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Generic theorems in the theory of cardinal invariants of topological spaces[1]

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Abstract

The main aim of this paper is to present a technical result, which provides an algorithm to prove several cardinal inequalities and relative versions of cardinal inequalities related. Moreover, we use this result and the weak Hausdorff number, H^*, introduced by Bonanzinga in [Houston J. Math. 39 (3) (2013), 1013–1030], to generalize some upper bounds on the cardinality of topological spaces.

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1. Introduction

Among the best known theorems concerning cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. Of course, one of these results is the famous Arhangel'skiĭ inequality, answering a 50 years old question posed by P.S. Alexandroff and P. Urysohn, namely: For each Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$.

The previous inequality generated a great development in the theory of topological cardinal functions, as well as new questions and open problems. The

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ideas employed in their proof became proving technique, which is now a line of research. At present, there is a wide range of results that attempt to capture the central ideas of Arhangel'skiĭ's proof, in order to obtain generic theorems (see [11]). The reader interested in knowing about Arhangel'skiĭ's inequality can consult the work of Hodel [11].

On the other hand, since the appearance of Arhangel'skiĭ's inequality, in 1969, to date, more results have been obtained that are either a generalization or a variation of Arhangel'skiĭ's result. In [6], Bonanzinga introduced the Hausdorff number and the weak Hausdorff number of a space X, denoted by $H(X)$ and $H^*(X)$, respectively, to analyze, among others, problems related to Arhangel'skiĭ's inequality. Bonanzinga's ideas have given new impetus to the theory of the topological cardinal invariants and today many authors have returned to the problems in this field.

In this paper we prove in Theorem 2.2 a general technical result, closely related to [2, Theorem 1], which provides an algorithm for proving a wide range of cardinal inequalities in absolute and relative versions. Moreover, we use Theorem 2.2 to prove other well-known generic theorems and we establish upper bounds on the cardinality of topological spaces which generalize some recently presented.

We recall the following. Let X be a topological space and let A be a subset of X. We denote by $\overline{A}$ or $cl_X(A)$ the closure of A in X.

If X is a set and κ is an infinite cardinal, then $[X]^{\leq\kappa}$ (respectively, $[X]^{<\kappa}$, $[X]^{\kappa}$ and $[X]^{\geq\kappa}$) denotes the collection of all subsets of X with cardinality $\leq \kappa$ (respectively, $< \kappa$, $= \kappa$ and $\geq \kappa$). Also, if X is a topological space and $Y \subseteq X$, then the *κ-closure of Y in X*, denoted by $cl_\kappa(Y)$ or $[Y]_\kappa$, is the set: $\bigcup\{\overline{D} : D \in [Y]^{\leq\kappa}\}$. We say that Y is a *κ-closed subset of* X if $Y = [Y]_\kappa$.

We refer the reader to [13] and [14] for definitions and terminology on cardinal functions not explicitly given. Let L, χ, ψ, ψ_c, c, t, nw, F and d denote the following standard cardinal functions: Lindelöf degree, character, pseudocharacter, closed pseudocharacter, cellularity, tightness, networkweight, free sequence number and density, respectively. The following definitions are known (see e.g. [7], compare also [12]). For $Y \subseteq X$, the *almost Lindelöf degree of Y relative to X*, denoted by $aL(Y,X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of Y, by open subsets of X, there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $Y \subseteq \bigcup\overline{\mathcal{V}} = \bigcup\{\overline{V} \mid V \in \mathcal{V}\}$. The *almost Lindelöf degree* of X, denoted by $aL(X)$ is $aL(X,X)$. The *almost Lindelöf degree relative to closed subsets* of X is $aL_c(X) = \sup\{aL(C,X) \mid C \text{ is a closed subset of } X\}$. The *$\kappa$-almost Lindelöf degree* of X [7] is $aL_\kappa(X) = \sup\{aL(C,X) \mid C \text{ is a } \kappa\text{-closed subset of } X\}$. For definitions of weak Lindelöf degree of X, $wL(X)$, and weak Lindelöf degree relative to closed subsets, $wL_c(X)$, see [12] (compare with [1, 5]).

2. Generic theorems

In what follows, τ and κ are infinite cardinals such that $\kappa < cf(\tau)$. Let X be a nonempty set. A *κ-sensor in X* is a pair $s = (\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a family of subsets of X and $\mathcal{F}$ is a collection of families of subsets of X such that: for every $A \in \mathcal{A}$, $|\mathcal{A}| \leq \kappa$ and $|A| \leq \kappa$; and, for every $\mathcal{C} \in \mathcal{F}$, $|\mathcal{F}| \leq \kappa$ and $|\mathcal{C}| \leq \kappa$. Given $H \subseteq X$ and $\mathcal{G} \subseteq \mathcal{P}(X)$, we say that a κ-sensor $s = (\mathcal{A}, \mathcal{F})$ in X is *generated by the pair* $(H, \mathcal{G})$, if $A \subseteq H$, for each $A \in \mathcal{A}$, and $\mathcal{C} \subseteq \mathcal{G}$, for each $\mathcal{C} \in \mathcal{F}$.

The proof of the following proposition is easy.

Proposition 2.1. *Let X be a set. If $H \subseteq X$ and $\mathcal{G} \subseteq \mathcal{P}(X)$, then the collection of κ-sensors in X generated by the pair $(H, \mathcal{G})$ has cardinality less than or equal to $|H|^{\kappa} \cdot |\mathcal{G}|^{\kappa}$.*

Let Θ denote a function such that each κ-sensor s in X, is associated with a subset $\Theta(s)$ of X, called the *Θ-closure of s*, and we say that the function Θ is a Θ-closure. Let Y be a nonempty subset of X. If s a κ-sensor in X, we say that s is *small for Y* if $Y \setminus \Theta(s) \neq \varnothing$. When $Y = X$, we only say that s is a *small* κ-sensor.

Let τ and κ be infinite cardinals. An operator $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ will be called *(τ, κ)-closing* if whenever $A \subseteq X$ such that $|A| \leq \tau^{\kappa}$, then $|\rho(A)| \leq \tau^{\kappa}$ and $A \subseteq \rho(A)$. It is clear that if $\tau = \kappa^{+}$, then the condition $|A| \leq \tau^{\kappa}$ implies $|\rho(A)| \leq \tau^{\kappa}$ in this definition is equivalent to $|A| \leq 2^{\kappa}$ implies $|\rho(A)| \leq 2^{\kappa}$.

