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SCATTERING NEAR A PENETRABLE FINITE PLANE

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Acoustic scattering due to a point source by a penetrable finite plane introducing the Kutta-Joukowski condition is studied. This investigation is important in the sense that point source is regarded as fundamental radiating device. Mathematical problem which is solved is an approximate model for a noise barrier which is not perfectly rigid and therefore transmits sound. Approximate boundary condition depends upon the thickness and material constants which constitute the finite plane. The problem is solved using integral transforms, the Wiener-Hopf technique and asymptotic methods. It is found that the diffracted field is sum of the fields produced by the two edges of the finite plane and an interaction field. It is once again found that the field produced by the Kutta-Joukowski condition will be substantially larger than the field produced in its absence when the source is near the edge. Finally, physical interpretation of the result is discussed.

1. Introduction

In recent years, noise reduction by means of barriers is a common method of reducing noise pollution in heavily built up areas [1, 2]. Traffic noise from motorways, railways and airports, and other outdoor noises from heavy construction machinery or stationery installations, such as large transformers or plants, can be shielded by a barrier which intercepts the line of sight from source to receiver. Noise in open plan offices can also be reduced by means of barrier partitions. In most of the calculations with noise barriers, the field in the shadow region of the barrier is assumed to be solely due to diffraction at the edge. This assumption supposes that the barrier is perfectly rigid and therefore, does not transmit sound. However, the barriers used for practical purpose are usually made of wood or plastic and will consequently transmit some of the noise through the barriers. YEH [3] considered the problem of diffraction by a penetrable parabolic cylinder and obtained the solution in the complicated form of infinite series of parabolic cylinder functions. Another approximate approach for parabolic cylinder coordinates was used by SHMOYS [4] to present the results in terms of Fresnel integrals. PISTOL'KORS [5] used the Kirchhoff-Huygens integral equation approach to solve the general problem of diffraction by a penetrable strip. Later on, KHREBET [6] extended this analysis to a dielectric half plane. The approximate boundary condition used by PISTOL'KORS [5] and KHREBET [6] was only good in describing a perfectly penetrable half plane, i.e. no loss within the material which comprises the half plane. For a smooth transition from a perfectly penetrable half plane to a non penetrable half plane, RAWLINS [7] used an alternative boundary condition and calculated the diffracted field due to a line source incidence. ASGHAR and HAYAT [8] also discussed the diffraction of a line source near an absorbing strip. In another paper, ASGHAR and HAYAT [9] studied the acoustic scattering from the coupling of soft and locally reacting half planes. Morerecently, ASGHAR *et al.* [10] examined the diffraction of an acoustic wave by a slit in an infinite, plane, porous barrier.

In 1970, it was shown by FFOWCS WILLIAMS and HALL [11] that the aerodynamic sound scattered by a sharp edge is proportional in intensity to the fifth power of the flow velocity and inversely to the cube of the distance of the source from the edge. Thus, the edge is likely to be the dominant sound source, especially when the source is very close to the edge. Their findings were however based upon the assumption of a potential flow near the sharp edge with velocity becoming infinite there. Instead of that if one wishes to prescribe that the velocity is finite, there are two possible points of view. One way is to abandon lighthill's theory and use linearized Navier-Stokes equation with source term as employed by ALBLAS [12]. Before discussing the second option, it is better to introduce the Kutta-Joukowski condition. JONES [13] introduced the wake condition to examine the effect of the Kutta-Joukowski condition at the edge of the half-plane. He calculated the field scattered from a line source parallel to a semi-infinite rigid plane attached to a wake. This problem was further extended to the point source excitation by BALASUBRAMANYAM [14] and to the diffraction of a cylindrical pulse by RIENSTRA [15].

The aim of the present paper is to analyze the diffraction of a spherical wave by a penetrable finite plane introducing the wake condition to examine the effect of the Kutta-Joukowski condition. This is important in the sense that point sources are regarded as better substitutes for real sources. The barrier is modelled as a rigid material filled with narrow pores, normal to the plane of the barrier, that provide sound damping. However, the barrier is thin enough that sound transmission takes place. The integral transforms and the Wiener-Hopf technique [16] are employed to obtain the integral representation of the diffracted field. These integrals are normally difficult to handle because of the presence of branch points and are only amenable to solution using asymptotic approximations. The analytic solution of these integrals is thus obtained using the steepest descent method [17].

