x_n \in f^{-1}(y_n)$.

(b) $f$ is a 1-sequence covering map if for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$, then there is a sequence $\{x_n\}$ in $X$ converging to $x$ with each $x_n \in f^{-1}(y_n)$.

Clearly every 1-sequence covering map is a sequence covering map. But the converse is not true, which is shown by the following example.

**Example 2.7.** Suppose $\{y_n\}$ is a convergent sequence of real numbers with its limit $y$. Let $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ and $\bigwedge = \{\alpha : \alpha : \mathbb{N} \to \mathbb{N}\}$ where $\alpha$ be an increasing mapping. Define $Y_\alpha = \{(y_{\alpha(k)}, \alpha) : k \in \mathbb{N}\} \cup \{(y, \alpha)\}$ for each $\alpha \in \bigwedge$. The metric $d_\alpha$ on $Y_\alpha$ is defined by $d_\alpha((p, \alpha), (q, \alpha)) = |p - q|$. Take $X = \bigoplus_{\alpha \in \bigwedge} Y_\alpha$ and the metric $D$ on $X$ is defined as $D(x_1, x_2) = \min\{d_\alpha(x_1, x_2), 1\}$ when $x_1, x_2 \in Y_\alpha$ for some $\alpha \in \bigwedge$ and $D(x_1, x_2) = 1$ when $x_1 \in Y_\alpha, x_2 \in Y_\beta$ for distinct $\alpha, \beta \in \bigwedge$. Moreover, the Topology of $Y$ is considered as: each $\{y_n\}$ is open and a basic open set $U$ containing $y$ is of the form $\{y\} \cup \{y_n : n \geq n_0\}$ for some $n_0$.
Now define a map $f : X \to Y$ by $f(y_{\alpha(k)}, \alpha) = y_{\alpha(k)}$ and $f(y, \alpha) = y$ for each $\alpha \in \bigwedge$. Clearly $f$ is continuous. Any convergent sequence $\{z_n\}$ in $Y$ is of the form $z_{s(k)} = y_{\alpha(k)}$ for $n = s(k), \alpha \in \bigwedge$ and $z_n = y$ otherwise. Then choose $r_n = (y_{\alpha(k)}, \alpha)$ for $n = s(k)$ and $r_n = (y, \alpha)$ otherwise. Clearly $r_n \in f^{-1}(z_n)$ and $r_n \to (y, \alpha) \in f^{-1}(y)$. So $f$ is a sequence covering map. But for $y \in Y$ choose any $(y, \alpha) \in f^{-1}(y)$. Then there is $y_{\beta(k)} \to y,\ \beta \neq \alpha$ but $(y_{\beta(k)}, \beta)$ does not converge to $(y, \alpha)$. So $f$ is not 1-sequence covering map.

**Definition 2.8** ([2]). If $X$ is a space that can be mapped onto a metric space by a one-to-one mapping, then $X$ is said to have weaker metric topology.

**Definition 2.9** ([6]). A sequence $\{x_n\}$ in a space $X$ is said to be $\mathcal{I}-$convergent to $x \in X$ (i.e $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ ) if for every neighbourhood $U$ of $x$, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

**Lemma 2.10** ([25]). *Let $X$ be a space with a weaker metric topology. Then there is a sequence $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of $X$ such that $\cap_{n\in\mathbb{N}} st(K, \mathcal{P}_i) = K$ for each compact subset $K$ of $X$.*

**Definition 2.11** ([10]). Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$ such that for each $x \in X$, following conditions $(a)$ and $(b)$ are satisfied:

(a) $\mathcal{P}_x$ is a *network* at $x$ in $X$, i.e., $x \in \cap\mathcal{P}_x$ and for each neighbourhood $U$ of $x$ in $X$, $P \subset U$ for some $P \in \mathcal{P}_x$.
(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$(i)$ $\mathcal{P}$ is called a *sn-network* of $X$ if each element of $\mathcal{P}_x$ is a sequential neighbourhood of $x$ for each $x \in X$, where $\mathcal{P}_x$ is called a $sn$-network at $x$ in $X$. A space $Y$ is termed as an $snf$-countable if $Y$ has a $sn$-network $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ such that $\mathcal{P}_y$ is countable and closed under finite intersections for each $y \in Y$.

$(ii)$ $\mathcal{P}$ is called a *weak base* of $X$ if whenever $G \subset X$, $G$ is open in $X$ if and only if for each $x \in G$, there exists a $P \in \mathcal{P}_x$ such that $P \subset G$.

**Definition 2.12** ([13]). Let $A \subset X$ and let $\mathcal{O}$ be a family of subsets of $X$. Then $\mathcal{O}$ is an *external base* of $A$ in $X$ if for each $x \in A$ and an open set $U$ with $x \in U$ there is a $V \in \mathcal{O}$ such that $x \in V \subset U$.

**Definition 2.13** ([9]). Let $f : X \to Y$ be a map.

(a) $f$ is a *boundary compact map* if $\partial f^{-1}(y)$ is compact in $X$ for each $y \in Y$.
(b) $f$ is called *sequentially quotient* if for each convergent sequence $\{y_n\}$ in $Y$ there is a convergent sequence $\{x_k\}$ in $X$ with $f(x_k) = y_{n_k}$ for each $k$, where $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$. See also [14, 16] for more details.

**Definition 2.14** ([9]). Let $X = \{0\}\cup(\mathbb{N}\times\mathbb{N})$. For every $n,\ m \in \mathbb{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$, let $W(n, m) = \{(n, k) \in \mathbb{N} \times \mathbb{N} : k \geq m\}$, and $L(f) = \cup\{W(n, f(n)) : n \in \mathbb{N}\}$. Then the set $X$ with the following topology is called a sequential fan and denoted briefly as $S_\omega$ : for each $x \in X$, take $\mathcal{N}_x = \{\{x\}\},\ if\ x \in \mathbb{N} \times \mathbb{N}$. $\mathcal{N}_x = \{\{x\} \cup L(f) : f \in \mathbb{N}^{\mathbb{N}}\}$, if $x = 0$ as a neighbourhood base of $x$.

*Remark* 2.15 ([9]). Sequential fan $S_\omega$ is the space obtained by identifying the limits of countably many convergent sequences.

**Definition 2.16** ([18]). Let $\mathcal{A}, \mathcal{B}$ be two non-empty collections of families of subsets of an infinite set $X$. Then $\mathcal{S}_1(\mathcal{A}, \mathcal{B})$ is defined as: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

## 3. Main results

3.1. $(\mathcal{I},\mathcal{J})$**-sequence covering map.** In this section we introduce a map namely, $(\mathcal{I},\mathcal{J})$-sequence covering map and investigate some of its properties.

**Definition 3.1.** Let $\mathcal{I}$ and $\mathcal{J}$ be two admissible ideals. Then

(a) $f : X \to Y$ is said to be a $(\mathcal{I},\mathcal{J})$-sequence covering map if for any given sequence $\{y_n\}$, $\mathcal{J}$-converging to $y$ in $Y$, there exists a sequence $\{x_n\}$, $\mathcal{I}$-convergent to $x$ in $X$ such that $x_n \in f^{-1}(y_n)$ for each $n$ and $x \in f^{-1}(y)$.

