Applied General Topology

Appl. Gen. Topol. 20, no. 1 (2019), 119-133
doi:10.4995/agt.2019.10360

Existence of fixed points for pointwise eventually asymptotically nonexpansive mappings

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Communicated by E. A. Sánchez-Pérez

Abstract

Kirk introduced the notion of pointwise eventually asymptotically nonexpansive mappings and proved that uniformly convex Banach spaces have the fixed point property for pointwise eventually asymptotically nonexpansive maps. Further, Kirk raised the following question: "Does a Banach space X have the fixed point property for pointwise eventually asymptotically nonexpansive mappings whenever X has the fixed point property for nonexpansive mappings?". In this paper, we prove that a Banach space X has the fixed point property for pointwise eventually asymptotically nonexpansive maps if X has uniform normal sturcture or X is uniformly convex in every direction with the Maluta constant $D(X) < 1$. Also, we study the asymptotic behavior of the sequence $\{T^n x\}$ for a pointwise eventually asymptotically nonexpansive map T defined on a nonempty weakly compact convex subset K of a Banach space X whenever X satisfies the uniform Opial condition or X has a weakly continuous duality map.

2010 MSC: *47H10; 47H09.*

Keywords: *fixed points; pointwise eventually asymptotically nonexpansive mappings; uniform normal structure; uniform Opial condition; duality mappings.*

Received 06 June 2018 – Accepted 12 December 2018

1. Introduction

Let K be a nonempty weakly compact convex subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{\alpha_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} \alpha_n = 1$ such that for each integer $n \geq 1$,

$$\|T^n x - T^n y\| \leq \alpha_n \|x - y\|, \text{ for all } x, y \in K. \tag{1.1}$$

If $\alpha_n = 1$ in (1.1) for all $n \in \mathbb{N}$, then T is said to be nonexpansive. If for each $x \in K$, the following inequality holds

$$\limsup_{n\to\infty} \left(\sup_{y\in K} \{\|T^n x - T^n y\| - \|x - y\|\} \right) \leq 0,$$

then T is said to be asymptotically nonexpansive type. Kirk [13] proved that if K is a nonempty weakly compact convex set in a Banach space X with normal structure, then every nonexpansive map T on K has a fixed point. Goebel and Kirk [9] further proved that if X is a uniformly convex Banach space, then every asymptotically nonexpansive map T on K has a fixed point. Later, this result was extended to mappings of asymptotically nonexpansive type by Kirk [14]. However, it remains open that whether normal structure condition on a Banach space X guarantees the existence of fixed points of asymptotically nonexpansive mapping.

Kim and Xu [12] proved that if X is a Banach space with uniform normal structure, then every asymptotically nonexpansive map T on K has a fixed point. Li and Sims [8] proved the existence of fixed points of asymptotically nonexpansive type mappings in the setting of Banach spaces having uniform normal structure. Gossez and Lami Dozo [11] studied the class of spaces which satisfies Opial's condition and observed that all such spaces have normal structure. Hence, if X is a Banach space satisfying Opial's condition, then every nonexpansive map T on K has a fixed point. However, it is not clear whether Opial's condition implies the existence of fixed points for asymptotically nonexpansive mappings.

In this direction, Lin et. al [18] proved that every asymptotically nonexpansive map T on K has a fixed point whenever X is a Banach space that satisfies the uniform Opial condition. Also, Lim and Xu [17] proved the existence of fixed points of an asymptotically nonexpansive map T on K in a Banach space X whenever the Maluta constant $D(X) < 1$ and T is weakly asymptotically regular on K.

In 2008, Kirk and Xu [16] introduced the notion of pointwise asymptotically nonexpansive mappings and studied the existence of fixed points in the setting of uniformly convex Banach spaces.

Definition 1.1 ([16])**.** A mapping $T : K \to K$ is said to be pointwise asymptotically nonexpansive if for each $x \in K$ there exists a sequence $\{\alpha_n(x)\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 1$ such that for each integer $n \geq 1$,

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|, \quad \text{for all } y \in K.$$

Theorem 1.2 ([16])**.** *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $T : K \to K$ be a pointwise asymptotically nonexpansive map. Then T has a fixed point in K.*

Recently, Kirk [15] introduced the notion of pointwise eventually asymptotically nonexpansive mappings as follows:

Definition 1.3 ([15])**.** A mapping $T : K \to K$ is said to be pointwise eventually asymptotically nonexpansive if for each $x \in K$ there exists a sequence $\{\alpha_n(x)\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 1$ and an integer $N(x) \in \mathbb{N}$ such that for $n \geq N(x)$,

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|, \text{ for all } y \in K.$$

Though the definition of pointwise asymptotically nonexpansive mappings and pointwise eventually asymptotically nonexpansive mappings are quite simple to understand, examples of such mappings are rare.

In [15] Kirk proved that Theorem 1.2 holds for pointwise eventually asymptotically nonexpansive mappings. Further, Kirk [15] raised the following question:

Does a Banach space X have the fixed point property for pointwise eventually asymptotically nonexpansive mappings whenever X has the fixed point property for nonexpansive mappings?

In this paper we give a partial answer to the above question. In section 3, we prove that a Banach space X have the fixed point property for pointwise eventually asymptotically nonexpansive mappings if X has uniform normal structure or X is uniformly convex in every direction with the Maluta constant $D(X) < 1$.