Throughout this paper, we put $\mathcal{L} = [X]^{\leq \tau^{\kappa}}$ and $\mathcal{Q} = [\mathcal{P}(X)]^{\leq \tau^{\kappa}}$. Moreover, if $g : \mathcal{L} \to \mathcal{Q}$ is a function and $\mathcal{E} \subseteq \mathcal{L}$, we put $\mathcal{U}_g(\mathcal{E}) = \bigcup \{g(F) \mid F \in \mathcal{E}\}$. When ρ is a (τ, κ)-closing operator we denote by $\rho(\mathcal{E})$ the set $\{\rho(E) \mid E \in \mathcal{E}\}$.

Let ρ be a (τ, κ)-closing operator and let Θ be a Θ-closure operator. Considering $\mathcal{E} \subseteq \mathcal{L}$ and a function $g : \mathcal{L} \to \mathcal{Q}$ we say that a κ-sensor s in X is *Θ-good for $\mathcal{E}$ with respect to Y* if s is generated by the pair $(\bigcup \rho(\mathcal{E}), \mathcal{U}_g(\mathcal{E}))$ and $Y \cap [\bigcup \rho(\mathcal{E})] \subseteq \Theta(s)$. When $Y = X$, we only say that s is Θ-good for $\mathcal{E}$.

Finally, a *(g, ρ, Θ)-quasi-propeller for Y* is a family $\mathcal{E} = \{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ such that no small κ-sensor for Y in X is Θ-good for $\mathcal{E}$ with respect to Y. When $Y = X$, we only say that $\mathcal{E} = \{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ is a *(g, ρ, Θ)-quasi-propeller* if no small κ-sensor is Θ-good for $\mathcal{E}$.

Now we are ready to prove our main result in Theorem 2.2. We mention that currently there are several results that are adapted to prove cardinal inequalities, but generally related to the inequality of Arhangel'skiĭ. Arhangel'skiĭ has a much more general result, an *algorithm*, for proving relative versions of cardinal inequalities (main Theorem from [2]). However, Arhangel'skiĭ mentions in [2] that he does not know a proof of Gryzlov's result [10] (see Theorem 2.6, here). Following the ideas of Arhangel'skiĭ in [2], we obtain in Theorem 2.2 a general technical result, which can be used to prove several well-known cardinal inequalities in relative and absolute version. Among others, the results given in [2], and Gryzlov's inequality.

Theorem 2.2. *Let X be a set, let Y be a nonempty subset of X, and let τ and κ be such that $\kappa < cf(\tau)$. If $g : \mathcal{L} \to \mathcal{Q}$ is a function, $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ is a (τ, κ)-closing operator, and Θ is a Θ-closure, then there exists a family $\{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$, such that:*

1. *for each $0 < \alpha < \tau$, $\bigcup\{\rho(E_\beta) \cap Y \mid \beta < \alpha\} \subseteq E_\alpha$, $\bigcup\{\rho(E) \cap Y \mid E \in [\bigcup_{\beta<\alpha} E_\beta]^{\leq\kappa}\} \subseteq E_\alpha$, and*
2. *$\mathcal{E} = \{E_\alpha \mid \alpha < \tau\}$ is a (g, ρ, Θ)-quasi-propeller for Y.*

Proof. We construct a sequence $\{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ and a collection of families of subsets of X, $\{\mathcal{U}_\alpha \mid 0 < \alpha < \tau\}$, such that:

(a) For each $0 \leq \alpha < \tau$,
$\bigcup\{\rho(E_\beta) \cap Y \mid \beta < \alpha\} \subseteq E_\alpha$ and $\bigcup\{\rho(E) \cap Y \mid E \in [\bigcup_{\beta<\alpha} E_\beta]^{\leq\kappa}\} \subseteq E_\alpha$;

(b) For each $0 < \alpha < \tau$, $\mathcal{U}_\alpha = \bigcup\{g(E_\beta) \mid \beta < \alpha\}$;

(c) For each $0 < \alpha < \tau$, if s is a κ-sensor such that is small for Y and is generated by the pair $(\bigcup\{\rho(E_\beta) \mid \beta < \alpha\}, \mathcal{U}_\alpha)$, then $(Y \cap E_\alpha) \setminus \Theta(s) \neq \varnothing$.

Fix $0 < \alpha < \tau$ and assume that E_β and $\mathcal{U}_\beta$ are already defined such that (a)-(c) hold for each $\beta < \alpha$. Note that $\mathcal{U}_\alpha$ has been defined by (b). We put $H_\alpha = \bigcup\{\rho(E_\beta) \cap Y \mid \beta < \alpha\}$. It is not difficult to prove that $|H_\alpha| \leq \tau^\kappa$. For each small for Y κ-sensor s generated by the pair $(\bigcup\{\rho(E_\beta) \mid \beta < \alpha\}, \mathcal{U}_\alpha)$, we choose one point $m(s) \in Y \setminus \Theta(s)$, and let F_α be the set of points chosen in this way. From Proposition 2.1, $|F_\alpha| \leq \tau^\kappa$. We put $H'_\alpha = \bigcup\{\rho(E) \cap Y \mid E \in [\bigcup_{\beta<\alpha} E_\beta]^{\leq\kappa}\}$. Note that $|H'_\alpha| \leq \tau^\kappa$.
Let $E_\alpha = H_\alpha \cup H'_\alpha \cup F_\alpha$. Clearly, $E_\alpha \in \mathcal{L}$ and E_α satisfies (c). This completes the construction. On the other hand, it is clear that the collection $\{E_\alpha \mid \alpha < \tau\}$ satisfies (1). Finally, the proof will be complete if we prove that $\mathcal{E} = \{E_\alpha \mid \alpha < \tau\}$ is a (g, ρ, Θ)-quasi-propeller for Y. To see this, suppose there is a κ-sensor $s_0 = (\mathcal{A}, \mathcal{F})$, which is small for Y, and s_0 is Θ-good for $\mathcal{E}$ with respect to Y. Thus, $Y \setminus \Theta(s_0) \neq \varnothing$, s_0 is generated by the pair $(\bigcup \rho(\mathcal{E}), \mathcal{U}_g(\mathcal{E}))$ and $Y \cap [\bigcup \rho(\mathcal{E})] \subseteq \Theta(s_0)$. Since $\kappa < cf(\tau)$, there exists $\alpha_0 < \tau$ such that for each $A \in \mathcal{A}$, $A \subseteq \bigcup\{\rho(E_\beta) \mid \beta < \alpha_0\}$, and for each $\mathcal{B} \in \mathcal{F}$, $\mathcal{B} \subseteq \mathcal{U}_{\alpha_0}$. Hence, s_0 is generated by the pair $(\bigcup\{\rho(E_\beta) \mid \beta < \alpha_0\}, \mathcal{U}_{\alpha_0})$ and satisfies $Y \setminus \Theta(s_0) \neq \varnothing$. Thus, by (c) there exists $m(s_0) \in E_{\alpha_0} \setminus \Theta(s_0)$, a contradiction. $\square$