2. Problem formulation

We consider the scattering of an acoustic wave due to a point source by a penetrable finite plane. A finite penetrable plane is assumed to occupy $y = 0$, $-l \leq x \leq 0$. The penetrable plane is assumed to be of negligible thickness and satisfying the penetrable boundary conditions on both sides of its surface. The geometry of the problem is shown in the Fig. 1. We consider a point source to be located at the position (x_0, y_0, z_0) and the time dependence is taken to be of harmonic nature $e^{-i\omega t}$ (ω is the low angular frequency).

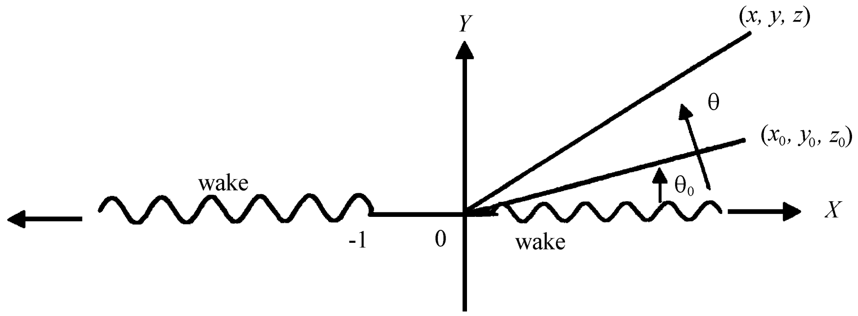

Fig. 1. Geometry of the problem.

Thus on suppressing the time harmonic factor the wave equation satisfied by the total velocity potential χ_t in presence of the point source is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\chi_t=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \tag{2.1}$$

where

$$k=\frac{\omega}{c}=k_1+ik_2 \tag{2.2}$$

is the free space wave number. In Eq. (2.2), k is complex and has a small positive imaginary part which has been introduced to ensure the convergence (regularity) of the Fourier transform integrals. The term k_2 is otherwise set to zero. The boundary conditions satisfied by χ_t on $y=0^{\pm}$, $-l\le x\le 0$ are

$$\pm\frac{\partial}{\partial y}\chi_t(x,0^{\pm},z)+ik\alpha\chi_t(x,0^{\pm},z)+ik\beta\chi_t(x,0^{\mp},z)=0, \tag{2.3}$$

where the parameters α and β will be defined shortly. The $0^{\pm}$ in Eq. (2.3) means that the field term is to be evaluated as $y\to 0$ through positive or negative values of y.

In order to satisfy the Kutta-Joukowski condition at the edge, JONES [13] introduced a discontinuity in the field at $0<x<\infty$ and postulated the existence of a wake condition. According to him, χ_t is discontinuous, while $(\partial\chi_t/\partial y)$ remains continuous for $y=0$, $x>0$. The boundary conditions can thus be expressed as

$$\frac{\partial}{\partial y}\chi_t(x,y^+,z)=\frac{\partial}{\partial y}\chi_t(x,y^-,z), \qquad (x<-l,x>0,y=0), \tag{2.4}$$

$$\begin{aligned}\chi_t(x,y^+,z)-\chi_t(x,y^-,z)&=a(z)e^{i\mu x}, \quad (x>0,\ y=0),\\ \chi_t(x,y^+,z)-\chi_t(x,y^-,z)&=a(z)e^{-i\mu x}, \quad (x<-l,\ y=0).\end{aligned} \tag{2.5}$$

In Eq. (2.5), constant μ is regarded as known i.e.,

$$\mu=k\cos\theta_1, \tag{2.6}$$

where $0\le\operatorname{Re}\theta_1<\pi$, $\operatorname{Im}\theta_1>0$; eventually we shall be concerned primarily with the case $\operatorname{Re}\theta_1=0$, $\operatorname{Im}\theta_1>0$.

In Eq. (2.5), $a(z)$ can be determined by means of a Kutta-Joukowski condition. We note that $a = 0$ corresponds to a no wake situation. It is appropriate to split χ_t as

$$\chi_t(x, y, z) = \chi_0(x, y, z) + \chi(x, y, z), \tag{2.7}$$

where $\chi_0(x, y, z)$ is the incident wave which accounts for the inhomogeneous source term and $\chi(x, y, z)$ is the solution of homogeneous wave equation (2.1) that corresponds to the diffracted field.

