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THE SOUND PRESSURE OF AN INDIVIDUAL MODE OF A CLAMPED-FREE ANNULAR PLATE

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The paper presents some theoretical results together with the corresponding numerical results of the directivity pattern of sound radiation of the clamped-free, free-clamped, and fully clamped annular plates. The results obtained are valid in the Fraunhoffer zone for some time-harmonic and axisymmetric processes. The results are compared for the three different plates of identical geometric and material parameters. The Kirchhoff-Love model of a linear perfectly elastic plate is used. The data for the free-clamped plate are presented here for the first time, whereas the data valid for clamped-free and fully clamped annular plates are partially quoted from the literature. This new inventory of data makes it possible to perform some interesting comparisons of the magnitudes valid for the three different boundary conditions. All the data presented in this paper make the basis for some further theoretical investigations of the sound power of the three plates.

1. Introduction

A number of studies deal with vibrations of circular and annular plates. On the other hand, there are relatively few reports on the plate sound radiation in the literature of the problem, which may be due to the high complexity of any theoretical investigations in this field.

The sound radiation of such basic sound sources as circular or annular pistons was analyzed by RAYLEIGH [1], STEPANISHEN [2], THOMPSON [3], RDZANEK [4], GREENSPON and SHERMAN [5], PRITCHARD [6], MALECKI [7], MORSE and INGARD [8], SVENSSON [9], and many others who have presented their results in the form of some integral formulations. The mode shapes of some non-homogeneous circular or annular membranes were presented by JABAREEN and EISENBERGER [10], while the directivity pattern of circular membranes was investigated by MALECKI [7], MORSE and INGARD [8] and RDZANEK [11], or in the case of circular plates by RDZANEK [12] and SHUYU [13].

The sound directivity pattern and the corresponding sound power of the fully clamped annular plate were investigated by RDZANEK JR. and ENGEL [14–16]. However, the analogous magnitudes valid for some combinations of the boundary conditions of annular plates were, to the author's knowledge, dealt with only in reference [17], where the rotatory effects of the computer disk on the generated sound field were reported.

With the exception of some integral formulae, no analytical results were reported for the sound radiation of annular plates, free on their inside and clamped on their outside. To fill this literature gap the author presents some formulations together with some sample numerical results for the directivity pattern of such plates further referred to as free-clamped plates. The data valid for the clamped-free and fully clamped plates are quoted herein to enable performing some interesting comparisons of the magnitudes under consideration.

2. Free vibrations

A thin annular plate is embedded in a perfectly rigid and infinite baffle. The two different boundary conditions of the plate are considered: (a) the internal edge is clamped and the external one is free (the clamped-free plate), (b) the opposite situation (the free-clamped plate). The Kirchhoff-Love theory of a perfectly elastic plate is used to analyse the plate free and axisymmetric vibrations. All the magnitudes presented below will depend on the radial variable r only. The plate's thickness h and vibration velocity $v(r)$ are small as compared with the other geometric dimensions of the plate. The n-th mode shape takes the form of $v_n(r) = -i\omega\eta_n(r)$ for some time-harmonic and axisymmetric processes, where $\eta_n(r,t) = \eta(r)_n \exp(-i\omega t)$ is the solution of the plate's equation of motion

$$(k_n^{-4}\nabla_r^4 - 1)\,\eta_n(r) = 0, \tag{1}$$

and $n = 0, 1, 2, \ldots$ is the mode number, $k_n^2 = \omega_n\sqrt{\varrho h/B}$ and k_n is the n-th structural wavenumber, ω_n is the n-th eigenfrequency, ϱ, E, ν are the density, Young's modulus and Poisson's ratio of the plate, respectively, $B = Eh^3/[12\,(1-\nu^2)]$ is the plate's bending stiffness and $\nabla_r^4 = (\mathrm{d}^2/\mathrm{d}r^2 + r^{-1}\mathrm{d}/\mathrm{d}r)^2$. The solution of equation (1) is predicted in the form of

$$\eta_n(r) = A_n\,[J_0(k_n r) + B_n I_0(k_n r) - C_n N_0(k_n r) - D_n K_0(k_n r)], \tag{2}$$