In section 4, we study the asymptotic behavior of the sequence $\{T^n x\}$ for a pointwise eventually asymptotically nonexpansive map T defined on a nonempty weakly compact convex set K in a Banach space X whenever X satisfies the uniform Opial condition or X has a weakly continuous duality mapping. For results about asymptotic behavior of nonexpansive mappings and asymptotically nonexpansive mappings, one may refer to [2, 7, 17, 18].

2. Preliminaries

Let X be a Banach space and $\mathcal{C}$ be the collection of all closed bounded convex sets in X. For $K \in \mathcal{C}$, define

(1) for $x \in X$, $\delta(x, K) = \sup\{\|x - y\| : y \in K\}$;
(2) $r(K) = \inf\{\delta(x, K) : x \in K\}$ and
(3) $\delta(K) = diam(K) = \sup\{\delta(x, K) : x \in K\}$.

Definition 2.1 ([3])**.** A Banach space X is said to have normal structure if every nonempty closed bounded convex subset K of X with $diam(K) > 0$ has a point $x_0 \in K$ such that $\delta(x_0, K) < diam(K)$.

Also, Brodskii and Milman [3] gave a characterization for normal structure in terms of sequences as follows:

Theorem 2.2 ([3]). *A Banach space X does not have normal structure if and only if there exists a nonconstant bounded sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} d\left(x_{n+1}, co\{x_1, \ldots, x_n\}\right) = diam(\{x_n\}).$$

Using this characterization of normal structure, Maluta [19] defined the constant $D(X)$ of a given Banach space as follows:

Definition 2.3 ([19]). Let X be a Banach space. The Maluta constant $D(X)$ of X is defined as

$$D(X) = \sup\left\{\frac{\limsup_{n\to\infty} d\left(x_{n+1}, co\{x_1, \ldots, x_n\}\right)}{diam(\{x_n\})}\right\}$$

where the supremum is taken over all non-constant bounded sequences in X.

It is known from [19] that a Banach space X with $D(X) < 1$ has normal structure.

In [5] Bynum defined the concept of uniform normal structure as follows:

Definition 2.4 ([5]). A Banach space X is said to have uniform normal structure if $N(X) < 1$, where $N(X) = \sup\left\{\frac{r(K)}{\delta(K)} : K \in \mathcal{C} \text{ with } \delta(K) > 0\right\}$.

Remark 2.5. The following facts are known from [19].

(1) For a Banach space X, $0 \le D(X) \le 1$ and $D(X) \le N(X)$.
(2) If X is nonreflexive Banach space, then $D(X) = 1$. Thus if $D(X) < 1$ then X is reflexive.

Definition 2.6 ([20]). A Banach space X is said to satisfy Opial's condition if for each weakly convergent sequence $\{x_n\}$ in X with limit $x_0 \in X$,

$$\limsup_{n\to\infty} \|x_n - x_0\| < \limsup_{n\to\infty} \|x_n - x\|, \text{ for all } x \in X \text{ with } x \neq x_0.$$

It is known that every Hilbert space, finite dimensional Banach spaces and the Banach space $l^p(\mathbb{N})$ for $1 < p < \infty$ satisfy Opial's condition.

In [21] Prus introduced the notion of the uniform Opial condtion:

Definition 2.7 ([21]). A Banach space X is said to satisfy the uniform Opial condition if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \le \limsup_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \ge c$ and each sequence $\{x_n\}$ in X such that $w - \lim_{n\to\infty} x_n = 0$ and $\limsup_{n\to\infty} \|x_n\| \ge 1$.

Further, Prus [21] defined the Opial modulus of X denoted by r_X, as follows

$$r_X(c) = \inf\left\{\limsup_{n\to\infty} \|x_n + x\| - 1\right\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and sequences $\{x_n\}$ in X such that $w - \lim_{n\to\infty} x_n = 0$ and $\limsup_{n\to\infty} \|x_n\| \geq 1$. It is easy to see that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$.

Definition 2.8 ([10]). A continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ is said to be gauge if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.9 ([10]). Let X be a Banach space and φ be a gauge function. Then we associate with X a generalized duality map $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\},$$

where $\langle ., . \rangle$ is the pairing between X and X^*.

Define $\Phi(t) = \int_0^t \varphi(s)ds$ for $t \geq 0$. Then it is easy to see that Φ is a convex and gauge function on $[0, \infty)$. Also, it is known from [4] that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|.\|)$ at $x \in X$.

Definition 2.10 ([10]). A Banach space X is said to have a weakly continuous duality map if there exists a gauge function φ such that the duality map J_φ is single-valued and continuous from X with the weak topology to X^* with the weak* topology.

It is clear from [10] that the Banach space $l^p(\mathbb{N})$ for $1 < p < \infty$ has a weakly continuous duality map with the gauge function $\varphi(t) = t^{p-1}$. Also, Browder [4] proved that a Banach space X with a weakly continuous duality map satisfies Opial's condition.

3. Existence Result

In this section, we prove that a pointwise eventually asymptotically nonexpansive mappings defined on a weakly compact convex subset K of a Banach space X always has a fixed point if X has uniform normal structure or X is uniformly convex in every direction with the Maluta constant $D(X) < 1$.