We apply Theorem 2.2 in Section 3 to obtain some results on cardinal invariants of topological spaces. However, Theorem 2.2 also can be used to obtain other generic theorems. For example, the next technical result due to Hodel captures the common core of several cardinal inequalities which are either a generalization or a variation of Arhangel'skiĭ's inequality: For each Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$.

Corollary 2.3 ([12]). *Let X be a set, let κ and λ be infinite cardinals with $\lambda \leq 2^\kappa$, let $c', d : \mathcal{P}(X) \to \mathcal{P}(X)$ be operators on X, and for each $x \in X$, let $\mathcal{B}_x = \{V(\gamma, x) \mid \gamma < \lambda\}$ a collection of subsets of X. Assume the following:*

(T) if $x \in c'(H)$, then there exists $A \in [H]^{\leq\kappa}$, such that $x \in c'(A)$;

(C) if $A \in [X]^{\leq\kappa}$, then $|c'(A)| \leq 2^\kappa$; and

(C-S) if $H \neq \varnothing$, $c'(H) \subseteq H$, and $q \notin H$, then there exist $A \in [H]^{\leq\kappa}$ and a function $f : A \to \lambda$ such that $H \subseteq d(\bigcup\{V(f(x), x) \mid x \in A\})$ and $q \notin d(\bigcup\{V(f(x), x) \mid x \in A\})$.

Then $|X| \leq 2^\kappa$.

Proof. Let $\tau = \kappa^+$. Let $\mathcal{L} = [X]^{\leq 2^\kappa}$, $\mathcal{Q} = [\mathcal{P}(X)]^{\leq 2^\kappa}$, and $g : \mathcal{L} \to \mathcal{Q}$ given by $g(F) = \bigcup\{\mathcal{B}_x \mid x \in F\}$, for every $F \in \mathcal{L}$. It is easy to see that $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $\rho(A) = c'(A)$ is a (τ, κ)-closing operator. For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$, we put $\Theta(s) = d(\bigcup\{\bigcup\mathcal{C} \mid \mathcal{C} \in \mathcal{F}\})$. Then, there exists a family $\{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ such that parts (1) and (2) of Theorem 2.2, hold. Let $P = \bigcup\mathcal{E}$. Clearly $P \neq \varnothing$ and $|P| \leq 2^\kappa$. Moreover, $c'(P) \subseteq P$.

The proof will be complete once we show that $X \subseteq P$. Suppose the contrary. Then, there exists $p \in X \setminus P$. Hence, by (C-S) there exist $A \in [P]^{\leq\kappa}$ and a function $f : A \to \lambda$ such that $P \subseteq d(\bigcup\{V(f(x), x) \mid x \in A\})$ and $p \notin d(\bigcup\{V(f(x), x) \mid x \in A\})$. Let $s_0 = (\varnothing, \{\{V(f(x), x) \mid x \in A\}\})$ and $\Theta(s_0) = \bigcup\{V(f(x), x) \mid x \in A\}$. Then s_0 is a small κ-sensor in X which is generated by the pair $(\bigcup\rho(\mathcal{E}), \mathcal{U}_g(\mathcal{E}))$ and $P \subseteq \Theta(s_0)$, a contradiction. Hence, $X \subseteq P$. Therefore, $|X| \leq 2^\kappa$. □

We observe that Cammaroto, Catalioto and Porter [7] use Corollary 2.3 to generalize the inequalities: $|X| \leq 2^{L(X)F(X)\psi(X)}$ due to Spadaro-Juhász [16], and $|X| \leq 2^{L(X)F_c(X)\psi(X)}$ due to Bella [4]. On the other hand, the next result improves a result by Cammaroto et al. [8, Main Theorem], which is also a unified approach to prove several cardinal inequalities.

Corollary 2.4. *Let X be a set, κ and τ infinite cardinals such that $\kappa < cf(\tau)$, $\rho : [X]^{\leq\tau^\kappa} \to \mathcal{P}(X)$ a function, and for $x \in X$, $\mathcal{B}_x = \{V(x, \alpha) \mid \alpha \in \kappa\}$ a collection of subsets of X such that:*

- (i) *For $A, B \in [X]^{\leq\tau^\kappa}$, $A \subseteq \rho(A)$ and if $A \subseteq B$, then $\rho(A) \subseteq \rho(B)$.*
- (ii) *For $A \in [X]^{\leq\tau^\kappa}$, $|\rho(A)| \leq \tau^\kappa$.*
- (iii) *If $H \neq \varnothing$, $\rho(H) \subseteq H$, and $q \notin H$, then there exist $A \in [H]^{\leq\kappa}$ and a function $f : A \to \kappa$ such that $H \subseteq \bigcup_{x \in A} V(x, f(x))$ and $q \notin \bigcup_{x \in A} V(x, f(x))$.*

Then $|X| \leq \tau^\kappa$.