where A_n, B_n, C_n, D_n are the constants to be derived, and J_0, I_0, N_0, K_0 are Bessel's, modified Bessel's, Neumann's and McDonald's functions of zero order, respectively (cf., references [18, 19]). The plate's boundary conditions (a) and (b) imply that the value of solution $\eta_n(r)$ and its first order radial derivative are equal to zero for the clamped edge, i.e. (a) $r' = r_1$, (b) $r' = r_2$:

$$\eta_n(r') = 0, \qquad \left.\frac{\mathrm{d}\eta_n(r)}{\mathrm{d}r}\right|_{r=r'}, \tag{3}$$

(cf., references [15, 17, 18, 20, 21]), whereas the bending moment m_r and the transversely acting force q_r are equal to zero for the free edge, i.e. (a) $r' = r_2$, (b) $r' = r_1$:

$$m_r(r') \;=\; -B\left[\frac{\mathrm{d}^2\eta_n(r)}{\mathrm{d}r^2} + \frac{\nu}{r}\,\frac{\mathrm{d}\eta_n(r)}{\mathrm{d}r}\right]_{r=r'} = 0, \tag{4a}$$

$$q_r(r') \;=\; -B\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left[r\,\frac{\mathrm{d}\eta_n(r)}{\mathrm{d}r}\right]\right)\right]_{r=r'} = 0, \tag{4b}$$

(c.f., references [17, 18, 20–23]). Inserting solution (2) into the boundary conditions (3) and (4) provides the following equation systems

$$\begin{bmatrix} J_0(x_n) & I_0(x_n) & -N_0(x_n) & -K_0(x_n) \\ -J_1(x_n) & I_1(x_n) & N_1(x_n) & K_1(x_n) \\ J_1(sx_n) & I_1(sx_n) & -N_1(sx_n) & K_1(sx_n) \\ J_0(sx_n) - 2DJ_1(sx_n) & -I_0(sx_n) & -N_0(sx_n) + 2DN_1(sx_n) & K_0(sx_n) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ B_n \\ C_n \\ D_n \end{bmatrix} = \mathbf{0}, \tag{5a}$$

$$\begin{bmatrix} J_0(sx_n) & I_0(sx_n) & -N_0(sx_n) & -K_0(sx_n) \\ -J_1(sx_n) & I_1(sx_n) & N_1(sx_n) & K_1(sx_n) \\ J_1(x_n) & I_1(x_n) & -N_1(x_n) & K_1(x_n) \\ J_0(x_n) - 2dJ_1(x_n) & -I_0(x_n) & -N_0(x_n) + 2dN_1(x_n) & K_0(x_n) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ B_n \\ C_n \\ D_n \end{bmatrix} = \mathbf{0}, \tag{5b}$$

valid for the boundary configurations (a) and (b), respectively, where $s = r_2/r_1$ is the plate's geometric parameter, $d = sD = (1-\nu)/x_n$ and $x_n = k_n r_1$ is the n-th eigenvalue of the plate (cf., table 1 and references [15, 17, 18, 20]). Some transformations of Eqs. (5) and the use of Wronsky's determinant $K_0(z)I_1(z) + K_1(z)I_0(z) = 1/z$ lead to the frequency equations and the corresponding values of constant C_n valid for both boundary configurations:

$$C_n = \frac{sS(sx_n) + S(x_n) - 2dI_1(sx_n)J_1(sx_n)}{sT(sx_n) + T(x_n) - 2dI_1(sx_n)N_1(sx_n)} = \frac{sN(sx_n) + N(x_n) + 2dJ_1(sx_n)K_1(sx_n)}{sR(sx_n) + R(x_n) + 2dN_1(sx_n)K_1(sx_n)}, \tag{6a}$$

$$C_n = \frac{sS(sx_n) + S(x_n) - 2dI_1(x_n)J_1(x_n)}{sT(sx_n) + T(x_n) - 2dI_1(x_n)N_1(x_n)} = \frac{sN(sx_n) + N(x_n) + 2dJ_1(x_n)K_1(x_n)}{sR(sx_n) + R(x_n) + 2dN_1(x_n)K_1(x_n)}, \tag{6b}$$

with the following denotations used

$$\begin{aligned} S(x) &= J_1(x)I_0(x) + J_0(x)I_1(x), & T(x) &= N_1(x)I_0(x) + N_0(x)I_1(x), \\ N(x) &= J_1(x)K_0(x) - J_0(x)K_1(x), & R(x) &= N_1(x)K_0(x) - N_0(x)K_1(x). \end{aligned} \tag{7}$$