In addition we insist that χ represents an outward travelling wave as $r = (x^2 + y^2 + z^2)^{1/2} \to \infty$.

3. The boundary condition

Figure 2 shows a porous barrier of thickness 2h extending to infinity in the $\pm x$-directions. The space is divided into three region. The regions V^+ and V^- are those above and below the barrier and are occupied by a gas having density ρ and sound speed c. The region V_0 is that occupied by the porous barrier. Following a formulation that is identical to that given in Section I.B of Harris *et al.* [17], the velocity potential χ_s scattered from this barrier is represented by

$$\kappa_s = -\int\limits_S [\chi_g(x', x)\,\nabla\chi_t(x') - \chi_t(x')\nabla'\chi_g(x', x)] \cdot \hat{\mathbf{n}}\, dS(x'), \quad x \in V^+ \cup V^-, \tag{3.1}$$

where χ_t is the total potential given by Eq. (2.7) and χ_g the three-dimensional, free-space Green's function. The surface S is comprised of the upper and lower surfaces of the barrier, $\hat{\mathbf{n}}$ is a unit normal vector pointing out of the barrier and ∇' indicates that the gradient is taken with respect to the argument x'. The vector x indicates the observation point and lies outside the barrier, while the vector x' indicates the source point and lies on the surface S.

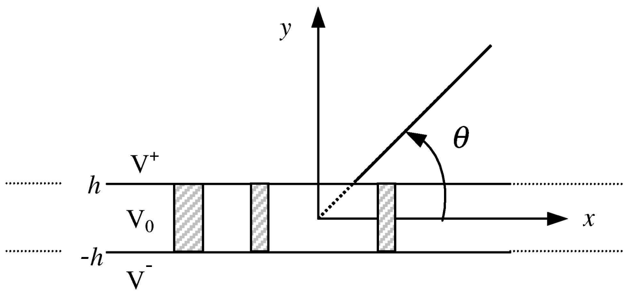

Fig. 2. The geometry of the barrier.

Asking that the unit normal $\hat{\mathbf{n}}$ now point only in the positive y-direction, we define the discontinuities

$$[\nabla\chi_t \cdot \hat{\mathbf{n}}] = \nabla\chi_t(x, h, z) \cdot \hat{\mathbf{n}} - \nabla\chi_t(x, -h, z) \cdot \hat{\mathbf{n}}, \tag{3.2}$$

and

$$[\chi_t] = \chi_t(x, h, z) - \chi_t(x, -h, z). \tag{3.3}$$

These are the sources of the scattered sound as can be seen by noting that, provided the discontinuities in Eqs. (3.2) and (3.3) are no larger than $O(1)$, then the integral representation (3.1) can be approximated to $O(kh)$ by evaluating the Green's terms at $y' = 0$. This leaves us with

$$\chi_t(x) = -\int\!\!\int_S \{\chi_g(x', 0, z', \mathbf{x})\,[\nabla\chi_t \cdot \hat{\mathbf{n}}] - [\chi_t]\nabla'\chi_g \cdot \hat{\mathbf{n}}\}\, dx'dz' + O(kh), \tag{3.4}$$

where $\mathbf{x}$ lies outside the volume enclosed by S. Note that we have approximated a function that we know and whose length scale is set by the wavenumber k and not by the wavenumber of the porous material. It is therefore the discontinuities, Eqs. (3.2) and (3.3), that (2.3) must mimic.

Returning to the Rawlins boundary condition, we note that if we take the limit $kh \to 0$ of the following:

$$\begin{aligned} [\nabla\chi_t \cdot \hat{\mathbf{n}}] &= -ik(\alpha + \beta)[\chi_t(x, h, z) + \chi_t(x, -h, z)], &(3.5)\\ [\chi_t] &= -\left[ik(\alpha - \beta)\right]^{-1}\left[\nabla\chi_t(x, h, z) \cdot \hat{\mathbf{n}} + \nabla\chi_t(x, -h, z) \cdot \hat{\mathbf{n}}\right], &(3.6)\end{aligned}$$

then by adding and subtracting Eqs. (3.5) and (3.6), we recover Eq. (2.3). Accordingly, by estimating the discontinuities, Eqs. (3.2) and (3.3), we may use Eqs. (3.5) and (3.6) to determine the parameters α and β.