(b) $f : X \to Y$ is said to be a $(\mathcal{I},\mathcal{J})$-1-sequence covering map if for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence $\mathcal{J}$-converging to $y$, there exists a sequence $\{x_n\}$, which is $\mathcal{I}$-convergent to $x$ in $X$ such that $x_n \in f^{-1}(y_n)$ for each $n$.

These are generalization of sequence covering map and 1-sequence covering map respectively. Clearly every $(\mathcal{I},\mathcal{J})$-1-sequence covering map is a $(\mathcal{I},\mathcal{J})$-sequence covering map. But the converse is not true which is shown by the next example.

**Example 3.2.** $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$. Topology of $Y$ is defined by each $\{y_n\}$ is open and a basic open set $U$ containing $y$ equals to $\{y_n : n \geq n_0\} \cup \{y\}$ for some $n_0 \in \mathbb{N}$. Suppose $\{p_n\}$ is a convergent sequence of real numbers with its limit $p$. Define $x_n = p_n$ if $n \in M$ where $M \in \mathcal{F}(\mathcal{I})$ and $x_n = n$, otherwise. Clearly $\{x_n\}$, $\mathcal{I}$-converges to $p$. Let $\bigwedge = \{\alpha : \alpha : \mathbb{N} \longrightarrow \mathbb{N}$ is an increasing mapping $\}$. Take $X_\alpha = \{(x_k,\alpha) : k \in \mathbb{N}\} \cup \{(p,\alpha)\}$ and metric $d_\alpha$ on $X_\alpha$ is defined by $d_\alpha((m,\alpha),(n,\alpha)) = |m-n|$. $X = \bigoplus_{\alpha\in\bigwedge} X_\alpha$ and the metric $D$ on $X$ is defined as $D(m_1,m_2) = \min\{d_\alpha(m_1,m_2),1\}$ when $m_1,m_2 \in X_\alpha$ for some $\alpha \in \bigwedge$ and $D(m_1,m_2) = 1$ when $m_1 \in Y_\alpha, m_2 \in Y_\beta$ for distinct $\alpha,\beta \in \bigwedge$. Now define a map $f : X \to Y$ by $f(x_k,\alpha) = y_{\alpha(k)}$ and $f(p,\alpha) = y$ for each $\alpha \in \bigwedge$. Clearly $f$ is continuous. Convergent sequence $\{z_n\}$ in $Y$ is of the form $z_{s(k)} = y_{\alpha(k)}$ $\alpha \in \bigwedge$ and $z_n = y$ for $n \in \{s(k) : k \in \mathbb{N}\}^c$. Then choose $r_{s(k)} = (x_k,\alpha)$ and $r_n = (p,\alpha)$ for $n \in \{s(k) : k \in \mathbb{N}\}^c$. Clearly $r_n \in f^{-1}(z_n)$ and $\{r_n\}$ $\mathcal{I}$-converges to $(p,\alpha) \in f^{-1}(y)$. So $f$ is $(\mathcal{I},\mathcal{J})$-sequence covering map. But for $y \in Y$ choose any $(p,\alpha) \in f^{-1}(y)$. Then there is $\{y_{\beta(k)}\} \to y,\ \beta \neq \alpha$ but $\{(x_k,\beta)\}$ does not converge to $(p,\alpha)$. So $f$ is not $(\mathcal{I},\mathcal{J})$-1-sequence covering map.

**Proposition 3.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be any two maps. Also let $\mathcal{I},\mathcal{J},\mathcal{K}$ be admissible ideals. Then the following hold:*

(a) *If $f$ is $(\mathcal{I},\mathcal{J})$-sequence covering map and $g$ is $(\mathcal{J},\mathcal{K})$-sequence covering map then $g \circ f$ is $(\mathcal{I},\mathcal{K})$-sequence covering map.*

(b) *If $g \circ f$ is $(\mathcal{I},\mathcal{K})$-sequence covering map then $g$ is $(\mathcal{I},\mathcal{K})$-sequence covering map.*

*Proof.* (a) Let $z \in Z$ and $\{z_n\}$ be a sequence $\mathcal{K}$-converging to $z$. Since $g$ is $(\mathcal{J},\mathcal{K})$-sequence covering map so there exists a sequence $\{y_n\}$, $\mathcal{J}$-converging to $y$ in $Y$ such that $y_n \in g^{-1}(z_n)$ and $y \in g^{-1}(z)$. Since $f$ is $(\mathcal{I},\mathcal{J})$-sequence

covering map so there exists a sequence $\{x_n\}$, $\mathcal{I}$-converging to $x$ in $X$ such that $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$. So we get $x_n \in (g \circ f)^{-1}(z_n)$ and $x \in (g \circ f)^{-1}(z)$. Hence $g \circ f$ is $(\mathcal{I}, \mathcal{K})$-sequence covering map.
(b) Since f is continuous $g$ is $(\mathcal{I}, \mathcal{K})$-sequence covering map. □

**Proposition 3.4.** *Let $\mathcal{I}, \mathcal{J}$ be two admissible ideals. Then,*

(a) *Finite product of $(\mathcal{I}, \mathcal{J})$-sequence covering mappings is $(\mathcal{I}, \mathcal{J})$-sequence covering map.*
(b) *$(\mathcal{I}, \mathcal{J})$-sequence covering mappings are hereditarily $(\mathcal{I}, \mathcal{J})$-sequence covering mappings.*

*Proof.* $(a)$ Let $\prod_{i=1}^{N} f_i : \prod_{i=1}^{N} X_i \to \prod_{i=1}^{N} Y_i$ be a map where each $f_i : X_i \to Y_i$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map for $i = 1, 2, ...N$. Let $\{(y_{i,n})\}_{n \in \mathbb{N}}$ be a sequence $\mathcal{J}$-converging to $(y_i)$ in $\prod_{i=1}^{N} Y_i$. Then $\{y_{i,n}\}$ $\mathcal{J}$-converging to $y_i$ in $Y_i$ for each $i = 1, 2, ...N$. Since each $f_i$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map there exists $\{x_{i,n}\}$ $\mathcal{I}$-converges to $x_i$ in $X_i$ such that $f_i(x_{i,n}) = y_{i,n}$ and $f_i(x_i) = y_i$, $i = 1, 2...N$. Now consider the sequence $\{(x_{i,n})\}_{n \in \mathbb{N}}$ which is $\mathcal{I}$-convergent to $(x_i)$. So $\prod_{i=1}^{N} f_i$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map.
$(b)$ Let $f : X \to Y$ be $(\mathcal{I}, \mathcal{J})$-sequence covering map and $H$ be subspace of $Y$. Take $g = f|_{f^{-1}(H)}$ such that $g : f^{-1}(H) \to H$ is a map. Let $\{y_n\}$ be a sequence $\mathcal{J}$-converges to $y$ in H. Then $\{y_n\}$, $\mathcal{J}$-converges to y in $Y$. Since $f$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map so there exists a sequence $\{x_n\}$, $\mathcal{I}$-converging to $x$ in $X$ such that $x_n \in f^{-1}(y_n) \subset f^{-1}(H)$ and $x \in f^{-1}(y) \subset f^{-1}(H)$. Hence $g$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map. □