Solving the equation system (5) provides the values of constants B_n and D_n, which can be expressed in terms of C_n as follows

$$\begin{aligned} B_n &= x_n\left[N(x_n) - C_n R(x_n)\right] = x_n\left[G_1(x_n)K_0(x_n) - G_0(x_n)K_1(x_n)\right],\\ D_n &= x_n\left[S(x_n) - C_n T(x_n)\right] = x_n\left[G_1(x_n)I_0(x_n) + G_0(x_n)I_1(x_n)\right], \end{aligned} \tag{8a}$$

$$\begin{aligned} B_n &= sx_n\left[N(sx_n) - C_n R(sx_n)\right] = sx_n\left[G_1(sx_n)K_0(sx_n) - G_0(sx_n)K_1(sx_n)\right],\\ D_n &= sx_n\left[S(sx_n) - C_n T(sx_n)\right] = sx_n\left[G_1(sx_n)I_0(sx_n) + G_0(sx_n)I_1(sx_n)\right], \end{aligned} \tag{8b}$$

where denotations $G_0(x) = J_0(x) - C_n N_0(x)$ and $G_1(x) = J_1(x) - C_n N_1(x)$ may be useful (cf., reference [15]). The last constant to be computed is A_n. It can be derived

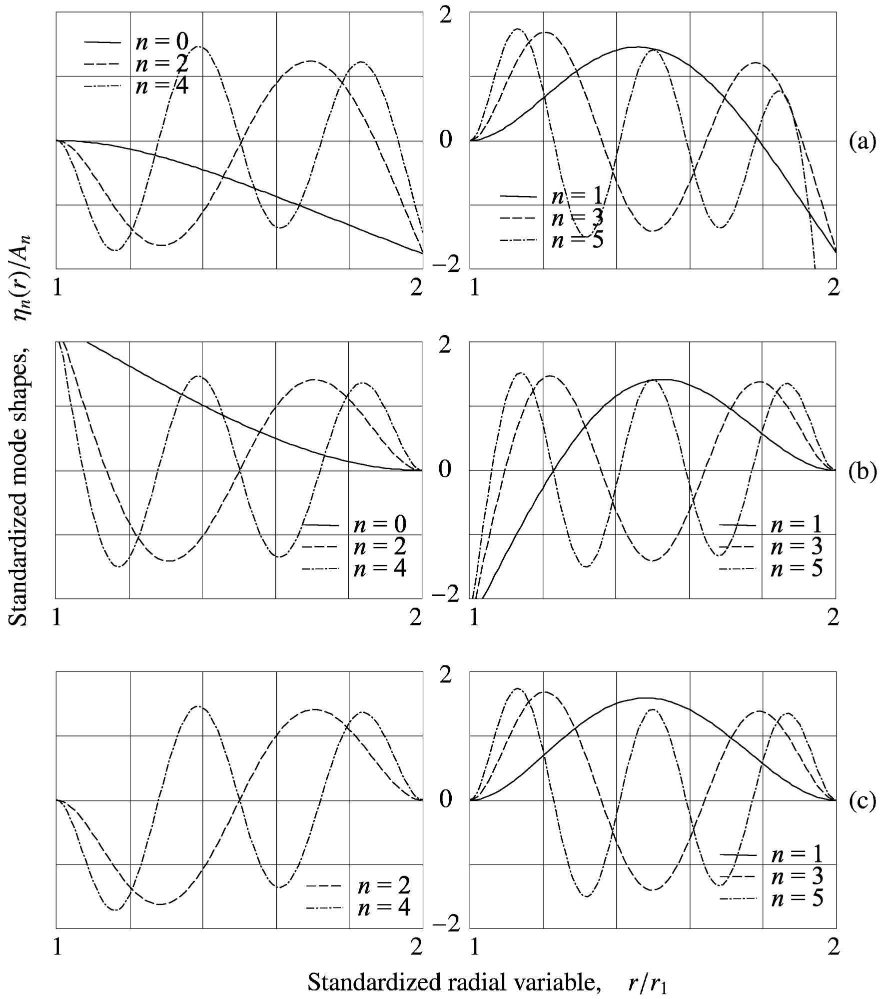

Fig. 1. The values of standardized mode shapes $\eta_n(r)/A_n$ of an annular plate for $r_2/r_1 = 2$ valid for the following edge configurations: (a) clamped-free, (b) free-clamped, (c) clamped-clamped, (cf. references [17, 18, 20, 24]).

from the standardization condition $\int_{r_1}^{r_2} \eta_n^2(r) r \, dr = r_1^2(s^2-1)/2$ to give

$$A_n^{-2} = \frac{2}{s^2-1}\left\{s^2 G_0^2(sx_n) - G_0^2(x_n) - 2dG_1(sx_n)\left[sG_0(sx_n) + \frac{\nu}{x_n} G_1(sx_n)\right]\right\}, \tag{9a}$$

$$A_n^{-2} = \frac{2}{s^2-1}\left\{s^2 G_0^2(sx_n) - G_0^2(x_n) + 2dG_1(x_n)\left[G_0(x_n) + \frac{\nu}{x_n} G_1(x_n)\right]\right\}. \tag{9b}$$

Based on the values of constants A_n, B_n, C_n, D_n, the standardized mode shapes $\eta_n(r)/A_n$ have been plotted in Figs. 1(a) and 1(b) for the boundary conditions (a) and (b), whereas Fig. 1(c) presents the data valid for the clamped-clamped boundaries quoted from references [17, 18, 20, 24]. It is necessary to remember that the mode shape value of the n-th mode does not represent any physical magnitude but the contribution of the mode to the plate's transverse deflection. Moreover, the