We use the following two Lemmas in the sequel.

Lemma 3.1 ([6]). *Let X be a Banach space with uniform normal structure. Then for every bounded sequence $\{x_n\}$ in X there exists a point $z \in \bigcap_{k=1}^{\infty} \overline{co}\{x_n : n \geq k\}$ such that*

(1) $\limsup_{n\to\infty} \|x_n - z\| \leq N(X)\delta(\overline{co}(\{x_n\}))$
(2) $\|z - y\| \leq \limsup_{n\to\infty} \|x_n - y\|$, *for all $y \in X$.*

Lemma 3.2 ([22]). *Let K be a nonempty weakly compact convex subset of a Banach space X and $T : K \to K$ be an asymptotically nonexpansive type mapping. Let K_0 be minimal with respect to being a nonempty closed convex subset of K such that for every $x \in K_0$ we have $\omega_w(x) \subseteq K_0$, where $\omega_w(x)$ is* *the set of all weak subsequential limit points of the sequence* $\{T^n x : n \in \mathbb{N}\}$. *Then there exists a constant* $\rho_0 \geq 0$ *such that* $\limsup_{n\to\infty} \|T^n x - y\| = \rho_0$ *for all* $x, y \in K_0$.

Remark 3.3. We infer that if K is bounded, then a pointwise eventually asymptotically nonexpansive mapping on K is of asymptotically nonexpansive type.

Theorem 3.4. *Let* K *be a nonempty weakly compact convex subset of a Banach space* X *with uniform normal structure and* $T : K \to K$ *be a pointwise eventually asymptotically nonexpansive map. Then* T *has a fixed point in* K.

Proof. Assume that K_0 is minimal with respect to being a nonempty closed convex subset of K with the property that for every $x \in K_0$, $\omega_w(x) \subseteq K_0$. By Lemma 3.2, there is a constant $\rho_0 \geq 0$ such that $\limsup_{n\to\infty} \|T^n x - y\| = \rho_0$, for all $x, y \in K_0$. First note that if $\rho_0 = 0$, then $\lim_{n\to\infty} \|T^n x - x\| = 0$ and $\lim_{n\to\infty} \|T^n x - Tx\| = 0$ for $x \in K_0$, and it follows that $K_0 = \{x\}$ with $Tx = x$. To see that $\rho_0 = 0$ we break the proof into two assertions.

Assertion I : If $\{T^n x\}$ has a convergent subsequence for some $x \in K_0$, then $\rho_0 = 0$.

Proof. Assume $\rho_0 > 0$, and suppose that there exists a $x \in K_0$ such that $\lim_{i\to\infty} T^{n_i} x = y$ for some $y \in K_0$, and choose $r_1 > 0$ so that $(1 + r_1)N(X) < 1$. Since $\alpha_n(y) \to 1$, there exists a natural number $N_1 \geq N(y)$ such that $\alpha_n(y) < 1 + r_1$, for all $n \geq N_1$.

Define $F = \overline{co}\{T^n y : n \geq N_1\}$. For $l, m \in \mathbb{N}$ with $l > m \geq N_1$,

$$\begin{aligned} \|T^l(y) - T^m(y)\| &= \lim_{i\to\infty} \|T^{l+n_i}(x) - T^m y\| \\ &\leq \limsup_{n\to\infty} \|T^n x - T^m y\| \\ &\leq \limsup_{n\to\infty} \alpha_m(y)\|T^{n-m} x - y\| \\ &= \alpha_m(y)\rho_0 < (1 + r_1)\rho_0. \end{aligned}$$

This gives that $\delta(F) \leq (1 + r_1)\rho_0$.

Now by Lemma 3.1, there exists a $z \in F \cap K_0$ such that

$$\begin{aligned} \rho_0 = \limsup_{n\to\infty} \|T^n y - z\| &\leq N(X)\delta(F) \\ &\leq N(X)(1 + r_1)\rho_0 < \rho_0. \end{aligned}$$

Hence assertion I is proved.

Assertion II : There exists a $x \in K_0$ such that $\{T^n x\}$ has a convergent subsequence.

Proof. Let $x_0 \in K_0$ and define $D_1 = \overline{co}\{T^n x_0 : n = 0, 1, 2, \dots\}$. By Lemma 3.1, there exists a $x_1 \in D_1 \cap K_0$ such that

$$0 \leq \beta_1 = \limsup_{n\to\infty} \|x_1 - T^n x_0\| \leq N(X)\delta(D_1).$$

Choose $r_0 > 0$ so that $r = (1 + r_0)^2 N(X) < 1$. Since $\alpha_n(x_1) \to 1$, there exists a natural number $n_1 \geq N(x_1)$ such that $\alpha_n(x_1) < 1 + r_0$, for all $n \geq n_1$.