**Lemma 3.5.** *Let $\bigwedge$ be any index set and let $X = \prod_{\alpha \in \bigwedge} X_\alpha$ has the product topology. Then $\{(x_{\alpha,i})\}_{i \in \mathbb{N}}$ is $\mathcal{I}$-converges to $(x_\alpha)$ if and only if sequence of each component $\{x_{\alpha,i}\}$ $\mathcal{I}$-converges to $x_\alpha$.*

*Proof.* Suppose $p_\alpha : X \to X_\alpha$ is a projection. So if $\{(x_{\alpha,i})\}_{i \in \mathbb{N}}$ $\mathcal{I}$-converges to $(x_\alpha)$ then $p_\alpha(\{(x_{\alpha,i})\}_{i \in \mathbb{N}}) = \{x_{\alpha,i}\}_{i \in \mathbb{N}}$ $\mathcal{I}$-converges to $x_\alpha$. Conversely assume that sequence of each component $\{x_{\alpha,i}\}$ $\mathcal{I}$-converges to $x_\alpha$. Let $U$ be a basic open set containing $(x_\alpha)$ then $p_\alpha(U) = X_\alpha$ for all $\alpha \in \bigwedge$ but finitely many. Take the finite set be $\{\alpha_1, \alpha_2, ...\alpha_k\}$. Since$\{x_{\alpha_j,i}\}$ $\mathcal{I}$-converges to $x_{\alpha_j}$ so we get $A_{\alpha_j} = \{i \in \mathbb{N} : x_{\alpha_j,i} \notin p_{\alpha_j}(U)\} \in \mathcal{I}$ for $j = 1, 2, ..k$. $\{i \in \mathbb{N} : (x_{\alpha,i}) \notin U\} \subset \cup_{j=1}^{k} A_{\alpha_j} \in \mathcal{I}$. Hence $\{(x_{\alpha,i})\}_{i \in \mathbb{N}}$ is $\mathcal{I}$-converges to $(x_\alpha)$. □

From Lemma 3.5 we can say that if $f : X = \prod_{\alpha \in \bigwedge} X_\alpha \to Y = \prod_{\alpha \in \bigwedge} Y_\alpha$ be the infinite product of $(\mathcal{I}, \mathcal{J})$-sequence covering mappings $f_\alpha : X_\alpha \to Y_\alpha$ and $X, Y$ has the product topology then $f$ is also $(\mathcal{I}, \mathcal{J})$-sequence covering map.

The following example shows that the concept of ideal sequence covering map generalizes the concept of statistical sequence covering map and hence sequence covering map.

**Example 3.6.** Let $\mathcal{F}$ be the set of all increasing mappings from $\mathbb{N}$ to $\mathbb{N}$. For each $f \in \mathcal{F}$ consider $S_f$ be a $\mathcal{I}$-convergent sequence with its $\mathcal{I}$-limit $x_f$. i.e

$S_f = \{x_{f,n} : n \in \mathbb{N}\} \cup \{x_f\}$. Topology of $S_f$ is defined as each $\{x_{f,n}\}$ is open and a basic open set $U$ containing $x_f$ is such that $A_u = \{n \in \mathbb{N} : x_{f,n} \notin U\} \in \mathcal{I}$ and there exists at least one $U$ such that $d(A_u) \neq 0$. Consider the topological sum $X = \oplus_{f \in \mathcal{F}} S_f$. Let $\{y_n\} \to y$ and $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$. Topology of $Y$ is defined as each $\{y_n\}$ is open and a basic open set $U$ containing $y$ equals to $\{y_n : n \geq n_0\} \cup \{y\}$ for some $n_0 \in \mathbb{N}$. Let $\phi : X \to Y$ defined as $\phi(x_{f,k}) = y_{f(k)}$, for all $k \in \mathbb{N}$ and $\phi(x_f) = y$. Consider any subsequence $\{y_{f(k)})\}$ of $\{y_n\}$. Let $z_k = y_{f(k)}$ then consider corresponding $x_{f,k}$ in $S_f$. Then $\{x_{f,n}\}$ $\mathcal{I}$-converging to $x_f$. Now $\phi(x_{f,k}) = y_{f(k)} = z_k$ and $\phi(x_f) = y$. Consider $\beta \in \mathcal{F}$ and $s : \mathbb{N} \to \mathbb{N}$ is an increasing function. Suppose $z_n = y_{\beta(k)}, n = s(k)$ *and* $z_n = y$, otherwise. Then clearly $z_n \to y$. Consider $p_{s(k)} = x_{\beta,k}$ *and* $p_n = x_\beta$, otherwise. Then $\phi(p_{s(k)}) = \phi(x_{\beta,k}) = y_{\beta(k)} = z_{s(k)}$ and $\phi(p_n) = y$ otherwise. Let $U$ be an open set containing $x_f$. Hence $\{n \in \mathbb{N} : p_n \notin U\} = \{s(k) : p_{s(k)} \notin U, k \in \mathbb{N}\}$. But $\{k \in \mathbb{N} : x_{\beta,k} \notin U\} \in \mathcal{I}$. If $\mathcal{I}$ has increasing function property i.e $B \in \mathcal{I} \Rightarrow f(B) \in \mathcal{I}$ for each increasing function $f$ from $\mathbb{N}$ to $\mathbb{N}$ then $\{p_n\}$ $\mathcal{I}$-converging to $x_f$. So $\phi$ is $(\mathcal{I}, \mathcal{I}_{fin})-$sequence covering map but not sequence covering map.

**Theorem 3.7.** *Let $f : X \to Y$ be a $(\mathcal{I}, \mathcal{J})-$sequence covering compact map where $\mathcal{I}, \mathcal{J}$ be two admissible ideals and suppose there exists a sequence $\{M_n\}$ of disjoint infinite subset of $\mathbb{N}$ such that $M_n \notin \mathcal{I}$ for each $n$. Then for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that whenever $U$ is an open neighbourhood of $x$, $f(U)$ is a sequential neighbourhood of $y$.*