Proof. Let $\mathcal{L} = [X]^{\leq\tau^\kappa}$, $\mathcal{Q} = [\mathcal{P}(X)]^{\leq\tau^\kappa}$, and $g : \mathcal{L} \to \mathcal{Q}$ given for every $F \in \mathcal{L}$ by $g(F) = \bigcup\{\mathcal{B}_x \mid x \in F\}$. Clearly, ρ is a (τ, κ)-closing operator. Now, for every κ-sensor $s = (\mathcal{A}, \mathcal{F})$, we put $\Theta(s) = \bigcup\{\bigcup\mathcal{C} \mid \mathcal{C} \in \mathcal{F}\}$. Hence, there exists a family $\{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ such that parts (1) and (2) of Theorem 2.2 hold.

We see $\rho(P) \subseteq P$. For this end, it suffices to note that if $B \in [P]^{\leq\kappa}$, then $\rho(B) \subseteq P$. Indeed, if $B \in [P]^{\leq\kappa}$, then by regularity of τ, there exists $\alpha_0 < \tau$ such that $B \subseteq \bigcup\{E_\beta \mid \beta < \alpha_0\}$. Thus, by second contention of part (1) in Theorem 2.2 $\rho(B) \subseteq P$.

We show that $X \subseteq P$. Suppose the contrary and fix $q \in X \setminus P$. By (iii) there exist $A \in [P]^{\leq\kappa}$ and a function $f : A \to \kappa$ such that $P \subseteq \bigcup_{x \in A}\{V(x, f(x))\}$

and $q \notin \bigcup_{x\in A}\{V(x,f(x))\}$. We consider the κ-sensor $s_0 = (\varnothing, \{\{V(x,f(x)) \mid x \in A\}\})$ and $\Theta(s_0) = \bigcup_{x\in A}\{V(x,f(x))\}$. We note that $q \in X \setminus \Theta(s_0)$. Thus, s_0 is a small κ-sensor which is Θ-good for $\mathcal{E}$, a contradiction. It follows that $X \subseteq P$. Therefore, $|X| \leq \tau^\kappa$. □

We conclude this section with a proof of Gryzlov's theorem using Theorem 2.2.

Lemma 2.5 ([10]). *Let X be a T_1 compact space with $\psi(X) \leq \kappa$. Let H be a subset of X such that every infinity subset of H of cardinality $\leq \kappa$ has a complete accumulation point in H. Then H is compact.*

Theorem 2.6. *If X is a T_1 compact space, then $|X| \leq 2^{\psi(X)}$.*

Proof. Let $\kappa = \psi(X)$ and $\tau = \kappa^+$. For each $x \in X$, let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq \kappa$. We consider the operator $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\rho(A) = A \cup A'$, where A' is the set defined as follows: For each infinite subset, $B \subseteq A$ with $|B| \leq \kappa$, we take a complete accumulation point of B in X and A' is the set formed by such points. Clearly ρ is a (τ,κ)-closing operator.

For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$ in X, we put $\theta(s) = \bigcup\{\bigcup\mathcal{C} \mid \mathcal{C} \in \mathcal{F}\}$. Consider $g : \mathcal{L} \to \mathcal{Q}$ defined by $g(F) = \bigcup\{\mathcal{B}_x \mid x \in \rho(F)\}$, for $F \in \mathcal{L}$. By Theorem 2.2, there is a family $\mathcal{E} = \{E_\alpha \mid \alpha \in \kappa^+\}$, which is a (g,ρ,Θ)-quasi-propeller in $\mathcal{L}$. Let $H = \bigcup\{\rho(E_\alpha) \mid \alpha \in \kappa^+\}$. It is not difficult to show, using Lemma 2.5, that H is compact. Moreover, $|H| \leq 2^\kappa$.

Let us show that $X \subseteq H$. Suppose not and let $p \in X \setminus H$. For each $x \in H$, let $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. Clearly the collection $\{V_x \mid x \in H\}$ cover H. Hence, there exist $x_1, \ldots, x_n \in H$ such that $H \subseteq \bigcup\{V_{x_i} \mid i \in \{1,\ldots,n\}\}$. Let $\mathcal{F} = \{V_{x_i} \mid i \in \{1,\ldots,n\}\}$. Then, we have that $s_0 = (\varnothing, \{\mathcal{F}\})$ is a small κ-sensor in X, which is Θ-good for $\mathcal{E}$. This is a contradiction, since $\mathcal{E}$ is a (g,ρ,Θ)-quasi-propeller. Thus, $X \subseteq H$. Therefore, $|X| \leq 2^{\psi(X)}$. □

3. Some applications in cardinal functions

In 1969, Arhangel'skiĭ [3] proved his famous result: For each Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$. This inequality has been generalized by some authors as Sapadaro [16], Bella [4], Cammaroto Catalioto and Porter [7], among others. Another generalization from Arhangel'skiĭ's result is due to Bonanzinga in 2013, namely she proved: For each T_1 space X with $H^*(X) \leq \omega$, $|X| \leq 2^{L(X)\chi(X)}$, the which is a positive partial answer to a question posed by Arhangel'skiĭ, that is: Is it true that if X is a T_1-space, then $|X| \leq 2^{L(X)\chi(X)}$? Like all the important results, Bonanzinga's inequality, in addition to solving a long-standing problem, introduces new techniques and generates new questions. Among the new concepts presented by Bonanzinga we find the Hausdorff number and the weak Hausdorff number of a space X, denoted by $H(X)$ and $H^*(X)$, respectively. Next, we use these cardinal functions and Theorem 2.2 in order to present some generalizations of bounds to the cardinality of topological spaces. Before this, we recall the notion of the weak Hausdorff number of a space.

Definition 3.1 ([6])**.** Let X be a topological space. The *weak Hausdorff number of X* is

$$H^*(X) = \min\{\tau \mid \text{ for each } A \in [X]^{\geq\tau}, \text{ there is } B \in [A]^{<\tau}, \text{ and for every } b \in B, \text{there exists an open subset } U_b \text{ such that } b \in U_b \text{ and } \bigcap_{b\in B} U_b = \varnothing\}.$$

The following concepts were introduced in [5]. If X is a T_1 topological space then for each $x \in X$ we put $Hw(x) = \bigcap\{\overline{U} \mid x \in U \text{ and } U \text{ is open in } X\}$. The *Hausdorff width* is $HW(X) = \sup\{|Hw(x)| \mid x \in X\}$ (see [5]). Moreover, for every $x \in X$, $\psi w(x) = \min\{|\mathcal{U}_x| \mid \mathcal{U}_x$ is a family of open neighbourhoods of x and $\bigcap\{\overline{U} \mid U \in \mathcal{U}_x\} = Hw(x)\}$. Thus, we have that $\psi w(X) = \sup\{\psi w(x) \mid x \in X\}$.

