

THE SELF POWER OF A CLAMPED CIRCULAR PLATE AN ANALYTICAL ESTIMATION

W. P. RDZANEK Jr., and W. J. RDZANEK

Department of Acoustics, University of Rzeszów
Al. Rejtana 16A, 35-310 Rzeszów, Poland
wprdzanek@univ.rzeszow.pl

A low frequency approximation for the self power of a clamped circular plate is presented in a form useful for some efficient numerical computations. The approximation is precise enough within nearly the whole low frequency range. It is assumed that the plate's vibrations are time harmonic and axisymmetric. The approximation can be used for some further computations of the total sound power radiated by the excited plate in an acoustic fluid.

1. Introduction

Finding the self power of a vibrating flat plate is important from the practical point of view since it makes it possible to theoretically analyze the sound field generated as the energy flow within the neighborhood of the sound source. Therefore, it is necessary to formulate the self power in a form which enables some efficient numerical computations. Any integral formulations would be time consuming and therefore not very useful for this purpose. Only some elementary formulations seem to be convenient for this purpose. In the literature such formulations in their exact form were found only for the simplest piston radiators [1]. For some more complex acoustic systems containing a flat plate it is also possible to obtain some approximate formulations for the self power.

CZARNECKI, ENGEL and PANUSZKA used an equivalent area method to estimate the radiation efficiency of a vibrating circular plate. The results obtained showed a good agreement with the experimental data [2]. ENGEL and STRYCZNIEWICZ determined the sound power radiated by a clamped circular plate analytically [3]. Some high frequency asymptotic formulations for the self power or mutual power of a clamped circular plate were found by LEVINE, LEPPINGTON, RDZANEK, ENGEL and RDZANEK Jr. [4-7]. LEE and SINGH used the Rayleigh-Ritz approximate method to derive the radiation efficiency of a spinning computer disk [8].

RDZANEK presented some low frequency approximations for the mutual power of a clamped circular plate. The approximation consists only of the zero expansion term which is enough only for some numerical computations for a narrow range of the lowest frequencies [6]. This paper presents an approximation of a considerably higher accuracy for the power of a clamped circular plate with three successive expansion terms. The problem was earlier signaled in [9].

2. Analysis assumptions

The acoustic system considered consists of a clamped circular plate embedded in a flat infinite rigid baffle. The plate vibrates and radiates some acoustic waves into the half-space located above it and filled with a lossless gaseous medium. It is assumed that the surrounding medium does not influence vibrations of the plate. The internal damping of the plate is assumed to be small and therefore not taken into account. The plate's vibrations are assumed to be timeharmonic and axisymmetric and therefore, it is convenient to express all the magnitudes in their amplitude forms.

The n -th mode shape of the plate is

$$W_n(r) = A_n [J_0(k_n r) + B_n I_0(k_n r)], \quad (1)$$

where r is the radial variable in the polar coordinates, J_0 and I_0 are the Bessel and modified Bessel functions, respectively, $k_n^2 = \omega_n \sqrt{M/D}$ is the n -th structural wavenumber of the plate, ω_n is its n -th eigenfrequency, M is its mass per surface unit, $D = Eh^3/[12(1 - \nu^2)]$ is the plate's stiffness (cf., reference [10]). The clamped plate satisfies the following boundary conditions

$$W_n(a) = 0, \quad \left(\frac{d^2}{dr^2} + \frac{\nu}{r} \frac{d}{dr} \right) W_n(r) \Big|_{r=a} = 0, \quad (2)$$

where a is the plate's radius and ν is its Poisson ratio. The eigenvalues of the plate are denoted by $\lambda_n = k_n a$ for $n = 0, 1, 2, \dots$. They are known from the literature e.g. reference [10], and they are necessary for any further computations concerning the system. The frequency equation of the plate assumes the form of

$$B_n = - \frac{J_0(\lambda_n)}{I_0(\lambda_n)} = \frac{J_1(\lambda_n)}{I_1(\lambda_n)}. \quad (3)$$

The value of constant A_n is

$$A_n^{-2} = 2J_0^2(\lambda_n), \quad (4)$$

where $\alpha_n \equiv J_1(\lambda_n)/J_0(\lambda_n)$ (cf., reference [10]).

The n -th eigenfunction of the system is

$$\psi_n(\vartheta) = -i\omega_n \int_0^a W_n(r) J_0(kr \sin \vartheta) r \, dr, \quad (5)$$

where $k = 2\pi/\lambda$ is the acoustic wavenumber, λ is the radiated wavelength, ϑ is the deflection angle between the main direction of the plate and the direction of measuring the pressure (cf., reference [11]). The eigenfunction can be evaluated analytically to produce

$$\psi_n(\vartheta) = -i\omega_n 2A_n a^2 \frac{J_0(\lambda_n)}{\lambda_n} \tilde{\psi}_n(ka \sin \vartheta), \quad (6)$$

where

$$\tilde{\psi}_n(u) = \frac{\alpha_n J_0(u) - (u/\lambda_n) J_1(u)}{1 - (u/\lambda_n)^4}, \quad (7)$$

and $u = ka \sin \vartheta$ (cf., reference [7]).