Adapting a simple theory of porous materials given in [18, pp. 252–256], the equations governing the acoustical behavior of the porous barrier are

$$i\omega\kappa_p\Omega p = \frac{du_2}{dy}\,, \tag{3.7}$$

and

$$\frac{dp}{dy} = i\omega\rho_p\left[1 + \frac{i\Phi}{\rho_p\omega}\right]u_2. \tag{3.8}$$

The particle velocity in the barrier u_2 is restricted to be in the normal direction only, the particle velocity in the tangential direction must be zero, and the acoustic pressure in the barrier is p. The parameters of the model are κ_p the compressibility of the gas in the pores, Ω the porosity or fraction of the volume occupied by the pores and hence by the gas, ρ_p the effective density of the gas in the pores and Φ the flow resistance. This last parameter determines the effective sound absorbing properties of the barrier. At the boundaries of the barrier the pressure and normal components of the particle velocity are continuous. No condition is placed on the tangential particle velocity components immediately outside the barrier. Integrating Eqs. (3.7) and (3.8), noting that p and u_2 are the total fields in the barrier and using the boundary conditions at the barrier walls gives

$$[\nabla\chi_t \cdot \hat{\mathbf{n}}] = -\omega^2\rho\kappa_p\Omega(-i\omega\rho)^{-1}\int_{-h}^{h} p\,dy, \tag{3.9}$$

and

$$[\chi_t] = i\omega\rho_p \left[1 + \frac{i\Phi}{\rho_p\omega}\right] (-i\omega\rho)^{-1} \int_{-h}^{h} u_2 \, dy. \tag{3.10}$$

The barrier is both thin and absorbing. We wish to capture both these features. Defining $\kappa_e = \kappa_p\Omega$, $\rho_e = \rho_p \left[1 + \dfrac{i\Phi}{\rho_p\omega}\right]$ and $c_e = (\rho_e\kappa_e)^{-1/2}$, the effective wavenumber in the barrier is $k_e = \omega/c_e$. We assume that p and u_2 vary slowly enough through the barrier to be approximated accurately by the first two terms of a Taylor series in the scaled thickness variable $|k_e|h(y/h)$. This assumes that the flow resistance is not so strong as to cause the wavefield in the barrier to decay very rapidly. We are therefore able to relate Eqs. (3.5) and (3.6) to the porous barrier model by noting that

$$\frac{1}{(-i\omega\rho)2h} \int_{-h}^{h} p \, dy = \frac{[\chi_t(x,h,z) + \chi_t(x,-h,z)]}{2} + O(|k_e|h)^2, \tag{3.11}$$

and

$$\frac{-1}{2h} \int_{-h}^{h} u_2 \, dy = \frac{[\nabla\chi_t(x,h,z)\cdot\hat{\mathbf{n}} + \nabla\chi_t(x,-h,z)\cdot\hat{\mathbf{n}}]}{2} + O(|k_e|h)^2. \tag{3.12}$$

Assuming that $(|k_e|h)^2$ is small, we find that

$$\alpha + \beta = -i\rho c^2 \kappa_p \Omega h k, \tag{3.13}$$

and

$$\alpha - \beta = \frac{i\rho}{\rho_p[1 + i\Phi/\rho_p\omega]} \,. \tag{3.14}$$

Note that only $(\alpha - \beta)$ contains the flow resistance term.

To estimate the sizes of these terms assume that κ_p and ρ_p are equal to the compressibility κ an density ρ of the surrounding gas, so that $\kappa_p\rho_p c^2 = 1$. This is not quite the case because ρ_p can be larger than ρ, and κ_p can be the isothermal compressibility rather than the adiabatic compressibility κ. Nevertheless, if the barrier is to absorb the incident sound, then $\Phi/\rho\omega$ must be moderately large. MORSE and INGRAD [18] suggest a value as high as 10 at 1000 Hz. We are therefore left with the following estimates:

$$\alpha + \beta = -i\Omega h k, \tag{3.15}$$

and

$$(\alpha - \beta)^{-1} = \frac{kh\Phi}{\rho\omega} \,. \tag{3.16}$$

For kh small, $(\alpha + \beta)$ is small because $\Omega < 1$, but $(\alpha - \beta)^{-1}$ need not be because, for effective sound absorption, $\Phi/\rho\omega > 1$. Moreover, $|k_e|h = kh(\Omega\Phi/\rho\omega)^{1/2}$. Examining the approximation in Eqs. (3.11) and (3.12), we note that, provided $kh\Phi/\rho\omega = O(1)$ or equivalently $h\Phi/\rho c = O(1)$, then the error leading to the approximate equivalence between Eqs. (3.5) and (3.6), and Eqs. (3.11) and (3.12) is, at least, $O(kh)$ throughout.