*Proof.* Suppose not, that is there exists $y \in Y$ such that for every $x \in f^{-1}(y)$ there exists an open neighbourhood $U_x$ of $x$ such that $f(U_x)$ is not a sequential neighbourhood of y. Since $f^{-1}(y) \subset \cup_{x \in f^{-1}(y)} U_x$ and $f$ is a compact map, so there exists $x_1, x_2, \ldots, x_{n_0} \in f^{-1}(y)$ such that $f^{-1}(y) \subset \cup_{i=1}^{n_0} U_{x_i}$. Since each $f(U_{x_m})$ is not a sequential neighbourhood of $y$, choose $\{y_{m,n}\}_{n=1}^{\infty}$ converging to $y$ such that $y_{m,n} \notin f(U_{x_m})$ for all $m \in \{1, 2, ...n_0\}, n \in \mathbb{N}$. Now define $y_k = y_{m,k}$ if $k \in M_m, m \in \{1, 2, ...n_0\}$ and $y_k = y$, otherwise. Then clearly $y_k \to y$, which shows that $\{y_k\}$, $\mathcal{J}$-converging to $y$. Since $f$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map so there exists $\{x_k\}$, $\mathcal{I}$-converging to $x$ such that $x_k \in f^{-1}(y_k)$ and $x \in f^{-1}(y)$. Now $x \in f^{-1}(y) \subset \cup_{i=1}^{n_0} U_{x_i}$. So there exists $m_0$ such that $x \in U_{x_{m_0}}$ and $\{k \in \mathbb{N} : x_k \notin U_{x_{m_0}}\} \in \mathcal{I}$. Thus $\{k \in \mathbb{N} : f(x_k) \notin f(U_{x_{m_0}})\} \in \mathcal{I}$ which shows that $\{k \in \mathbb{N} : y_k \notin f(U_{x_{m_0}})\} \in \mathcal{I}$. But $M_{m_0} \subset \{k \in \mathbb{N} : y_k \notin f(U_{x_{m_0}})\} \in \mathcal{I}$, which contradicts that $M_{m_0} \notin \mathcal{I}$. Thus $f(U)$ is a sequential neighbourhood of $y$. □

**Theorem 3.8.** *Let $\mathcal{I}$ be an admissible ideal and suppose there exists a sequence $\{M_n\}$ of disjoint infinite subset of $\mathbb{N}$ such that $M_n \notin \mathcal{I}$ for each $n$. Then the following conditions are equivalent for a space $Y$ :*

(a) *$Y$ is a $(\mathcal{I}, \mathcal{I}_{fin})$-1-sequence covering compact image of a weaker metric topology.*
(b) *$Y$ is a $(\mathcal{I}, \mathcal{I}_{fin})$-sequence covering compact image of a weaker metric topology.*

(c) *$Y$ has a sequence $\{\mathcal{F}_i\}_{i\in\mathcal{N}}$ of point-finite sn-covers such that $\cap_{i\in\mathbb{N}}st(y,\mathcal{F}_i) = \{y\}$ for each $y \in Y$.*
(d) *$Y$ has a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs-covers such that $\cap_{i\in\mathbb{N}}st(y,\mathcal{F}_i) = \{y\}$ for each $y \in Y$.*

*Proof.* It is clear that $(a) \Rightarrow (b), (c) \iff (d), (c) \Rightarrow (a)$.[15]
$(b) \Rightarrow (c)$ Suppose $f : X \to Y$ is a $(\mathcal{I},\mathcal{I}_f)-$sequence covering compact mapping. As $X$ being a space with a weaker metric topology, there is a sequence $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ of locally finite open covers of $X$ such that $\cap_{i\in\mathbb{N}}st(K,\mathcal{P}_i) = K$ for each compact subset $K \subset X$, by Lemma 3.5. For each $i \in \mathbb{N}$, put $\mathcal{F}_i = f(\mathcal{P}_i)$. Then $\mathcal{F}_i$ is a point finite cover of $Y$, since $f$ is compact. By Theorem 3.7 for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that for every open neighbourhood $U_x$ of $x$, $f(U_x)$ is a sequential neighbourhood of $y$. Since each $\mathcal{P}_i$ is an open cover of $X$, there exists $P \in \mathcal{P}_i$ such that $x \in P$ and so $F = f(P)$ is a sequential neighbourhood of $y$. Choose $\mathcal{F}'_i \subset \mathcal{F}_i$ which are sequential neighbourhood of $y$. Thus $\mathcal{F}'_i$ is a point finite sn-cover of $Y$. For each $y \in Y$, $f^{-1}(y)$ is compact subset of $X$ and $\cap_{i\in\mathbb{N}}st(f^{-1}(y),\mathcal{P}_i) = f^{-1}(y)$. Thus $\cap_{i\in\mathbb{N}}st(y,\mathcal{F}_i) = \{y\}$. □

**Theorem 3.9.** *Let $X$ be a strongly Fréchet space with property $\omega D$. If $f : X \to Y$ is a closed and $(\mathcal{I},\mathcal{J})-$sequence covering map, then $Y$ is strongly Fréchet.*

*Proof.* Clearly $Y$ is a Fréchet space, since it is a closed image of a strongly Fréchet. Suppose $Y$ is not strongly Fréchet. Then $Y$ contains a homomorphic copy of the sequential fan $S_\omega$ and the copy can be closed in $Y$ [9]. Consider $S_\omega \subset Y$ as a closed set where $S_\omega = \{y\} \cup \{y_{m,n} : m,n \in \omega\}$. Take each $S_m = \{y_{m,n}\}_{n\in\omega}$ is a sequence converges to $y$. Take $A \in \mathcal{F}(\mathcal{I})$. For each $m \in \mathbb{N}$, choose $y_{m_k} = y_{0,k}$ if $k \in A^c$ and $y_{m_k} = y_{m,k}$, if $k \in A$. Then the sequence $\{y_{m_k}\}$ converges to $y$. Since $f$ is $(\mathcal{I},\mathcal{J})$-sequence covering map there exists $x_m \in f^{-1}(y)$ and a sequence $Q_m$, $\mathcal{I}$-converging to $x_m$ such that $f(Q_m) = \{y_{m_k}\}$. Now for each $k \in \omega$ take $T_k = \cup\{f^{-1}(S_m) : m \geq k\}$. Suppose that there exists $z \in X$ such that for every open neighbourhood $U$ of $z$, $\{n \in \mathbb{N} : x_n \in U\}$ is infinite. Then $z \in \cap_{k\in\mathbb{N}}\overline{T_k}$. Since $X$ is strongly Fréchet, there exists a sequence $\{z_k\}$ converges to $z$ where $z_k \in T_k$. But $\{f(z_k)\}_{k\in\mathbb{N}}$ does not converges to $y$, which is a contradiction. So for every $z \in X$ there is an open neighbourhood $U_z$ of $z$ such that $\{n \in \mathbb{N} : x_n \in U_z\}$ is finite. So $z$ cannot be a limit point of $\{x_n\}$. Hence $\{x_n\}$ is closed. Suppose the set $\{x_n\}$ is finite. Let it be $z = x_n,\ n \in N'$ where $N'$ is a infinite subset of $\mathbb{N}$. Then for every open neighbourhood $U$ of z, $\{n \in \mathbb{N} : x_n \in U\}$ is infinite which is a contradiction. Therefore the set $\{x_n\}_{n\in\mathbb{N}}$ is infinite, closed and discrete in $X$.
Since $X$ has the property $\omega$D there exists an infinite subset $\{x_{n_j}\}_{j\in\mathbb{N}}$ and a discrete open family $\{U_j\}_{j\in\omega}$ such that $U_j \cap \{x_{n_j}\}_{j\in\omega} = \{x_{n_j}\}$. Recall that $Q_{n_j}$, $\mathcal{I}$-converges to $x_{n_j}$ and $f(Q_{n_j}) = \{y_{n_j}\}$. Therefore we can take $u_j \in U_j \cap Q_{n_j}$ such that $\{f(u_j)\}_{j\in\omega}$ is infinite and contained in $\{y_{0,n} : n \in \mathbb{N}\}$. Since $\{u_j\}_{j\in\omega}$ is closed in $X$, $\{f(u_j)\}_{j\in\omega}$ is closed in $S_w$, which is a contradiction. Thus $Y$ is strongly Fréchet. □