physical magnitudes are to be expressed by some infinite eigenfunction series that must be convergent and make it possible to take into account the influence of the plate's internal friction and the air column's damping on the plate's vibrational behavior (cf., references [17, 25, 26]). Thus, all the data presented in this study make it possible to investigate such physical magnitudes of the plates as the transverse deflection and the sound pressure in the Fraunhoffer's zone, which are not discussed in this paper.

3. Sound radiation

For some time-harmonic and axisymmetric processes the sound pressure in Fraunhoffer's zone $p_n(r)$ associated with the n-th mode can be expressed by the well-known formula

$$p_n(R,\vartheta) = -i\omega_n \varrho_0 W_n(k\sin\vartheta)\, \frac{e^{ikR}}{R}, \tag{10}$$

(cf., references [15, 23]) where ϱ_0 is the density of the surrounding air column, c is the sound velocity in air, $k = 2\pi/\lambda$ is the acoustic wavenumber, λ is the radiated wavelength, R is the distance between the far field point and the plate's center, ϑ is the deflection angle between the main direction of the plate and the radiation direction, and $W_n(\tau)$ is a function characterizing the sound radiation associated with the n-th mode

$$W_n(\tau) = \int_{r_1}^{r_2} v_n(r) J_0(\tau r) r \mathrm{d}r, \tag{11}$$

(cf., references [15, 23]) where τ is the complex wavenumber. For further analysis the impedance approach will be used to compute integral (11) and express it in its elementary form by substituting $k\sin\vartheta$ for τ and by denoting $u = \beta\sin\vartheta$ and $\beta = kr_1$:

$$W_n(k\sin\vartheta) = i\omega_n A_n 2r_1^2 w_n(u), \tag{12}$$

where

$$w_n(u) = \frac{x_n^2}{x_n^4 - u^4}\left[sx_nG_1(sx_n)J_0(su) - suG_0(sx_n)J_1(su) \right.$$
$$\left. - x_nG_1(x_n)J_0(u) + uG_0(x_n)J_1(u)\right]$$
$$- \frac{1}{x_n^2 + u^2}\left\{sx_nG_1(sx_n)J_0(su) - \left[G_0(sx_n) - DG_1(sx_n)\right]suJ_1(su)\right\}, \tag{13a}$$