Define $D_2 = \overline{co}\{T^{kn_1}(x_1) : k = 0, 1, 2, \dots\}$. For $k \geq 1$, we have

$$\|T^{kn_1}(x_1) - T^n(x_0)\| \leq \alpha_{kn_1}(x_1)\|x_1 - T^{n-kn_1}(x_0)\|.$$

Letting $\limsup\limits_{n\to\infty}$ on both sides, we get

$$\begin{aligned}\limsup_{n\to\infty} \|T^{kn_1}(x_1) - T^n(x_0)\| &\leq \alpha_{kn_1}(x_1) \limsup_{n\to\infty} \|x_1 - T^{n-kn_1}(x_0)\| \\ &= \alpha_{kn_1}(x_1) \limsup_{n\to\infty} \|x_1 - T^n(x_0)\| \\ &= \alpha_{kn_1}(x_1)\beta_1\end{aligned}$$

and for $l, m \in \mathbb{N}$ with $l > m$,

$$\begin{aligned}\|T^{mn_1}(x_1) - T^{ln_1}(x_1)\| &\leq \alpha_{mn_1}(x_1)\|x_1 - T^{(l-m)n_1}(x_1)\| \\ &\leq \alpha_{mn_1}(x_1) \limsup_{n\to\infty} \|T^{(l-m)n_1}(x_1) - T^n x_0\| \\ &= \alpha_{mn_1}(x_1)\alpha_{(l-m)n_1}(x_1)\beta_1 \\ &< (1 + r_0)^2\beta_1 \\ &\leq (1 + r_0)^2 N(X)\delta(D_1) = r\delta(D_1).\end{aligned}$$

This gives that $\delta(D_2) \leq r\delta(D_1)$.

Now by Lemma 3.1, there exists a $x_2 \in D_2 \cap K_0$ such that

$$0 \leq \beta_2 = \limsup_{k\to\infty} \|x_2 - T^{kn_1}(x_1)\| \leq N(X)\delta(D_2).$$

Since $\alpha_{kn_1}(x_2) \to 1$ as $k \to \infty$, we can choose $k_1 \in \mathbb{N}$ such that $n_2 = k_1 n_1 \geq N(x_2)$ and $\alpha_{kn_1}(x_2) < 1 + r_0$, for all $k \geq k_1$.

Define $D_3 = \overline{co}\{T^{kn_2}(x_2) : k = 0, 1, 2, \dots\}$. For $l \geq 1$, we have

$$\begin{aligned}\|T^{ln_2}(x_2) - T^{kn_1}(x_1)\| &\leq \alpha_{ln_2}(x_2)\|x_2 - T^{kn_1 - ln_2}(x_1)\| \\ &= \alpha_{ln_2}(x_2)\|x_2 - T^{(k-lk_1)n_1}(x_1)\|.\end{aligned}$$

This implies that $\limsup\limits_{k\to\infty} \|T^{ln_2}(x_2) - T^{kn_1}(x_1)\| \leq \alpha_{ln_2}(x_2)\beta_2$.

So for $l, m \in \mathbb{N}$ with $l > m$,

$$\begin{aligned}\|T^{mn_2}(x_2) - T^{ln_2}(x_2)\| &\leq \alpha_{mn_2}(x_2)\|x_2 - T^{(l-m)n_2}(x_2)\| \\ &\leq \alpha_{mn_2}(x_2) \limsup_{k\to\infty} \|T^{(l-m)n_2}(x_2) - T^{kn_1}(x_1)\| \\ &\leq \alpha_{mn_2}(x_2)\alpha_{(l-m)n_2}(x_2)\beta_2 \\ &< (1 + r_0)^2\beta_2 \\ &\leq (1 + r_0)^2 N(X)\delta(D_2) \\ &\leq r^2\delta(D_1).\end{aligned}$$

This gives that $\delta(D_3) \leq r^2\delta(D_1)$.

By continuing the above process, we obtain a sequence $\{x_m\}$ and a sequence of sets $\{D_m\}$ with the following properties:

(1) There exists a $x_m \in D_m \cap K_0$ such that

$$0 \leq \beta_m = \limsup_{k\to\infty} \|x_m - T^{kn_{m-1}}(x_{m-1})\| \leq N(X)\delta(D_m)$$

where $D_m = \overline{co}\{T^{kn_{m-1}}(x_{m-1}) : k = 0, 1, 2, \dots\}$ and $n_{m-1} = k_{m-2}n_{m-2} \geq N(x_{m-1})$ for some $k_{m-2} \in \mathbb{N}$.

(2) $\delta(D_m) \leq r\delta(D_{m-1})$ and hence $\delta(D_m) \leq r^{m-1}\delta(D_1)$.

Note that $x_{m-1}, x_m \in D_m$ and $\|x_m - x_{m-1}\| \leq \delta(D_m) \leq r^{m-1}\delta(D_1)$. This implies that $\{x_m\}$ is a Cauchy sequence in K_0.

Thus, there exists a $x \in K_0$ such that $x = \lim_{m\to\infty} x_m$. Now, for $k \geq N(x)$,

$$\begin{aligned}\|T^k x - x\| &\leq \|T^k x - T^k x_m\| + \|T^k x_m - x\| \\ &\leq \alpha_k(x)\|x - x_m\| + \|T^k x_m - x_{m+1}\| + \|x_{m+1} - x\|\end{aligned}$$

Letting $\liminf_{k\to\infty}$ on both sides, we get

$$\begin{aligned}\liminf_{k\to\infty} \|T^k x - x\| &\leq \|x - x_m\| + \liminf_{k\to\infty} \|T^k x_m - x_{m+1}\| + \|x_{m+1} - x\| \\ &\leq \|x - x_m\| + \liminf_{k\to\infty} \|T^{kn_m} x_m - x_{m+1}\| + \|x_{m+1} - x\| \\ &\leq \|x - x_m\| + \beta_{m+1} + \|x_{m+1} - x\|.\end{aligned}$$

As $\beta_m \to 0$ and $x_m \to x$, we have $\liminf_{k\to\infty} \|T^k x - x\| = 0$.
Thus, there exists a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ such that $\lim_{i\to\infty} T^{n_i}x = x$. Hence assertion II is proved.