**Corollary 3.10.** *Every closed and $(\mathcal{I},\mathcal{J})$-sequence covering image of a metric space is metrizable.*

*Proof.* Suppose $f : X \to Y$ is closed and a $(\mathcal{I},\mathcal{J})$-sequence covering map. Let $X$ be a metric space. Since every metrizable space has $\omega$D property [13], $Y$ is strongly Fréchet by Theorem 3.9. Also every strongly Fréchet space which is a closed image of a metric space is metrizable [8, 24]. Hence $Y$ is metrizable. □

3.2. $(\mathcal{I},\mathcal{J},\mathcal{L})$**- *sequentially quotient map*.** In this section we introduce the concept of ideal sequentially quotient map and investigate some of its properties. Also we investigate under what condition the mapping will coincide with the ideal sequence covering map.

**Definition 3.11.** Let $\mathcal{I},\mathcal{J},\mathcal{L}$ be three admissible proper ideals of $\mathbb{N}$. A map $f : X \to Y$ is said to be $(\mathcal{I},\mathcal{J},\mathcal{L})$- *sequentially quotient map* if $\{y_n\}$ is a sequence, $\mathcal{J}$-converging to $y$ there is a sequence $\{x_k\}$, $\mathcal{I}$-converging to $x$ such that $x_k \in f^{-1}(y_{n_k}),\ x \in f^{-1}(y)$ and $\{n \in \mathbb{N} : x_k \notin f^{-1}(y_n) \text{ for all } k \in \mathbb{N}\} \in \mathcal{L}$.

*Remark* 3.12. If $\mathcal{I} = \mathcal{I}_{fin} = \mathcal{J}$ and $\mathcal{L} = \mathcal{I}_d$ then each $(\mathcal{I},\mathcal{J},\mathcal{L})$-sequentially quotient map is a statistically sequentially quotient map.

We observe following implications.
sequence covering map$\Longrightarrow$ statistically sequentially quotient map and $(\mathcal{I},\mathcal{J},\mathcal{L})$-sequentially quotient map$\Longrightarrow$ sequentially quotient map.
The reverse implications need not be true. We consider the following examples:

**Example 3.13.** Let $\mathfrak{f} = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \text{ is infinite and } d(A) < 1\}$. Let $\mathcal{I}$ be an admissible maximal ideal of $\mathbb{N}$ then $\mathcal{F}(\mathcal{I})$ is an ultrafilter on $\mathbb{N}$. For each $\alpha \in \mathfrak{f}$ consider a sequence $\{x_{\alpha,i} : i \in \mathbb{N}\}$ converging to $x_\alpha$ and let $\{y_n : n \in \mathbb{N}\}$ be a sequence converging to $y$. Define $S_\alpha = \{x_{\alpha,i} : i \in \alpha\} \cup \{x_\alpha\}$ and $Y = \{y_n : n \in \mathbb{N}\}$. Topologies of $S_\alpha,\ \alpha \in \mathfrak{f}$ and $Y$ are defined below. Each $\{x_{\alpha,i}\}$ is open and basic open set containing $x_\alpha$ is of the form $\{x_{\alpha,i} : i \geq n_0, i \in \alpha\} \cup \{x_\alpha\}$. Each $\{y_n\}$ is open and basic open set containing $y$ is of the form $\{y_n : n \geq n_0\} \cup \{y\}$. Let $X = \bigoplus_{\alpha\in\mathfrak{f}} S_\alpha$. Define a map $\phi : X \to Y$ by $\phi(x_{\alpha,i}) = y_i$ and $\phi(x_\alpha) = y$ for all $\alpha \in \mathfrak{f}$.
Now there is $\beta \in \mathcal{F}(\mathcal{I})$ with $d(\beta) < 1$. Suppose $\beta = \{n_k : k \in \mathbb{N}\}$ and let $x_k = x_{\beta,n_k}$. Then $\{x_k\}$ converges to $x_\beta$ and $\phi(x_k) = y_{n_k}$. Also $\{n_k : k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$. Consider a subsequence $\{y_{m_i}\}$ of $\{y_n\}$. Let $A = \{m_i : i \in \mathbb{N}\}$. Now two cases may appear.
*Case 1*: Let $\mathbb{N}\setminus A$ be infinite. Name $z_i = y_{m_i}$. Consider a set, $B$ which is in $\mathcal{F}(\mathcal{I})$ and of density less than 1. Let $M_B$ be the set of all indices of corresponding elements of $\{y_{m_i}\}$. Clearly $d(M_B) < 1$ and $\mathbb{N} \setminus M_B$ is infinite. Then there is a sequence converging to $x_{M_B}$. Define a sequence $\{r_j\}$ where $r_j = x_{M_B,m_j}$ for $j \in B$ and $r_j = x_{M_B}$ for $j \notin B$ then $r_j \to x_{M_B}$. Also $B \subset \{n_k : \phi(r_j) = z_{n_k}\}$.
*Case 2*: Let $\mathbb{N}\setminus A$ be finite. Consider a set $B \in \mathcal{F}(\mathcal{I})$ and of density less than 1. Let $M_B$ be the set of all corresponding elements of $\{y_{m_i}\}$. Clearly $d(M_B) < 1$ and $\mathbb{N} \setminus M_B$ is infinite. Then there is a sequence converging to $x_{M_B}$. Define a sequence $\{r_j\}$ where $r_j = x_{M_B,m_j}$ for $j \in B$ and $r_j = x_{M_B}$ for $j \notin B$ then