In the following results, if $Y \subseteq X$, we denote by Y^* the set $\bigcup\{Hw(x) : x \in Y\}$. The next application of Theorem 2.2, generalizes [17, Theorem 2.2], [5, Theorem 2.22] and [6, Theorem 31] (see Corollary 3.3 parts (a), (b) and (c), respectively).

Theorem 3.2. *Let X be a T_1-space and for every infinite cardinal κ assume that:*

(i) $aL_\kappa(X)\psi w(X) \leq \kappa$;
(ii) *For each $A \in [X]^{\leq\kappa}$, $|\overline{A}| \leq 2^\kappa$.*

Then $|X| \leq HW(X)2^\kappa$.

Proof. Let $\tau = \kappa^+$. For every $x \in X$ we fix a collection $\mathcal{B}_x$ of open subsets of X containing x such that $|\mathcal{B}_x| \leq \kappa$ and $\bigcap\overline{\mathcal{B}}_x = Hw(x)$. We consider the operator $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\rho(A) = [A]_\kappa$. By (ii), we have that ρ is a (τ, κ)-closing operator. For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$ in X, we put $\Theta(s) = \bigcup\{\bigcup\overline{\mathcal{C}} \mid \mathcal{C} \in \mathcal{F}\}$ and we take $g : \mathcal{L} \to \mathcal{Q}$ defined by $g(F) = \bigcup\{\mathcal{B}_x \mid x \in \rho(F)\}$, for $F \in \mathcal{L}$. By Theorem 2.2, there is a family $\mathcal{E} = \{E_\alpha \mid \alpha < \kappa^+\}$, which is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$.

Let $H = \bigcup\mathcal{E}$. We have that $\rho(H) = \bigcup\{\rho(E_\alpha) \mid \alpha \in \kappa^+\}$. Indeed, if $p \in \rho(H)$, then there exists $C \in [H]^{\leq\kappa}$ such that $p \in \overline{C}$. Because $H = \bigcup\mathcal{E}$, there exists $\alpha_p < \kappa^+$ such that $C \subseteq \bigcup\{E_\beta \mid \beta < \alpha_p\}$. By hypothesis in (i), $[C]_\kappa = \rho(C) \subseteq E_{\alpha_p}$. Thus, $p \in \bigcup\{\rho(E_\alpha) \mid \alpha \in \kappa^+\}$. Moreover, it is clear that $|H| \leq 2^\kappa$. Hence $|\rho(H)| \leq 2^\kappa$. Then $|\rho(H)^*| \leq HW(x)2^\kappa$.

Let us show that $X \subseteq \rho(H)^*$. Assume the contrary, and fix $p \in X \setminus \rho(H)^*$. For each $x \in \rho(H)$, choose $U_x \in \mathcal{B}_x$ such that $p \notin \overline{U}_x$. Hence $\mathcal{V} = \{U_x \mid x \in \rho(H)\}$ is a collection of open subsets of X which cover $\rho(H)$. Thus, there exists $A \in [\rho(H)]^{\leq\kappa}$ such that $\rho(H) \subseteq \bigcup\{\overline{U}_x \mid x \in A\}$.

It is clear that $p \in X \setminus \bigcup\{\overline{U}_x \mid x \in A\}$. Let $\mathcal{F}_0 = \{U_x \mid x \in A\}$ and we put $s_0 = (\varnothing, \{\mathcal{F}_0\})$. Then, we have that s_0 is a small κ-sensor in X, which is Θ-good for $\mathcal{E}$. This is a contradiction, since $\mathcal{E}$ is a (g, ρ, Θ)-quasi-propeller. Thus, $X = \rho(H)^*$ and therefore, $|X| \leq HW(X)2^\kappa$. □

Corollary 3.3. *Let X be a T_1-space with $H^*(X) \leq \omega$. Then*

(a) $|X| \leq HW(X)2^{aL_\kappa(X)\chi(X)}$.
(b) ([5]) $|X| \leq HW(X)2^{aL_c(X)\chi(X)}$.
(c) ([6]) $|X| \leq HW(X)2^{L(X)\chi(X)}$.

Proof. (a) By [5, Note 2.21], we have that $aL_\kappa(X)\psi w(X) \leq aL_\kappa(X)\chi(X)$. Thus, part (i) from Theorem 3.2 holds. Moreover, by [6, Proposition 2.8], we obtain that, for every $A \in [X]^{\leq aL_\kappa(X)\chi(X)}$, $|\overline{A}| \leq 2^{aL_\kappa(X)\chi(X)}$. Thus, part (ii) from Theorem 3.2 holds. We have shown that $|X| \leq HW(X)2^{aL_\kappa(X)\chi(X)}$.

(b) Let $\kappa = aL_c(X)\chi(X)$. Since $t(X) \leq \chi(X) \leq \kappa$, we have that each κ-closed subset is a closed subset; hence, $aL_\kappa(X) \leq \kappa$. Moreover, $\psi w(X) \leq \chi(X) \leq \kappa$; thus, $aL_\kappa(X)\psi w(X) \leq \kappa$. It is easy to see that part (ii) from Theorem 3.2 holds. Therefore, $|X| \leq HW(X)2^{aL_c(X)\chi(X)}$. □

The following definition is from [2]. Let X be a T_1 space. A subspace Y of X is said to be *Lindelöf in X* if for each open cover $\mathcal{U}$ of X, there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\omega}$ such that $Y \subseteq \bigcup\mathcal{V}$. For a cardinal number κ, we say that Y is *initially κ-Lindelöf in X*, if for every open cover $\mathcal{U}$ of X of cardinality less than κ, there is a subcolection $\mathcal{V} \in [\mathcal{U}]^{\leq\omega}$ such that $Y \subseteq \bigcup\mathcal{V}$.