Therefore $\rho_0 = 0$ and K_0 is singleton. Hence T has a fixed point in K. □

Next, we use the ultrafilter techniques to prove the existence of fixed points for a pointwise eventually asymptotically nonexpansive mapping T on K in a Banach space X with the Maluta constant $D(X) < 1$. This result (Theorem 3.6) is motivated by the following result of Lim and Xu [17]: If T is an asymptotically nonexpansive map defined on a nonempty weakly compact convex set K in a Banach space with the Maluta constant $D(X) < 1$ and T is weakly asymptotically regular on K, then T has a fixed point in K. In the following remark, we recall some facts about ultrafilter which are used in the proof of Theorem 3.6. For more information about ultrafilters, one may refer to [1, 10].

Remark 3.5 ([1]). A filter $\mathcal{F}$ on $\mathbb{N}$ is a nonempty collection of subsets of $\mathbb{N}$ satisfying

(1) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
(2) if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

If $\mathcal{F}$ is a filter on $\mathbb{N}$ and $\varnothing \in \mathcal{F}$, then $\mathcal{F} = 2^{\mathbb{N}}$ and is called an improper filter.

By usual set inclusion, the family $\mathcal{P}$ of all proper filters on $\mathbb{N}$ is a partially ordered set and every chain in $\mathcal{P}$ has an upper bound. By Zorn's lemma, we get a maximal proper filter in $\mathcal{P}$. A maximal filter in $\mathcal{P}$ is called an ultrafilter on $\mathbb{N}$.

A sequence $\{x_n\}$ in a Banach space X converges to $x \in X$ over the filter $\mathcal{F}$ if for every neighbourhood V of x, the set $\{n \in \mathbb{N} : x_n \in V\}$ belongs to $\mathcal{F}$ and it is denoted by $\lim_{\mathcal{F}} x_n = x$.

A trivial ultrafilter $\mathcal{F}_{n_0}$ on $\mathbb{N}$ is the collection of subsets of $\mathbb{N}$ which contains an element $n_0 \in \mathbb{N}$, where $n_0 \in \mathbb{N}$ is fixed and all other ultrafilters on $\mathbb{N}$ are said to be non-trivial. It is known that nontrivial ultrafilters always exist (Zorn's lemma), and a sequence $\{x_n\}$ in a compact set always converges relative to any nontrivial ultrafilter over $\mathbb{N}$.

Theorem 3.6. *Let K be a nonempty weakly compact convex subset of a Banach space X with the Maluta constant $D(X) < 1$ and $T : K \to K$ be a pointwise eventually asymptotically nonexpansive map. Further, assume that T is weakly asymptotically regular on $K\Big($ i.e., $w - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$ for all $x \in K\Big)$. Then T has a fixed point in K.*

Proof. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. Define a mapping S on K by $S(x) = w-\lim_{\mathcal{U}} T^n x$, for $x \in K$. Since K is weakly compact, $S(x)$ is well defined and there exists a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ such that $\{T^{n_i}x\}$ converges weakly to $S(x)$. For $x, y \in K$, we have

$$\begin{aligned}\|Sx - Sy\| &\leq \liminf_{k\to\infty} \|T^{n_k}x - T^{n_k}y\| \\ &\leq \limsup_{n\to\infty} \|T^n x - T^n y\| \\ &\leq \limsup_{n\to\infty} \alpha_n(x)\|x - y\| \\ &= \|x - y\|.\end{aligned}$$

Hence S is a nonexpansive map on K.

Since $D(X) < 1$, X has normal structure. Therefore, the nonexpansive map $S : K \to K$ has a fixed point in K. That is, there exists a $x \in K$ such that $w - \lim_{\mathcal{U}} T^n x = x$. This implies that there exists a subsequence $\{T^{n'_k}(x)\}$ of $\{T^n x\}$ converges weakly to x.

Without loss of generality, we may assume that $n'_k \geq N(x)$ for all $k \in \mathbb{N}$. Choose $r > 0$ so that $(1+r)^2 D(X) < 1$. Since $\alpha_n(x) \to 1$ as $n \to \infty$, we can find a subsequence $\{n_i\}$ of $\{n'_k\}$ such that $\alpha_{n_i}(x) < 1 + r$ and $\alpha_{n_{i+1}-n_i}(x) < 1 + r$ for all $i \in \mathbb{N}$. From the definition of $D(X)$,

$$\limsup_{i\to\infty} \|T^{n_i}x - x\| \leq D(X)\delta(\{T^{n_i}x\}).$$

Since T is weakly asymptotically regular at $x \in K$, for fixed $i > j$ it must be the case that $T^{n_t+(n_i-n_j)}(x)$ converges weakly to x as $t \to \infty$.