The active self power in its integral form is (cf., references [6, 7])

$$\mathcal{P}_{a,n} = 4(k/k_n) \int_0^{\pi/2} \tilde{\psi}_n^2(ka \sin \vartheta) \sin \vartheta \, d\vartheta, \quad (8)$$

which is considerably time consuming and therefore not very useful for further numerical computations.

3. The active self power estimation

It is impossible to analytically evaluate the integral in equation (8). Therefore, the term containing the denominator has been expanded into the following series

$$[1 - (k/k_n)^4 \sin^4 \vartheta]^{-2} = 1 + 2(k/k_n)^4 \sin^4 \vartheta + 3(k/k_n)^8 \sin^8 \vartheta + \mathcal{O}[(k/k_n)^{12} \sin^{12} \vartheta], \quad (9)$$

where the term denoted by symbol \mathcal{O} represents the approximation error. The expansion is convergent for $k/k_n < 1$ and it is substituted into equation (8) instead the denominator term.

Further, it is necessary to compute the following integrals

$$I_{A_\kappa} = \int_0^{\pi/2} (\sin \vartheta)^{2\kappa-1} J_0^2(ka \sin \vartheta) d\vartheta, \quad (10)_1$$

$$I_{B_\kappa} = \int_0^{\pi/2} (\sin \vartheta)^{2\kappa} J_0(ka \sin \vartheta) J_1(ka \sin \vartheta) d\vartheta, \quad (10)_2$$

$$I_{C_\kappa} = \int_0^{\pi/2} (\sin \vartheta)^{2\kappa+1} J_1^2(ka \sin \vartheta) d\vartheta, \quad (10)_3$$

for $\kappa = 1, 2, 3, \dots$, which requires applying the following expansion series

$$J_0^2(z) = \sum_{r=0}^{+\infty} (-1)^r \frac{(2r)!}{(r!)^4} \left(\frac{1}{2}z\right)^{2r}, \quad (11)_1$$

$$J_0(z) J_1(z) = \sum_{r=0}^{+\infty} (-1)^r \frac{(2r+1)!}{(r!)^2 [(r+1)!]^2} \left(\frac{1}{2}z\right)^{2r+1}, \quad (11)_2$$

$$J_1^2(z) = \sum_{r=0}^{+\infty} (-1)^r \frac{(2r+2)!}{r! [(r+1)!]^2 (r+2)!} \left(\frac{1}{2}z\right)^{2r+2}, \quad (11)_3$$

for $n, m = 0, 1, \dots$, where z is the complex variable, together with the following integral (cf., reference [12])

$$\int_0^{\pi/2} (\sin \vartheta)^{2n+1} d\vartheta = \frac{2^{2n}(n!)^2}{(2n+1)!}. \quad (12)$$

This leads to the following analytical formulae for the integrals

$$I_{A_\kappa} = \sum_{r=0}^{+\infty} \frac{(-1)^r (ka)^{2r} \varepsilon_{r,-1}}{(2r+1)(r!)^2}, \quad (13)_1$$

$$I_{B_\kappa} = \sum_{r=0}^{+\infty} \frac{(-1)^r (ka)^{2r+1} \varepsilon_{r,0}}{(2r+3)r!(r+1)!}, \quad (13)_2$$

$$I_{C_\kappa} = 2ka \sum_{r=0}^{+\infty} \frac{(-1)^r (ka)^{2r+1} \varepsilon_{r,+1}}{(2r+3)(2r+5)r!(r+1)!}, \quad (13)_3$$

where

$$\varepsilon_{r,i} = \begin{cases} 1, & \text{for } \kappa = 1, \\ \prod_{s=i+2}^{\kappa+i} \frac{2r+2s}{2r+2s+1}, & \text{for } \kappa > 1. \end{cases} \quad (14)$$