$r_j \to x_{M_B}$. Also $\phi(r_j) = y_{m_j} = z_j$ for $j \in B$ and $\phi(r_j) = x_{M_B},\ j \notin B$.
Next consider a sequence $\{z_i\}$ where $z_{n_k} = y_k,\ k \in \mathbb{N}$ and $z_i = y,\ i \notin \{n_k : k \in \mathbb{N}\}$. Then $z_i \to y$. As $\mathcal{F}(\mathcal{I})$ is an ultrafilter on $\mathbb{N}$ then either $\{n_k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$ or its complement belongs to $\mathcal{F}(\mathcal{I})$. Let $\mathcal{F} = \{n_k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$ and consider a proper subset of $\mathcal{F}$, $\mathcal{F}'$ that belongs to $\mathcal{F}(\mathcal{I})$ and whose density is less than 1. Let $\mathcal{F}' = \{n_{k_i} : i \in \mathbb{N}\}$ and let $\mathcal{C} = \{k_i : i \in \mathbb{N}\}$. Also $d(\mathcal{C}) < 1$. Define $r_{n_{k_i}} = x_{\mathcal{C},k_i}$ and $r_j = x_{\mathcal{C}}$ for $j \neq n_{k_i}$. Then $\mathcal{F}' \subset \{n_k : \phi(r_k) = z_{n_k}\}$. Consider a sequence $\{z_j\}$ where $z_{n_k} = y_{\alpha(k)}$ and $z_j = y$ for $j \neq \alpha(k)$ where $\alpha : \mathbb{N} \to \mathbb{N}$ is an increasing function. Let $\mathcal{F} = \{n_k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$. Consider an infinite subset $B$ of $\alpha(\mathbb{N})$ where $\mathbb{N} \setminus M_B$ is infinite. Then there is an infinite subset, $\mathcal{F}'$ of the set of all indices of corresponding elements of $z_{n_k}$ belonging to $\mathcal{F}(\mathcal{I})$. Let $\mathcal{F}' = \{n_{k_i} : i \in \mathbb{N}\}$ and $\mathcal{C} = \{\alpha(k_i) : i \in \mathbb{N}\}$. Consider a sequence $\{r_j : j \in \mathbb{N}\}$ where $r_j = x_{\mathcal{C},\alpha(k_i)},\ j = n_{k_i}$ and $r_j = x_{\mathcal{C}}$. Therefore $r_j \to x_{\mathcal{C}}$ and $\mathcal{F}' \subset \{n_k : \phi(r_k) = y_{n_k}\}$. Again let $\mathbb{N} \setminus \mathcal{F} = \mathcal{F}^{\mathsf{C}} \in \mathcal{F}(\mathcal{I})$, as $\mathcal{F}(\mathcal{I})$ is an ultrafilter on $\mathbb{N}$ there is an infinite subset of $\mathcal{F}^{\mathsf{C}}$, $\mathcal{F}^{\mathsf{C},\prime}$, which is a member of $\mathcal{F}(\mathcal{I})$ and of density less than 1. Let us construct a sequence $\{r_j\}$ where $r_j = x_{\mathcal{F}^{\mathsf{C},\prime}}$ if $j \in \mathcal{F}^{\mathsf{C},\prime}$ and put elements of $\{x_{\mathcal{F}^{\mathsf{C},\prime},i}\}$ in rest of the places. But as there is no set in $\mathfrak{f}$ with density equals to 1 and so $\phi$ cannot be a statistically sequentially quotient map but it is an $(\mathcal{I}_{fin}, \mathcal{I}_{fin}, \mathcal{I})$-sequentially quotient map.

**Example 3.14.** Let there be an infinite set $A \subset \mathbb{N}$ with $A \notin \mathcal{I}$ and $\mathbb{N} \setminus A \notin \mathcal{I}$. Also let $Y$ be a convergent sequence with its limit $\{y\}$. Consider $X_\alpha = \{x_i : i \in A\} \cup \{x_\alpha\}$ and $X_\beta = \{z_i : i \in \mathbb{N} \setminus A\} \cup (x_\beta)$ where $x_i \to x_\alpha$ and $z_i \to x_\beta$. Topology of $X_\alpha$ is defined as follows: $\{x_i\}$ are open and open set containing $x_\alpha$ is of the form $\{x_i : i \geq i_0, i \in A\} \cup \{x_\alpha\}$. Similarly Topologies of $X_\beta$ and $Y$ are defined. Define a map $f : X_\alpha \bigoplus X_\beta \to Y$ by $f(x_i) = y_i, f(z_i) = y_i, f(x_\alpha) = f(x_\beta) = y$. Then $f$ is a sequentially quotient map but $f$ is not an $(\mathcal{I}_{fin}, \mathcal{I}_{fin}, \mathcal{I})$-sequentially quotient map.

**Proposition 3.15.**

(1) *Let $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$ be four ideals of $\mathbb{N}$ and let $\mathcal{F}(\mathcal{L})$ satisfy the property: If $\{n_k : k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{L})$ and $\{k_i : i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{L})$ then $\{n_{k_i} : i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{L})$. $f : X \to Y$ and $g : Y \to Z$ be $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- and $(\mathcal{J}, \mathcal{K}, \mathcal{L})$-sequentially quotient maps. Then $g \circ f : X \to Z$ is an $(\mathcal{I}, \mathcal{K}, \mathcal{L})$-sequentially quotient map.*

(2) *If $g \circ f$ is $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient map then $g$ is so.*

*Proof.* (1) Let $\{z_n\}$ be a sequence, $\mathcal{K}$-converging to $z$ in $Z$. So there is a sequence $\{y_k\}$, $\mathcal{J}$-converging to $y$ so that for each $k \in \mathbb{N}$, $g(y_k) = z_{n_k}$, $g(y) = z$ and $\{n_k : g(y_k) = z_{n_k}\} \in \mathcal{F}(\mathcal{L})$. Now as $f : X \to Y$ is an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$-sequentially quotient map there is a sequence $\{x_i\}$, $\mathcal{I}$-converging to $x$ such that $f(x_i) = y_{k_i}$ for each $i$, $y = f(x)$ and $\{k_i \in \mathbb{N} : y_{k_i} = f(x_i)\} \in \mathcal{F}(\mathcal{L})$. Now $(g \circ f)(x) = z$ for each $i$. Thus $(g \circ f)(x_i) = z_{n_{k_i}}$ and $\{n_{k_i} : i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{L})$(by our assumptions). Therefore $g \circ f : X \to Z$ is an $(\mathcal{I}, \mathcal{K}, \mathcal{L})$- sequentially quotient map.
(2) Let $\{z_n\}$ be a sequence $\mathcal{J}$-converging to $z$. Then there is a sequence $\{x_k\}$, $\mathcal{I}$-converging to $x$. Now $(g \circ f)(x_k) = z_{n_k}$ and $(g \circ f)(x) = z$. Also

$\{n_k \in \mathbb{N} : (g \circ f)(x_k) = z_{n_k}\} \in \mathcal{F}(\mathcal{L})$. By continuity of $f$, it follows that $\{f(x_k)\}$, $\mathcal{I}$-converges to $f(x)$ and hence $g$ is $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient map. □

**Proposition 3.16.**

(1) $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- *sequentially quotient maps is preserved by finite products.*
(2) $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- *sequentially quotient maps are hereditarily* $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- *sequentially quotient maps.*