Now observe that

$$\begin{aligned}
\|T^{n_i}x - T^{n_j}x\| &\leq \alpha_{n_j}(x)\|T^{n_i-n_j}(x) - x\| \\
&\leq (1+r)\limsup_{t\to\infty}\|T^{n_i-n_j}(x) - T^{n_t+(n_i-n_j)}(x)\| \\
&\leq (1+r)\alpha_{n_i-n_j}(x)\limsup_{t\to\infty}\|x - T^{n_t}(x)\| \\
&\leq (1+r)^2\limsup_{t\to\infty}\|x - T^{n_t}(x)\|
\end{aligned}$$

Therefore

$$\limsup_{i\to\infty}\|x - T^{n_i}(x)\| \leq (1+r)^2 D(X)\limsup_{t\to\infty}\|x - T^{n_t}(x)\|$$

and since $(1+r)^2 D(X) < 1$, we conclude $\limsup_{i\to\infty}\|x - T^{n_i}(x)\| = 0$.

As T^m is continuous at x for all $m \geq N(x)$, $\{T^{n_i+m}(x)\}$ converges weakly to $T^m(x)$ and since T is weakly asymptotically regular at x this in turn implies $x = T^m(x)$ for all $m \geq N(x)$. Therefore $x = T^{m+1}x = T(T^m x) = Tx$. □

Theorem 3.7. *Let K be a nonempty weakly compact convex subset of a Banach space X, which is uniformly convex in every direction and $T : K \to K$ be a pointwise eventually asymptotically nonexpansive map. Further, assume that the Maluta constant $D(X) < 1$. Then T has a fixed point in K.*

Proof. Assume that K_0 is minimal with respect to being a nonempty closed convex subset of K with the property that for every $x \in K_0$, $\omega_w(x) \subseteq K_0$. By Lemma 3.2, there is a constant $\rho_0 \geq 0$ such that $\limsup_{n\to\infty}\|T^n x - y\| = \rho_0$, for all $x, y \in K_0$. Since X is uniformly convex in every direction, K_0 is singleton, say $K_0 = \{x\}$ and so the sequence $\{T^n x\}$ converges weakly to x.

Now choose $r > 0$ so that $(1+r)^2 D(X) < 1$. Since $\alpha_n(x) \to 1$ as $n \to \infty$, we can find a subsequence $\{n_i\}$ of $\{N(x), N(x)+1, \dots\}$ such that $\alpha_{n_i}(x) < 1+r$ and $\alpha_{n_{i+1}-n_i}(x) < 1+r$ for all $i \in \mathbb{N}$.

From the definition of $D(X)$,

$$\limsup_{i\to\infty}\|x - T^{n_i}x\| \leq D(X)\delta(\{T^{n_i}x\}).$$

Now for $i > j$, we have

$$\begin{aligned}
\|T^{n_i}x - T^{n_j}x\| &\leq \alpha_{n_j}(x)\|T^{n_i-n_j}(x) - x\| \\
&\leq (1+r)\limsup_{t\to\infty}\|T^{n_i-n_j}(x) - T^{n_t+(n_i-n_j)}(x)\| \\
&\leq (1+r)\alpha_{n_i-n_j}(x)\limsup_{t\to\infty}\|x - T^{n_t}(x)\| \\
&\leq (1+r)^2\limsup_{t\to\infty}\|x - T^{n_t}(x)\|
\end{aligned}$$

Thus, $\limsup_{i\to\infty}\|x - T^{n_i}(x)\| \leq (1+r)^2 D(X)\limsup_{t\to\infty}\|x - T^{n_t}(x)\|$. Since $(1+r)^2 D(X) < 1$, we have $\limsup_{i\to\infty}\|x - T^{n_i}(x)\| = 0$.

Finally, by the definition of T, we have $\{T^{n_i+m}(x)\}$ converging weakly to $T^m(x)$ for all $m \geq N(x)$. But, we know that the sequence $\{T^n x\}$ converges weakly to x. This implies that $x = T^m(x)$ for all $m \geq N(x)$. Hence x is a fixed point of T. □

4. Asymptotic Behavior

In this section, we investigate the asymptotic behavior of the sequence $\{T^n x\}$ for a pointwise eventually asymptotically nonexpansive map T defined on a nonempty weakly compact convex subset K of a Banach space X whenever X satisfies the uniform Opial condition or X has a weakly continuous duality map.

Lemma 4.1. *Let K be a nonempty weakly compact convex subset of a Banach space X satisfying Opial's condition and $T : K \to K$ be a pointwise eventually asymptotically nonexpansive map. Further, assume that T is weakly asymptotically regular at some $x \in K$. For $m \in \mathbb{N}$, define $b_m = \limsup_{i\to\infty} \|T^{n_i+m}(x) - y\|$ where $y = w - \lim_{i\to\infty} T^{n_i} x$. Then the sequence $\{b_m\}$ converges.*

Proof. Let $y = w - \lim_{i\to\infty} T^{n_i} x$. Since T is weakly asymptotically regular at $x \in K$, we have $y = w - \lim_{i\to\infty} T^{n_i+m}(x)$ for $m \in \mathbb{N}$.