After a considerable amount of mathematical manipulation it is possible to formulate them briefly

$$I_{A_1} = A_1, \quad I_{A_2} = (A_1 + A_3)/2, \quad I_{A_3} = (3A_1 + 2A_3 + 3A_5)/8, \quad (15)_1$$

$$I_{B_1} = B_3, \quad I_{B_2} = (B_3 + B_5)/2, \quad I_{B_3} = (3B_3 + 2B_5 + 3B_7)/8, \quad (15)_2$$

$$I_{C_1} = 2ka(B_3 - B_5)/2, \quad I_{C_2} = 2ka(3B_3 - 2B_5 - B_7)/8, \\ I_{C_3} = 2ka(5B_3 - 3B_5 - B_7 - B_9)/16, \quad (15)_3$$

where the following recurrence formulae are necessary

$$A_{2\kappa-1} = \sum_{r=0}^{+\infty} \frac{(-1)^r (ka)^{2r}}{(2r+2\kappa-1)(r!)^2} \\ = \begin{cases} J_0(2ka) + (\pi/2)[J_1(2ka)S_0(2ka) - J_0(2ka)S_1(2ka)], & \text{for } \kappa = 1, \\ [J_1(2ka) - (2\kappa-3)B_{2\kappa-1}]/2ka, & \text{for } \kappa > 1, \end{cases} \quad (16)_1$$

$$\begin{aligned}
 B_{2\kappa-1} &= \sum_{r=0}^{+\infty} \frac{(-1)^r (ka)^{2r+1}}{(2r+2\kappa-1)r!(r+1)!} \\
 &= - [J_0(2ka) - (2\kappa-3) A_{2\kappa-3}] / 2ka, \quad \text{for } \kappa > 1,
 \end{aligned} \tag{16}_2$$

and S_ν denotes the Struve function of order ν . The use of the Struve function to briefly express some expansion series has been signalled in reference [9]. Finally, the n -th active self power assumes the form of

$$\mathcal{P}_{a,n} = 4 \sum_{\kappa=1}^3 k \left(\frac{k}{k_n} \right)^{4\kappa-2} \left[\alpha_n^2 I_{A\kappa} - 2\alpha_n \frac{k}{k_n} I_{B\kappa} + \left(\frac{k}{k_n} \right)^2 I_{C\kappa} \right] + \mathcal{O} \left[\left(\frac{k}{k_n} \right)^{12} \right], \tag{17}$$

which is useful for some efficient numerical computations within the range of the low frequencies, i.e. when $k/k_n < 1$.

4. Numerical analysis

The curves plotted in Fig. 1 illustrate the self power for the two sample mode numbers n obtained from the low frequency approximation together with those obtained from the integral formulation (8) valid within the whole frequency spectrum, but since it is considerably time consuming it is used in this paper only to test the accuracy of the self power approximation within the low frequency range. Formulation (9) represents the series expanded around $(k/k_n) \sin \vartheta = 0$ and consists of the three expansion terms. Therefore, the final result given in equation (17) should be valid for a considerably wide range of the dimensionless acoustic wavenumber k/k_n which is also confirmed by Fig. 1. The integral and approximated curves obtained from both formulations are nearly identical for $0 < k/k_n < 0.8$.

Figure 2 illustrates both: the approximation error order $(k/k_n)^{12}$ plotted with a dashed line and the approximation error estimation formulated as $E_n = |\mathcal{P}_{I,n} - \mathcal{P}_{A,n}|$, where $\mathcal{P}_{I,n}$ and $\mathcal{P}_{A,n}$ represent the n -th self power obtained from integral (8) and its approximation (17), respectively. The error estimation does not exceed its theoretical value of the error order within the whole low frequency spectrum, i.e. when $0 < k/k_n < 1$ which shows that the theoretical formulation of the approximation error $(k/k_n)^{12}$ is valid for the whole spectrum.

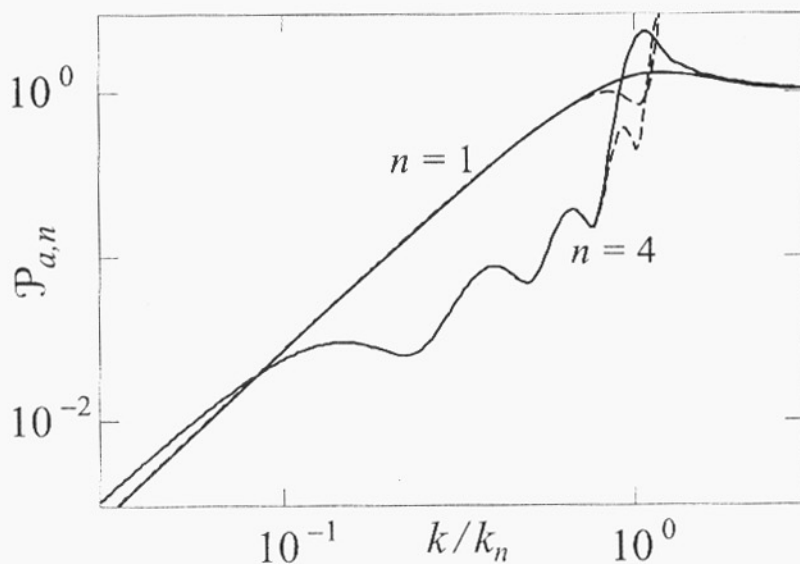


Fig. 1. The self power of a clamped circular plate $\mathcal{P}_{a,n}$ within the low frequency range, i.e. $0 < k/k_n < 1$. The curves obtained from approximation (17) are solid and those obtained from integral (8) are dashed.