*Proof.* (1) Let $f_i : X_i \to Y_i$ be $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient map for $i = 1, 2, \cdots, N$. We define a map $f : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ by $f(x_1, x_2, \cdots, x_N) = (f_1(x_1), f_2(x_2), \cdots, f_N(x_N))$. Then $f$ is continuous and onto. Let $\{(y_{i,n}) : n \in \mathbb{N}\}$ be a sequence $\mathcal{J}$-converge to $(y_i)$ in $\prod_{i=1}^N Y_i$. Then $\{y_{i,n}\}$, $\mathcal{J}$-converging to $y_i$ in $Y_i$ for $i = 1, 2, \cdots, N$. Now there is a sequence $\{x_{i,k}\}$, $\mathcal{I}$-converging to $x_i$ such that $f_i(x_{i,k}) = y_{i,n_k}$, $k \in \mathbb{N}$ and $\{n \in \mathbb{N} : x_{i,k} \notin f_i^{-1}(y_{i,n})$ for all $k \in \mathbb{N}\} \in \mathcal{L}$ for $i = 1, 2, \cdots, N,$ . Put $x = (x_i) \in \prod_{i=1}^N X_i$. Then $\{(x_{i,k}) : k \in \mathbb{N}\}$ $\mathcal{I}$-converges to $(x_i)$. Let $\mathcal{N}_i = \{n \in \mathbb{N} : x_{i,k} \in f_i^{-1}(y_{i,n})$ for all $k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{L})$. Put $\mathcal{N} = \bigcap_{i=1}^{N} \mathcal{N}_i \in \mathcal{F}(\mathcal{L})$. Then $\mathcal{N} \subset \{n \in \mathbb{N} : (x_{i,k}) \in f^{-1}(y_{i,n})$ for all $k \in \mathbb{N}\}$. Therefore $f$ is an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient map.
(2) Let $f : X \to Y$ be an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient map and let $H$ be a subspace of $Y$. Suppose that $\{y_n\}$ is a sequence in $H$, $\mathcal{J}$-converging to $y$ in $H$. Then there is a sequence $\{x_n\}$, $\mathcal{I}$-converging to $x$ in $X$ where $x_k \in f^{-1}(y_{n_k})$. Clearly $x_k \in f^{-1}(H)$ for each $k \in \mathbb{N}$ and $x \in f^{-1}(H)$. □

Recall that there is a class of ideals of $\mathbb{N}$, $\beta$ 'say' that satisfies: if $A \in \mathcal{I}$ then for every strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, $f(A) \in \mathcal{I}$. Several known ideals are $\beta$ ideals, for example

(1) The (analytic $P$-ideal) $\mathcal{I}_d$ of natural density zero sets,
(2) The ($F_\sigma$ $P$-ideal) summable ideal $\mathcal{I}_{\frac{1}{n}} = \{A \subset \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty\}$,
(3) The (analytic $P$-ideal)

$$\mathcal{I}_{\log} = \{A \subset \mathbb{N} : \lim_{n \to \infty} (\sum_{i \in A \cap \{1,2,\cdots,n\}} \frac{1}{i}) / (\sum_{i \in \{1,2,\cdots,n\}} \frac{1}{i}) = 0\}$$

of logarithmic density zero sets.

**Theorem 3.17.** *Let* $f : X \to Y$ *be an* $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- *sequentially quotient and boundary compact map where* $\mathcal{I} \in \beta$. *Also let there be an infinite countable partition* $\{M_i : i \in \mathbb{N}\}$ *of* $\mathbb{N}$ *such that* $M_i \notin \mathcal{L}$ *for each* $i \in \mathbb{N}$ *and* $\mathcal{I} \subset \mathcal{L}$. *Moreover, let* $Y$ *be snf-countable. Then for each non-isolated point* $y \in Y$, *there is a point* $x_y \in \partial f^{-1}(y)$ *such that whenever* $U$ *is an open subset with* $x_y \in U$, *there exists a* $P \in \mathcal{P}_y$ *satisfying* $P \subset f(U)$.

*Proof.* Let it be false. Then there exists a non-isolated point $y \in Y$ such that for each $x \in \partial f^{-1}(y)$ there exists an open neighbourhood $U_x$ of $x$ so that for all $P \in \mathcal{P}_y, P \not\subset f(U_x)$. Therefore $\partial f^{-1}(y) \subset \{U_x : x \in \partial f^{-1}(y)\}$. Since $\partial f^{-1}(y)$ is compact, there exists a finite subfamily $\mathcal{U}$ of $\{U_x : x \in \partial f^{-1}(y)\}$ that covers $\partial f^{-1}(y)$. Name $\mathcal{U} = \{U_i : 1 \leqslant i \leqslant n_0\}$. Let $\mathcal{P}_y = \{P_n : n \in \mathbb{N}\}$ and $\mathcal{W}_y = \{F_n = \bigcap_{i=1}^{n} P_i : n \in \mathbb{N}\}$. Then $\mathcal{W}_y \subset \mathcal{P}_y$. Also $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. Now for each $1 \leq m \leq n_0,\ n \in \mathbb{N}$ there exists $x_{n,m} \in F_n \setminus f(U_m)$. Put $y_n = x_{n,1},\ n \in M_1,\ y_n = x_{n,2},\ n \in M_2, \cdots, y_n = x_{n,n_0},\ n \in M_{n_0}$ and $y_n = y,\ n \in \mathbb{N} \setminus (\bigcup_{i=1,2,\cdots,n_0} M_i)$.

Then $\{y_n\}$ converges to $y$. As $f$ is $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequential quotient map there is a sequence $\{x_k\}$, $\mathcal{I}$-converging to $x \in \partial f^{-1}(y)$ in $X$ such that for each $k \in \mathbb{N},\ f(x_k) = y_{n_k}$ and $\{n \in \mathbb{N} : x_k \notin f^{-1}(y_n)$ for all $k\} \in \mathcal{L}$. Now there is $m_0 \in \{1, 2, \cdots, n_0\}$ such that $x \in U_{m_0}$ as $x \in \partial f^{-1}(y) \subset \cup\mathcal{U}$. Hence $\{k \in \mathbb{N} : x_k \notin U_{m_0}\} \in \mathcal{I}$ which shows that $\{n_k \in \mathbb{N} : y_{n_k} \notin f(U_{m_0})\} \in \mathcal{I}$. Thus $\{n_k \in \mathbb{N} : y_{n_k} \notin f(U_{m_0})\} \in \mathcal{L}$ as $\mathcal{I} \subset \mathcal{L}$. Therefore $\{n_k \in \mathbb{N} : f(x_k) = y_{n_k}\} \in \mathcal{F}(\mathcal{L})$ and hence $\{n_k \in \mathbb{N} : y_{n_k} \in f(U_{m_0})\} \in \mathcal{F}(\mathcal{L})$ which shows that $\{n \in \mathbb{N} : y_n \in f(U_{m_0})\} \in \mathcal{F}(\mathcal{L})$. Thus $\{n \in \mathbb{N} : y_n \notin f(U_{m_0})\} \in \mathcal{L}$. But $M_{m_0} \notin \mathcal{L}$ and for each $n \in M_{m_0}, y_n = x_{n,m_0} \notin f(U_{m_0})$, which is a contradiction. Hence the theorem. □

The following result gives the relation between ideal sequentially quotient map and ideal sequence covering map.