$$w_n(u) = \frac{x_n^2}{x_n^4 - u^4}\left[sx_nG_1(sx_n)J_0(su) - suG_0(sx_n)J_1(su) \right.$$
$$\left. - x_nG_1(x_n)J_0(u) + uG_0(x_n)J_1(u)\right]$$
$$+ \frac{1}{x_n^2 + u^2}\left\{x_nG_1(x_n)J_0(u) - \left[G_0(x_n) - dG_1(x_n)\right]uJ_1(u)\right\}, \tag{13b}$$

for both boundary conditions under consideration. For the main direction of the plate, i.e. when $\vartheta = 0$, equations (13) gets reduced to

$$w_n(0) = -G_1(x_n)/x_n, \tag{14a}$$
$$w_n(0) = sG_1(sx_n)/x_n. \tag{14b}$$

Thus, equations (12) and (13) represent the directivity patterns for the sound radiation in the Fraunhoffer zone, which can be expressed by the definition

$$\mathcal{R}_n(\vartheta) = \left|\frac{p_n(R,\vartheta)}{p_n(R,0)}\right| = \left|\frac{w_n(\beta\sin\vartheta)}{w_n(0)}\right| \tag{15}$$

(cf., reference [27]) with the criterion $kr \ll 2R/r$ satisfied (cf., references [14, 27, 28]).

4. Numerical results and their discussion

Several curves of the mode shapes and the directivity patterns valid for the three different boundary conditions, namely (a) clamped-free, (b) free-clamped, and (c) clamped-clamped, have been plotted to enable the comparison of their vibro-radiational behavior. The curves have been prepared for $r_2/r_1 = 2$, however a number of other curves can be obtained using the formulae presented herein. Figure 1 shows that a mode shape can approach a very high value in the vicinity of the free edge, especially for the clamped-free plate. This may be caused by the fact that no damping forces are taken into account but, obviously, the mode shapes represent their contributions to the infinite eigenfunction series for the plate's transverse deflection which must be finite and small enough to satisfy the assumptions made.

Figure 2 shows the curves for the directivity patterns. They are quite regular for the boundary conditions (a) and (b) in the sense that the higher the mode number is, the higher its sound radiation directivity will be. In the case of the clamped-clamped boundary conditions (cf., Fig. 2(c)) the sound radiation of an odd mode can be even more directional as compared with the sound radiation of an even mode of a higher mode number valid for the clamped-free and free-clamped plates (cf., reference [14]). The directivity patterns are also quite regular in the sense that the direction of their maximal