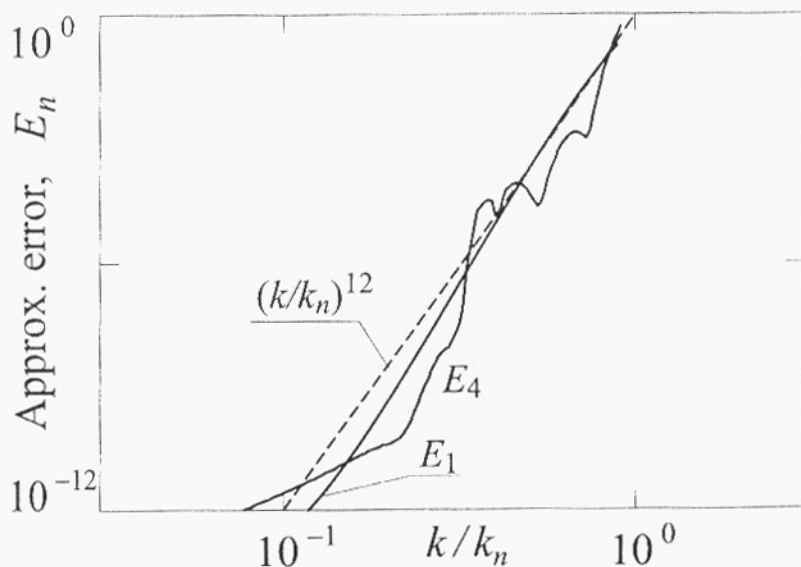


Fig. 2. The approximation error order $(k/k_n)^{12}$ plotted with the dashed line and the numerical estimation of the error $E_n = |\mathcal{P}_{i,n} - \mathcal{P}_{a,n}|$ for two sample mode-numbers $n = 1, 4$ plotted with solid lines. The symbols $\mathcal{P}_{i,n}$ and $\mathcal{P}_{a,n}$ represent the n -th self power expressed by integral (8) and that expressed by approximation (17), respectively.

5. Final remarks

The low frequency approximation of the active self power of a clamped circular plate has been derived. It is valid within almost the whole low frequency spectrum. The approximation is presented as an elementary formulation easy to express in terms of some computer code useful for some engineering computations of the radiation efficiency of the plate or of the total sound power radiated by the plate for the low frequencies.

Acknowledgement

The authors should like to acknowledge their gratitude to the Polish State Committee for Scientific Research, who have partially supported the investigations presented herein by grant no. 7T07B-051-18.

References

- [1] I. MALECKI, *Physical foundations of technical acoustics*, Pergamon Press, Oxford 1969.
- [2] S. CZARNECKI, Z. ENGEL, AND R. PANUSZKA, *Sound power and radiation efficiency of a circular plate*, Archives of Acoustics, **16**(4), 339–357 (1981).
- [3] Z. ENGEL AND L. STRYCZNIWICZ, *The determination of the acoustic power radiated by a vibrating plate* [in Polish], Mechanics–Quarterly at UMM, **9**(1), 5–12 (1990).
- [4] W. RDZANEK, *The sound power of a circular plate for high-frequency wave radiation*, Archives of Acoustics, **3**(4), 331–340 (1983).
- [5] H. LEVINE AND F. G. LEPPINGTON, *A note on the acoustic power output of a circular plate*, J. Sound and Vibration, **121**(2), 269–275 (1988).
- [6] W. RDZANEK, *Mutual impedance of a circular plate for axially-symmetric free vibrations at high frequency of radiating waves*, Archives of Acoustics, **17**(3), 439–448 (1992).
- [7] W. RDZANEK, Z. ENGEL AND W. P. RDZANEK JR., *Asymptotic formulas for the acoustic power output of a simply-supported circular plate*, Acustica/Acta Acustica, **87**(2), 206–214 (2001).
- [8] M.-R. LEE AND R. SINGH, *Analytical formulations for annular disk sound radiation using structural modes*, J. Acoustical Society of America, **95**(6), 3311–3323 (1994).
- [9] W. P. RDZANEK JR., Z. ENGEL AND W. RDZANEK, *The sound power of a simple-supported circular plate for low frequencies* [in Polish], [In:] Proc. of the 17-th Symposium on Hydroacoustics, 225–236, Gdynia, Poland, May 2000.
- [10] A. W. LEISSA, *Vibration of plates*, Vol. SP-160. NASA, Washington 1969.
- [11] P. M. MORSE AND K. U. INGARD, *Theoretical acoustics*, Princeton, Univ. Press 1986.
- [12] N. W. McLACHLAN, *Bessel functions for engineers*, Clarendon Press, Oxford 1955.