**Theorem 3.18.** *Let $f : X \to Y$ be an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient and boundary compact map where $X$ is first countable. Suppose $\mathcal{J} \subset \mathcal{I}$ and $X$ satisfies $S_1(\mathcal{F}(\mathcal{I}), \mathcal{F}(\mathcal{I}))$. Then $f$ is $(\mathcal{I}, \mathcal{J})$-sequence covering map provided $Y$ is snf-countable.*

*Proof.* Let $y$ be a non-isolated point in $Y$. As $Y$ is sn$f$-countable by Theorem 3.17, there exists a point $x_y \in \partial f^{-1}(y)$ such that whenever $U$ is an open neighbourhood of $x_y$ there exists $P \in \mathcal{P}_y,\ P \subset f(U)$. Let $\{B_n : n \in \mathbb{N}\}$ be a countable neighbourhood base at $x_y : B_{n+1} \subset B_n,\ n \in \mathbb{N}$. Now for each $B_n$, there exists a $P_n \in \mathcal{P}_y : P_n \subset f(B_n)$ which shows that $f(B_n)$ is a sequential neighbourhood of $y \in Y$ as each $P \in \mathcal{P}_y$ is a sequential neighbourhood of $y$. Let $\{y_i\}$ be a sequence, $\mathcal{J}$-converging to $y$ in $Y$, i.e. $\{i \in \mathbb{N} : y_i \notin P_n\} \in \mathcal{J}$. Now For each $n$, let $A_n = \{i \in \mathbb{N} : y_i \in f(B_n)\}$. Choose $i_n \in A_n,\ n \in \mathbb{N}$ such that $\{i_n : n \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$. Set $x_j = f^{-1}(y_j)$ if $j \neq i_n,\ n \in \mathbb{N}$ and $x_j = f^{-1}(y_j) \cap B_n$ if $j = i_n$. Let $U$ be an open neighbourhood of $x_y$. Then there is $B_n \subset U$. Therefore $\{x_j\}$, $\mathcal{I}$-converging to $x_y$ and for each $j \in \mathbb{N},\ f(x_j) = y_j,\ y = f(x_y)$.

□

The following Lemma exhibits the nature of the image of ideal sequentially quotient boundary compact map.

**Lemma 3.19.** *Let $\Omega$ be the set of all topological spaces such that each compact subset $K \subset X$ is metrizable and has a countable neighborhood base in $X$ and*

*$f : X \to Y$ be an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient and boundary compact map. If $X \in \Omega$ then $Y$ is snf-countable.*

*Proof.* Consider a non-isolated point $y \in Y$. $\partial f^{-1}(y) \neq \varnothing$, compact and since $X \in \Omega$ there is a countable external base $\mathcal{U}$ for $\partial f^{-1}(y)$ in $X$. Let $\mathcal{V} = \{\cup \mathcal{U}' : \mathcal{U}' \subset \mathcal{U}$ is finite and $\partial f^{-1}(y) \subset \cup \mathcal{U}'\}$. Now $f(\mathcal{V})$ is countable as $\mathcal{V}$ is so. Now we have to show that $f(\mathcal{V})$ is a *sn*-network at $y$.
Let $\mathcal{N}$ be a neighbourhood of $y$. Then clearly $\partial f^{-1}(y) \subset f^{-1}(\mathcal{N})$. Now for each $x \in \partial f^{-1}(y)$ there is $U_x \in \mathcal{U} : \; x \in U_x \subset f^{-1}(\mathcal{N})$. So $\{U_x : x \in \partial f^{-1}(y)\}$ covers $\partial f^{-1}(y)$. There is a finite subfamily $\mathcal{K}$ of $\{U_x : x \in \partial f^{-1}(y)\}$ that covers $\partial f^{-1}(y)$ and $\partial f^{-1}(y) \subset \cup \mathcal{K} \subset f^{-1}(\mathcal{N})$. Clearly $\cup \mathcal{K} \in \mathcal{V}$ and $y \in f(\cup \mathcal{K}) \subset \mathcal{N}$. Let $U_1 = f(U_1')$ and $U_2 = f(U_2')$ where $U_1'$ and $U_2'$ are elements of $\mathcal{V}$. Now $\partial f^{-1}(y) \subset U_1' \cap U_2'$ then as in above we get $U' \in \mathcal{V}$ such that $\partial f^{-1}(y) \subset U' \subset U_1' \cap U_2'$. Name $f(U') = V'$. Then $V' \subset U_1 \cap U_2$. Next we show that each $V \in f(\mathcal{V})$ is a sequential neighbourhood of $y$. Let $\{y_n\}$ be a sequence converging to $y$ in $Y$. Then there is a sequence $\{x_k\}$, $\mathcal{I}$-converging to $x$ such that $x \in \partial f^{-1}(y)$ and $f(x_k) = y_{n_k}, \; k \in \mathbb{N}$. Now $V = f(U) \in f(\mathcal{V})$. So $\{k \in \mathbb{N} : x_k \notin U\} \in \mathcal{I}$ which shows that $\{k \in \mathbb{N} : y_{n_k} \notin V\} \in \mathcal{I}$. Suppose that $B = \{n \in \mathbb{N} : y_n \notin V\}$ is an infinite set. Then $\{y_n' : n \in B\}$ is a subsequence of $\{y_n\}$ converging to $y$. There is a sequence $\{x_n'\}$, $\mathcal{I}$-converging to $x' \in \partial f^{-1}(y) \subset U$ and its image under $f$ is a subsequence of $\{y_n' : n \in B\}$, hence there is a $m \in B$ so that $y_m \in V$, which is a contradiction. So $B$ must be a finite set and hence the result. $\square$

**Theorem 3.20.** *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$ be three ideals of $\mathbb{N}$ that satisfy the following conditions.*

(1) *$\mathcal{I} \in \beta$ and $\mathcal{I}$ satisfies $S_1(\mathcal{F}(\mathcal{I}), \mathcal{F}(\mathcal{I}))$.*
(2) *$\mathcal{J} \subset \mathcal{I}$*
(3) *There is a countable infinite partition $\{M_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that $M_i \notin \mathcal{L}$ for each $i$.*

*Then each $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient and boundary compact map $f : X \to Y$, is an $(\mathcal{I}, \mathcal{J})$-sequence covering map if $X \in \Omega$.*

*Proof.* Let $f : X \to Y$ be an $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient and boundary compact map and let $X \in \Omega$. By Lemma 3.19, it follows that $Y$ is $snf$-countable. As $\partial f^{-1}(y)$ is compact hence applying Theorem 3.18, we have $f$ is an $(\mathcal{I}, \mathcal{J})$-sequence covering map. $\square$

**Corollary 3.21.** *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$ be three ideals of $\mathbb{N}$ that satisfy the following conditions.*

(1) *$\mathcal{I} \in \beta$ and $\mathcal{I}$ satisfies $S_1(\mathcal{F}(\mathcal{I}), \mathcal{F}(\mathcal{I}))$.*
(2) *$\mathcal{J} \subset \mathcal{I}$*
(3) *There is a countable infinite partition $\{M_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that $M_i \notin \mathcal{L}$ for each $i$.*

*Then each $(\mathcal{I}, \mathcal{J}, \mathcal{L})$- sequentially quotient and boundary compact map $f : X \to Y$, is an $(\mathcal{I}, \mathcal{J})$-sequence covering map if at least one of the following conditions holds.*

(1) *$X$ has a point-countable base.*
(2) *$X$ is a developable space.*

*Proof.* It is known that $X \in \Omega$ either if $X$ has a point-countable base or if $X$ is a developable space. Then the result follows from Theorem 3.20 □

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