Theorem 3.4. *Let X be a T_1-space with $H^*(X) \leq \omega$, and let Y be a subspace dense in X and initially $2^{\chi(X)}$-Lindelöf in X, then $|X| \leq 2^{\chi(X)}$ and Y is Lindelöf in X.*

Proof. Let $\kappa = \chi(X)$ and $\tau = \kappa^+$. For every $x \in X$ fix $\mathcal{B}_x$ a local base of x in X such that $|\mathcal{B}_x| \leq \kappa$. We consider the operator $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\rho(A) = \overline{A}$. Note that, by [6, Proposition 28], ρ is a (κ^+, κ)-closing operator. For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$ in X, we put $\Theta(s) = \bigcup\{\bigcup\mathcal{C} \mid \mathcal{C} \in \mathcal{F}\}$. Let $g : \mathcal{L} \to \mathcal{Q}$ be given by $g(F) = \bigcup\{\mathcal{B}_x \mid x \in \rho(F)\}$, for $F \in \mathcal{L}$. By Theorem 2.2, there is a family $\mathcal{E} = \{E_\alpha \mid \alpha < \kappa^+\}$, which is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$.

Let $H = \bigcup\mathcal{E}$. Since $\chi(X) \leq \kappa$, $\overline{H} = \bigcup\{\overline{E}_\alpha \mid \alpha \in \kappa^+\}$. Thus $|\rho(H)| \leq 2^\kappa$.

We show that $Y \subseteq \rho(H)$. Assume the contrary, and fix $p \in Y \setminus \rho(H)$. For each $x \in \rho(H)$, choose $U_x \in \mathcal{B}_x$ such that $p \notin U_x$. Hence, $\mathcal{V} = \{U_x \mid x \in \rho(H)\} \cup \{X \setminus \rho(H)\}$ is an open cover of X such that $|\mathcal{V}| \leq 2^\kappa$. Since Y is 2^κ-Lindelöf in X there exists $A \in [\rho(H)]^{\leq\omega}$ such that $Y \subseteq \bigcup\{U_x \mid x \in A\} \cup (X \setminus \rho(H))$. Clearly, if $x \in Y \cap \rho(H)$, then $x \notin X \setminus \rho(H)$. On the other hand, $p \in Y \setminus \bigcup\{U_x \mid x \in A\}$.

We put $\mathcal{F}_0 = \{U_x \mid x \in A\}$ and we put $s_0 = (\varnothing, \{\mathcal{F}_0\})$. Then, we have that s_0 is a κ-sensor in X, s_0 is generated by the pair $(\bigcup\rho(\mathcal{E}), \mathcal{U}_g(\mathcal{E}))$ and $Y \cap \bigcup\rho(\mathcal{E}) \subseteq \Theta(s_0)$. Thus, we conclude that s_0 is a κ-sensor in X, which is small for Y and is Θ-good for $\mathcal{E}$ with respect to Y, a contradiction, because $\mathcal{E}$ is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$. Hence, $Y \subseteq \rho(H)$. Therefore, $|Y| \leq 2^\kappa$. Finally, since Y is dense in X, from [6, Proposition 28], we conclude that $|X| \leq 2^\kappa$. In consequence, and since Y is 2^κ-Lindelöf in X, it follows that Y is Lindelöf in X. □

From Theorem 3.4, we obtain the following result due to Arhangel'skiĭ (see [2, Corollary 1]).

Corollary 3.5 ([2]). *Let X be a Hausdorff space, and let Y is a subspace dense in X and initially $2^{\chi(X)}$-Lindelöf in X, then $|X| \leq 2^{\chi(X)}$ and Y is Lindelöf in X.*

For the third application of Theorem 2.2, we recall that given a topological space and κ an infinite cardinal, we say that a subset $A \in [X]^{\leq 2^\kappa}$ is *κ-quasi-dense in X* if for every open cover $\mathcal{U}$ of X, there exist $B \in [A]^{\leq\kappa}$ and $\mathcal{V} \in [\mathcal{U}]^\kappa$ such that $X = cl_X(B) \cup \bigcup \mathcal{V}$. Moreover, $qL(X) = \min\{\kappa \mid$ there is a κ-quasi-dense subset A in $X\}$.

Theorem 3.6. *If X is a T_1-space with $H^*(X) \leq \omega$, then $|X| \leq 2^{qL(X)\chi(X)}$.*

Proof. Let $\kappa = qL(X)\chi(X)$, $\tau = \kappa^+$, for each $x \in X$, let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq \kappa$. Since $qL(X) \leq \kappa$, there exist $A \in [X]^{\leq 2^\kappa}$ which is a κ-quasi-dense in X. Let $g(F) = \bigcup\{\mathcal{B}_x \mid x \in \overline{F}\}$, for every $F \in \mathcal{L}$. We consider $\rho(A) = \overline{A}$. Then, since $H^*(X) \leq \omega$, we obtain by [6, Proposition 28] that ρ is a (κ^+, κ)-closing operator.

For every sensor $s = (\mathcal{A}, \mathcal{F})$, we put $\Theta(s) = cl_X(\bigcup\mathcal{A}) \cup \bigcup\{\bigcup\mathcal{C} \mid \mathcal{C} \in \mathcal{F}\}$. By Theorem 2.2, there is a family $\mathcal{E} = \{E_\alpha \mid \alpha < \kappa^+\}$, which is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$.

Let $H = \bigcup\mathcal{E}$. Note that $|H| \leq 2^\kappa$. Hence, $|\rho(H)| \leq 2^\kappa$. Observe that, $\rho(H) = \bigcup\{\rho(E_\alpha) \mid \alpha \in \kappa^+\}$.