For any $j \geq N(y)$, we have

$$\begin{aligned} b_{m+j} &= \limsup_{i\to\infty} \|T^{n_i+m+j}(x) - y\| \\ &\leq \limsup_{i\to\infty} \|T^{n_i+m+j}(x) - T^j y\| \\ &\leq \alpha_j(y) \limsup_{i\to\infty} \|T^{n_i+m}(x) - y\| \\ &= \alpha_j(y) b_m. \end{aligned}$$

We claim that $\lim_{m\to\infty} b_m$ exists. Note that there exists two subsequences $\{m_i\}$ and $\{m'_i\}$ of $\mathbb{N}$ such that $\lim_{i\to\infty} b_{m_i} = \limsup_{m\to\infty} b_m$ and $\lim_{i\to\infty} b_{m'_i} = \liminf_{m\to\infty} b_m$.

This gives that there exists $k_0 \in \mathbb{N}$ such that $m_j > m'_1 + N(y)$, for all $j \geq k_0$. i.e., $m_j = m'_1 + N(y) + n_j$, for some $n_j \in \mathbb{N}$.

For any $j \geq k_0$, we have

$$\begin{aligned} b_{m_j} &= b_{m'_1+N(y)+n_j} \\ &\leq \alpha_{N(y)+n_j}(y) b_{m'_1} \\ &= \alpha_{m_j - m'_1}(y) b_{m'_1} \end{aligned}$$

So $\limsup_{m\to\infty} b_m = \lim_{j\to\infty} b_{m_j} \leq b_{m'_1}$.

Similarly, for each $i \in \mathbb{N}$, we have $\limsup_{m\to\infty} b_m \leq b_{m'_i}$ and we get $\limsup_{m\to\infty} b_m \leq \liminf_{m\to\infty} b_m$. Hence the sequence $\{b_m\}$ converges. □

Theorem 4.2. *Let K be a nonempty weakly compact convex subset of a Banach space X satisfying the uniform Opial condition and $T : K \to K$ be a pointwise* *eventually asymptotically nonexpansive map. Then given an $x \in K$, $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Proof. If $\{T^n x\}$ converges weakly to a fixed point of T, then $\{T^n x\}$ is obviously weakly asymptotically regular at x.

Conversely, assume that T is weakly asymptotically regular at $x \in K$. Then we claim that $\omega_w(x) \subseteq F(T)$, where $F(T)$ is the set of all fixed point of T and $\omega_w(x)$ is singleton. To see that $\omega_w(x) \subseteq F(T)$, let $y \in \omega_w(x)$. Then there exists a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that $y = w - \lim_{i\to\infty} T^{n_i} x$. Since T is weakly asymptotically regular at x, we have $y = w - \lim_{i\to\infty} T^{n_i+m}(x)$ for $m \in \mathbb{N}$.

Now, let $b_m = \limsup_{i\to\infty} \|T^{n_i+m}(x) - y\|$. By Lemma 4.1, the sequence $\{b_m\}$ converges to $b \geq 0$. Assume $b = 0$. For $m \geq N(y)$, we have

$$\begin{aligned} \|T^m y - y\| &\leq \|T^m y - T^{n_i+2m}(x)\| + \|T^{n_i+2m}(x) - y\| \\ &\leq \limsup_{i\to\infty} \|T^m y - T^{n_i+2m}(x)\| + \limsup_{i\to\infty} \|T^{n_i+2m}(x) - y\| \\ &\leq \alpha_m(y) b_m + b_{2m}. \end{aligned}$$

Letting $m \to \infty$ on both sides, we get $T^m y \to y$.

Now, for $m \geq n \geq N(y)$, we have $\|T^m y - T^n y\| \leq \alpha_n(y)\|y - T^{m-n} y\|$.

It follows that if $n \geq N(y)$, then $T^m y \to T^n y$ as $m \to \infty$. This implies that $y = T^n y$, for $n \geq N(y)$ and hence $Ty = y$.

Now, suppose $b > 0$ and let $z_i^{(m)} = \dfrac{T^{n_i+m}(x) - y}{b_m}$. Then for each $m \geq 0$, $w - \lim_{i\to\infty} z_i^{(m)} = 0$ and $\limsup_{i\to\infty} \|z_i^{(m)}\| = 1$. By the uniform Opial condition of X, we have

$$1 + r_X(c) \leq \limsup_{i\to\infty} \|z_i^{(m)} + z\|, \text{ for all } z \in X \text{ with } \|z\| \geq c.$$

Take $z = \dfrac{y - T^m y}{b_{2m}}$. Then for $m \geq N(y)$,

$$1 + r_X\left(\frac{\|y - T^m y\|}{b_{2m}}\right) \leq \limsup_{i\to\infty} \left\|\frac{T^{n_i+2m}(x) - T^m y}{b_{2m}}\right\|$$

and hence

$$b_{2m}\left(1 + r_X\left(\frac{\|y - T^m y\|}{b_{2m}}\right)\right) \leq \limsup_{i\to\infty} \|T^{n_i+2m}(x) - T^m y\| \leq \alpha_m(y) b_m.$$

Letting $m \to \infty$, we get

$$b\left(1 + r_X\left(\frac{\limsup_{m\to\infty} \|y - T^m y\|}{b}\right)\right) \leq b;$$

$$r_X\left(\frac{\limsup_{m\to\infty} \|y - T^m y\|}{b}\right) \leq 0.$$

Since X satisfies the uniform Opial condition, we have $\lim_{m\to\infty} \|T^m y - y\| = 0$. This implies that $Ty = y$ and thus $\omega_w(x) \subseteq F(T)$.