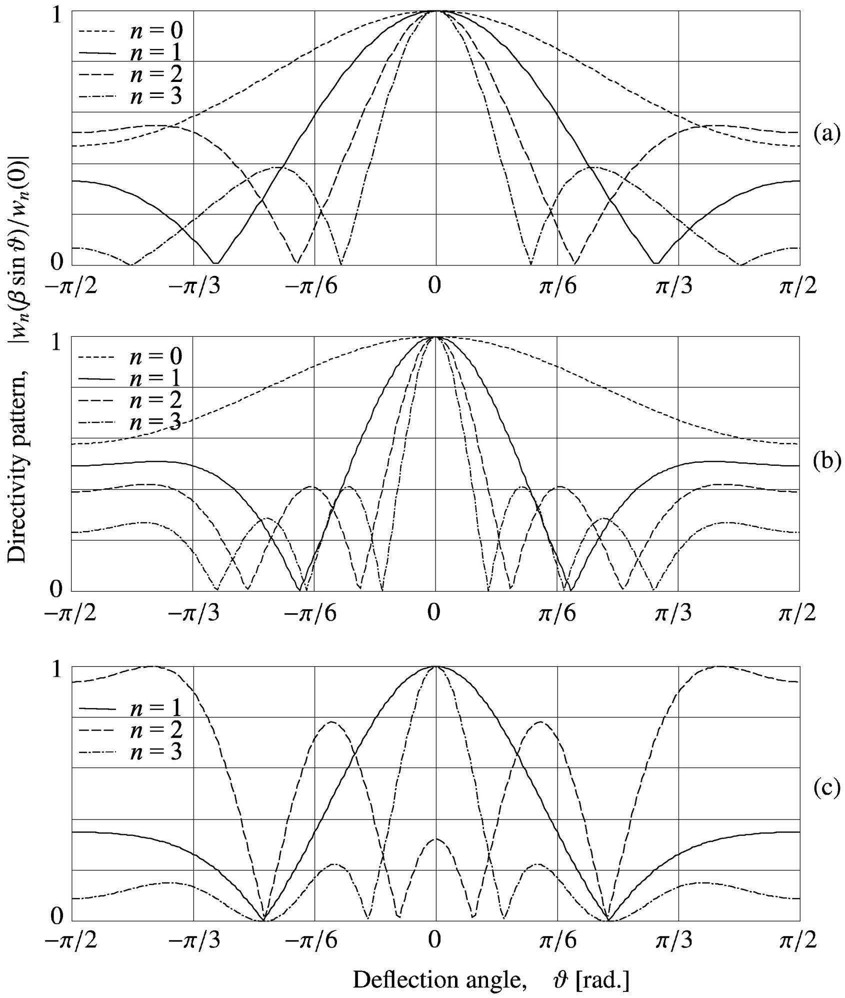

Fig. 2. The directivity pattern of the sound radiation $|w_n(\beta \sin\vartheta)/w_n(0)|$ of an annular plate for $r_2/r_1 = 2$ and for the value $x_n/2$ of the dimensionless wavenumber β with the following edge configurations: (a) clamped-free, (b) free-clamped, (c) clamped-clamped, (cf., reference [14]).

radiation is equal to the main direction of the plates, i.e. $\vartheta = 0$. The situation is not valid for the even modes of the clamped-clamped plates. The sound radiation of a clamped-free plate is essentially more in-directive than the sound radiation of a free-clamped plate. The zero mode is the most in-directive one for both plates, while the zero mode does not appear for the fully clamped plate (cf., reference [14]). Generally, there are some essential differences in the mode shapes and the corresponding directivity patterns of the three boundary conditions considered herein, and with all the geometric and material parameters being identical for the three different plates. However, the eigenvalues of all the plates are located very close to one another, with the exception of their zero mode (cf., Table 1).

Table 1. Eigenvalues $x_n = kr_1$ of the frequency equation (6) for three different values of the geometric parameter r_2/r_1 (cf., references [15, 17, 18, 20) and three different boundary configurations, i.e. clamped-free (c–f), clamped-clamped (c–c), and free-clamped (f–c).

	$r_2/r_1 = 1.2$			$r_2/r_1 = 2$			$r_2/r_1 = 5$		
n	c–f	c–c	f–c	c–f	c–c	f–c	c–f	c–c	f–c
0	9.220	—	9.589	1.804	—	2.104	0.455	—	0.645
1	23.34	23.65	23.62	4.611	4.724	4.844	1.137	1.177	1.312
2	39.20	39.26	39.36	7.805	7.848	7.940	1.934	1.957	2.050
3	54.92	54.98	55.04	10.96	10.99	11.06	2.731	2.743	2.810
4	70.64	70.68	70.73	14.11	14.13	14.18	3.520	3.529	3.581
5	86.36	86.39	86.43	17.26	17.28	17.32	4.307	4.315	4.358

5. Conclusions

A theoretical analysis of the sound pressure of an individual mode of the three different plates, namely clamped-free, free-clamped, and clamped-clamped, has been presented in this study and the results have been compared. The results obtained are valid in the Fraunhoffer zone for some time-harmonic and axisymmetric processes. Such data were not reported before in the literature on the free-clamped plates. However, the Rayleigh-Ritz approach to deal with the clamped-free annular disk was presented by LEE and SINGH in reference [17]. The results valid for the fully clamped plates are quoted from reference [14]. The formulations presented herein can make a basis for the analysis of the sound power of the three plates (cf., reference [16]).

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