We prove that $X \subseteq \rho(H)$. Suppose that there exists $p \in X \backslash \rho(H)$. For every $x \in \rho(H)$, let $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. It is clear that the collection $\{V_x \mid x \in \rho(H)\} \cup \{X \backslash \rho(P)\}$ cover X. Since $qL(X) \leq \kappa$, there exist $D \in [A]^{\leq\kappa}$ and $B \in [\rho(H)]^{\leq\kappa}$ such that $X = cl_X(D) \cup \bigcup\{V_x \mid x \in B\} \cup X \backslash \rho(H)$. Let $\mathcal{A} = \{D\}$, $\mathcal{F} = \{V_x \mid x \in B\}$ and let $s_0 = (\mathcal{A}, \mathcal{F})$. Clearly $p \notin \Theta(s_0)$ and $H \subseteq \rho(H) \subseteq \Theta(s_0)$. Then, we conclude that s_0 is a small κ-sensor in X, which is Θ-good for $\mathcal{E}$, a contradiction, because $\mathcal{E}$ is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$. □

Corollary 3.7. *For every Hausdorff space X,*

(1) $|X| \leq 2^{qL(X)\chi(X)}$.
(2) ([15]) $|X| \leq 2^{qL(X)\psi(X)t(X)}$.

Problem 3.8. *If X is a T_1-space with $H^*(X) \leq \omega$, then $|X| \leq 2^{qL(X)\psi(X)t(X)}$.*

For another application of Theorem 2.2 we consider the following notion introduced by Arhangel'skiĭ in [2] for $\kappa = \omega$. Given a topological space X and κ an infinite cardinal, we say that X *is strictly quasi-κ-Lindelöf* if for every closed subset P of X and every collection $\{\mathcal{U}_\alpha \mid \alpha \in \kappa\}$ of families of open subsets in X sucht that $P \subseteq \overline{\bigcup\{\bigcup\mathcal{U}_\alpha \mid \alpha \in \kappa\}}$, there exists, for each $\alpha \in \kappa$, $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^\omega$ such that $P \subseteq \bigcup\{\overline{\bigcup\mathcal{V}_\alpha} \mid \alpha \in \kappa\}$. It is easy to see that for every κ, if X is Lindelöf, then X is strictly quasi-κ-Lindelöf.

Theorem 3.9. *Let X be a T_1-space with $H^*(X) \leq \omega$. Let κ be an infinite cardinal such that:*

(i) *$\chi(X) \leq \kappa$,*
(ii) *X is strictly quasi-κ-Lindelöf.*

Then $|X| \leq HW(X)2^\kappa$.

Proof. Let $\tau = \kappa^+$. For every $x \in X$, we fix a collection $\mathcal{B}_x$ of open subset of X containing x such that $|\mathcal{B}_x| \leq \kappa$ and $\bigcap \overline{\mathcal{B}}_x = Hw(x)$. Let $g(F) = \bigcup\{\mathcal{B}_x \mid x \in \overline{F}\}$, for every $F \in \mathcal{L}$. We consider the operator $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\rho(A) = \overline{A}$. Note that, from [6, Proposition 28], ρ is a (κ^+, κ)-closing operator.

For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$ in X, we put $\Theta(s) = \bigcup\{\overline{\bigcup \mathcal{C}} : \mathcal{C} \in \mathcal{F}\}$. By Theorem 2.2, there is a family $\mathcal{E} = \{E_\alpha \mid \alpha < \kappa^+\}$, which is a (g, ρ, Θ)-quasi-propeller in $\mathcal{L}$.

Let $H = \bigcup \mathcal{E}$. Note that $|H| \leq 2^\kappa$. Moreover, since $\rho(H) = \bigcup\{\rho(E_\alpha) \mid \alpha \in \kappa^+\}$, then $|\rho(H)| \leq 2^\kappa$. Hence, $|\rho(H)^*| \leq HW(X)2^\kappa$. Because $\chi(X) \leq \kappa$, we obtain that $\rho(H)$ is a closed subset of X.

We prove that $X \subseteq \rho(H)^*$. For this end, suppose that there exists $p \in X \setminus \rho(H)^*$ and let $\mathcal{U}_p = \{U_\alpha \mid \alpha \in \kappa\}$ a local base of p in X. For every $\alpha \in \kappa$, let $\mathcal{U}_\alpha = \{V \in \mathcal{U}_g(\mathcal{E}) \mid V \cap U_\alpha = \varnothing\}$.

We claim that $\rho(H) \subseteq \bigcup\{\bigcup \mathcal{U}_\alpha \mid \alpha \in \kappa\}$. Indeed, let $x \in \rho(H)$. Since $p \notin \rho(H)^* = \bigcup\{Hw(h) : h \in \rho(H)\}$, then $p \notin Hw(x)$. Thus, since $Hw(x) = \bigcap\{\overline{U} \mid U \in \mathcal{B}_x\}$, there exists $V \in \mathcal{B}_x$ such that $p \notin \overline{V}$. Observe that there is $U_\beta \in \mathcal{U}_p$ such that $U_\beta \cap V = \varnothing$. Thus, $x \in \bigcup \mathcal{U}_\beta$. Hence, $x \in \bigcup\{\bigcup \mathcal{U}_\alpha \mid \alpha \in \kappa\}$, therefore, $\rho(H) \subseteq \bigcup\{\bigcup \mathcal{U}_\alpha \mid \alpha \in \kappa\}$.

Now, since X is strictly quasi-κ-Lindelöf y $\rho(H)$ is a closed subset, for every $\alpha \in \kappa$, there is $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{\leq \omega}$ such that $\rho(H) \subseteq \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} \mid \alpha \in \kappa\}$. Let $\mathcal{F} = \{\mathcal{V}_\alpha \mid \alpha \in \kappa\}$ and $s_0 = (\varnothing, \mathcal{F})$. By construction, we note that s is generated by $(\bigcup \mathcal{E}, \mathcal{U}_g(\mathcal{E}))$. Moreover, $p \notin \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} \mid \alpha \in \kappa\}$; that is, $p \in X \setminus \Theta(s_0)$. Hence, we conclude that s_0 is a small κ-sensor in X which is Θ-good for $\mathcal{E}$, a contradiction, because $\mathcal{E}$ is a (g, ρ, Θ)-quasi-propeller. Thus, we obtain that $X \subseteq \rho(H)^*$. Therefore, $|X| \leq HW(X)2^\kappa$. □

From Theorem 3.9, we obtain the following result due to Arhangel'skii (see [2, Corollary 22]).