Now, it remains to prove that $\omega_w(x)$ is a singleton. First to observe that $\lim_{n\to\infty} \|T^n x - p\|$ exists for every $p \in F(T)$. For this, let $p \in F(T)$. For $m \geq N(p)$ and $n \geq 1$, we have

$$\begin{aligned} \|T^{n+m}x - p\| &= \|T^{n+m}x - T^m p\| \\ &\leq \alpha_m(p)\|T^n x - p\| \end{aligned}$$

Letting $\limsup_{m\to\infty}$ on both sides, we get $\limsup_{m\to\infty} \|T^m x - p\| \leq \|T^n x - p\|$ for all $n \geq 1$. Thus, we have $\lim_{n\to\infty} \|T^n x - p\|$ exists for all $p \in F(T)$.

Suppose that there exists two subsequences $\{n_i\}$ and $\{m_i\}$ of $\mathbb{N}$ such that $w - \lim_{i\to\infty} T^{n_i}(x) = p_1$ and $w - \lim_{i\to\infty} T^{m_i}(x) = p_2$ for $p_1 \neq p_2$.

$$\begin{aligned} \lim_{n\to\infty} \|T^n x - p_1\| &= \lim_{i\to\infty} \|T^{n_i}(x) - p_1\| \\ &< \lim_{i\to\infty} \|T^{n_i}(x) - p_2\| \\ &= \lim_{i\to\infty} \|T^{m_i}(x) - p_2\| \\ &< \lim_{i\to\infty} \|T^{m_i}(x) - p_1\| \\ &= \lim_{n\to\infty} \|T^n x - p_1\| \end{aligned}$$

which is a contradiction. Hence $\omega_w(x)$ is singleton and the sequence $\{T^n x\}$ converges weakly to a fixed point of T. □

Theorem 4.3. *Let K be a nonempty weakly compact convex subset of a Banach space X with a weakly continuous duality map J_φ and $T : K \to K$ be a pointwise eventually asymptotically nonexpansive map. Then given an $x \in K$, $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Proof. If $\{T^n x\}$ converges weakly to a fixed point of T, then $\{T^n x\}$ is obviously weakly asymptotically regular at x.

Conversely, assume that T is weakly asymptotically regular at $x \in K$. Then we claim that $\omega_w(x) \subseteq F(T)$ and $\omega_w(x)$ is singleton. Since X has weakly continuous duality map J_φ, so it satisfies Opial's condition. By Theorem 4.2, it is enough to show that $w_\omega(x) \subseteq F(T)$. Let $y \in \omega_w(x)$. Then there exists a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that $y = w - \lim_{i\to\infty} T^{n_i} x$. Since T is weakly asymptotically regular at x, we have $y = w - \lim_{i\to\infty} T^{n_i+m}(x)$ for $m \in \mathbb{N}$.

Let $b_m = \limsup_{i\to\infty} \|T^{n_i+m}(x) - y\|$. By Lemma 4.1, the sequence $\{b_m\}$ converges to b.

For $m \geq N(y)$, we have

$$\begin{aligned}\Phi\left(\|T^{n_i+2m}(x)-y\|\right) &= \Phi\left(\|T^{n_i+2m}(x)-T^m y+T^m y-y\|\right)\\ &= \Phi\left(\|T^{n_i+2m}(x)-T^m y\|\right)+\\ &\quad \int_0^1 \left\langle T^m y-y, J_\varphi(T^{n_i+2m}(x)-T^m y+t(T^m y-y))\right\rangle dt\\ &\le \Phi\left(\alpha_m(y)\|T^{n_i+m}(x)-y\|\right)+\\ &\quad \int_0^1 \left\langle T^m y-y, J_\varphi(T^{n_i+2m}(x)-T^m y+t(T^m y-y))\right\rangle dt\end{aligned}$$

Letting $\limsup_{i\to\infty}$ on both sides, we get

$$\begin{aligned}\Phi\left(b_{2m}\right) &\le \Phi\left(\alpha_m(y)b_m\right)+\\ &\quad \int_0^1 \left\langle T^m y-y, J_\varphi(y-T^m y+t(T^m y-y))\right\rangle dt\\ &= \Phi\left(\alpha_m(y)b_m\right)-\int_0^1 \|y-T^m y\|\varphi\left((1-t)\|y-T^m y\|\right)dt\\ &= \Phi\left(\alpha_m(y)b_m\right)-\Phi\left(\|y-T^m y\|\right)\end{aligned}$$

Letting $\limsup_{m\to\infty}$ on both sides, we get

$$\Phi\left(\limsup_{m\to\infty}\|y-T^m y\|\right)\le\Phi(b)-\Phi(b).$$

Thus, $T^m y\to y$ as $m\to\infty$ and $Ty=y$. Hence $\omega_w(x)\subseteq F(T)$ and the sequence $\{T^n x\}$ converges weakly to a fixed point of T. □

Acknowledgements. *The authors would like to thank the anonymous referee for the comments and suggestions. The first author acknowledges the University Grants Commission, New Delhi, for providing financial support in the form of project fellow through Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai.*

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