Corollary 3.10 ([2])**.** *Let X be a T_2-space strictly quasi-$\chi(X)$-Lindelöf. Then $|X| \leq 2^{\chi(X)}$.*

It is easy to see that if $\kappa = c(X)$ or $\kappa = L(X)$, then X is strictly quasi-κ-Lindelöf. Hence, Theorem 3.9 is a common generalization of the inequalities $|X| \leq 2^{c(X)\chi(X)}$ and $|X| \leq 2^{L(X)\chi(X)}$, where X is a Hausdorff space.

Problem 3.11. *Let X be a T_1-space with $H^*(X) \leq \omega$ and strictly quasi-κ-Lindelöf.*

(1) *If $\psi(X)t(X) \leq \kappa$, then $|X| \leq 2^\kappa$?*
(2) *If $\psi(X)F(X) \leq \kappa$, then $|X| \leq 2^\kappa$?*

Finally, we present the last application of Theorem 2.2 to prove some upper bounds to density, netweight and cardinality of Hausdorff spaces, which are inspired by the bounds obtained by Charlesworth in [9]. Before this, we recall that if X is a Hausdorff space, then a collection of open subsets of $X, \mathcal{U}$, is called *closed separating cover of* X, if $X = \bigcup \mathcal{U}$ and $\bigcap \{\overline{U} \mid U \in \mathcal{U} \text{ and } x \in U\} = \{x\}$, for each $x \in X$. Moreover, the *closed point separating weight of* X, denoted $psw_c(X)$, is the smallest infinite cardinal κ such that X has a closed separating cover $\mathcal{U}$, such that $|\mathcal{U}_x| \leq \kappa$, where $\mathcal{U}_x = \{U \in \mathcal{U} \mid x \in U\}$. We have the following result.

Theorem 3.12. *If X is a Hausdorff space, then $d(X) \leq psw_c(X)^{aL_c(X)}$.*

Proof. Let $\kappa = aL_c(X)$ and $\tau = psw_c(X)$. Let $\mathcal{U}$ be a closed separating cover of X with $|\mathcal{U}_x| \leq \kappa$, where $\mathcal{U}_x = \{U \in \mathcal{U} \mid x \in U\}$. Let $\mathcal{L} = [X]^{\leq \tau^\kappa}$, $\mathcal{Q} = [\mathcal{P}(X)]^{\leq \tau^\kappa}$, and $g : \mathcal{L} \to \mathcal{Q}$ given by $g(F) = \bigcup \{\mathcal{U}_x \mid x \in F\}$. We consider the operator identity ρ. Clearly, ρ is (τ, κ)-closing. For each κ-sensor $s = (\mathcal{A}, \mathcal{F})$, we put $\Theta(s) = \bigcup(\overline{\bigcup \mathcal{F}})$. Thus, there exists a family $\mathcal{E} = \{E_\alpha \mid \alpha < \tau\} \subseteq \mathcal{L}$ such that (1) and (2) of Theorem 2.2 hold. Clearly $|P| \leq \tau^\kappa$, where $P = \bigcup \mathcal{E}$.

We show that $X \subseteq \overline{P}$. Indeed, suppose $p \in X \setminus \overline{P}$. Since $\mathcal{U}$ is closed separating cover of X, for each $x \in \overline{P}$, there exists $U_x \in \mathcal{U}_x$ such that $p \notin \overline{U}_x$. Clearly, the collection $\mathcal{U} = \{U_x \mid x \in \overline{P}\}$ covers $\overline{P}$. Hence, there exists $A \in [\overline{P}]^{\leq \kappa}$ such that $\overline{P} \subseteq \bigcup \{\overline{U_x} \mid x \in A\}$.

Note that each $x \in A$ may be replaced by an $x' \in P$. Indeed, since $x \in A$, then $x \in \overline{P}$. Hence, $U_x \cap P \neq \varnothing$. Thus, there exists $x' \in U_x \cap P$. Hence, $U_x \in \mathcal{U}_{x'}$. It follows $\overline{P} \subseteq \bigcup \{\overline{U_x^{x'}} \mid x' \in A'\}$, where $U_x^{x'} = U_x$. Then, $s_0 = (\varnothing, \{\{U_x^{x'} \mid x' \in A'\}\})$ is a small κ-sensor which is Θ-good for $\mathcal{E}$, a contradiction. Thus, $X \subseteq \overline{P}$. Therefore, $d(X) \leq psw_c(X)^{aL_c(X)}$. □

Corollary 3.13. *If X is a Hausdorff space, then $nw(X) \leq psw_c(X)^{aL_c(X)}$.*

Proof. Let $\kappa = aL_c(X)$ and let $\mathcal{U}$ be a closed separating cover of X with $|\mathcal{U}| \leq psw_c(X)$. By Theorem 3.12, there exists a dense subset D of X with $|D| \leq psw_c(X)^\kappa$. Let $\mathcal{N} = \{X \setminus \bigcup \overline{\mathcal{V}} \mid \mathcal{V} \in [\mathcal{U}]^{\leq \kappa}\}$. Notice that $|\mathcal{N}| \leq |[D]^{\leq \kappa}| \leq psw_c(X)^\kappa$. Thus $|\mathcal{N}| \leq psw_c(X)^\kappa$.

We claim that $\mathcal{N}$ is a network on X. Indeed, let $p \in X$ and let U be an open subset of X such that $p \in U$. For each $x \in X \setminus U$, we fix $U_x \in \mathcal{U}$ such that $p \notin \overline{U}_x$. Clearly $\{U_x \mid x \in X \setminus U\}$ covers $X \setminus U$. Hence, there exists $A \in [X \setminus U]^{\leq \kappa}$ such that $X \setminus U \subseteq \bigcup \{\overline{U}_x \mid x \in A\}$. Then $p \in X \setminus \bigcup \{\overline{U}_x \mid x \in A\} \subseteq U$. Thus, $\mathcal{N}$ is a network on X. The proof is complete. □

Corollary 3.14. *If X is a Hausdorff space, then $|X| \leq psw_c(X)^{aL_c(X)\psi(X)}$.*

Proof. It follows from [13, Theorem 4.1] that $|X| \leq nw(X)^{\psi(X)}$. Since

$$nw(X)^{\psi(X)} \leq (psw_c(X)^{aL_c(X)})^{\psi(X)},$$

then, by Corollary 3.13, we obtain that $|X| \leq psw_c(X)^{aL_c(X)\psi(X)}$. □

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