As we continue with the calculation we shall find that some terms are proportional to $(\alpha+\beta)$ and can be dropped, while others contain $(\alpha-\beta)$ or $(\alpha-\beta)^{-1}$ and cannot. We could just set $(\alpha+\beta)$ to zero at this point, but by carrying it through the calculation, the different roles of the barrier thickness and absorption become clearer. Moreover, though we are assuming that $(\alpha-\beta)$ is not small, it can be set to zero to recover the case of a rigid barrier.

The reflection R and transmission T coefficients for the velocity potential using the boundary condition equation (2.3) are given in [7]. Neglecting the $(\alpha+\beta)$, they are

$$R(\theta)=\frac{\sin\theta}{[\sin\theta+(\alpha-\beta)]}\,, \tag{3.17}$$

and

$$T(\theta)=\frac{(\alpha-\beta)}{[\sin\theta+(\alpha-\beta)]}\,. \tag{3.18}$$

Note that $\alpha\approx-\beta$ and thus $-2\beta\approx(\alpha-\beta)$. For normal incidence, using the previous estimates $T(\pi/2)$ is approximately $-(\rho c/2h\Phi)$ so that the barrier allows weak transmission of sound. The coefficients have no poles on the real θ axis $(0<\theta<\pi)$.

4. The Wiener-Hopf problem

The Fourier transform and its inverse over the variable z is defined as

$$\begin{aligned}\phi_t(x,y,\zeta) &= \int\limits_{-\infty}^{\infty}\chi_t(x,y,z)e^{-ik\zeta z}dz,\\ \chi_t(x,y,z) &= \frac{k}{2\pi}\int\limits_{-\infty}^{\infty}\phi_t(x,y,\zeta)e^{ik\zeta z}d\zeta.\end{aligned} \tag{4.1}$$

In Eq. (4.1), the transform parameter is taken as $k\zeta$ and ζ is non-dimensional. Transforming Eqs. (2.1), (2.7) and the boundary conditions (2.4) and (2.5) with respect to z by using Eq. (4.1) and after using the resulting equation of Eq. (2.7), we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\lambda^2\right)\phi_0(x,y,\zeta) = a_1\delta(x-x_0)\delta(y-y_0), \tag{4.2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\lambda^2\right)\phi(x,y,\zeta) = 0, \tag{4.3}$$

$$\frac{\partial}{\partial y}\phi_t(x,0^\pm,\zeta)\pm ik\alpha\phi_t(x,0^\pm,\zeta)\pm ik\beta\phi_t(x,0^\mp,\zeta)=0 \qquad (-l<x<0), \tag{4.4}$$

$$\frac{\partial}{\partial y}\phi(x,0^+,\zeta)=\frac{\partial}{\partial y}\phi(x,0^-,\zeta) \qquad (x<-l,\ x>0) \tag{4.5}$$

$$\begin{aligned}\phi(x,0^+,\zeta)-\phi(x,0^-,\zeta) &= \widetilde{a}(\zeta)e^{i\mu x} \quad (x>0),\\ \phi(x,0^+,\zeta)-\phi(x,0^-,\zeta) &= \widetilde{a}(\zeta)e^{-i\mu x} \quad (x<-l),\end{aligned} \tag{4.6}$$

where

$$\lambda^2 = 1 - \zeta^2, \qquad a_1 = e^{-ik\zeta z_0}. \tag{4.7 a, b}$$

Following the method in reference [8], the Wiener-Hopf functional equation is given by

$$\begin{aligned}
&\frac{d\overline{\phi}_+}{dy}(\nu,0,\xi) + e^{-i\nu l}\frac{d\overline{\phi}_-}{dy}(\nu,0,\xi) - i\gamma N(\nu)J_1(\nu,0,\xi) \\
&\qquad + \frac{\tilde{a}\gamma N(\nu)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu} - \frac{e^{-i(\nu-\mu)}l}{\nu-\mu}\right] - \frac{\tilde{a}k(\alpha-\beta)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu} - \frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right] \\
&\qquad\qquad = \frac{-k\lambda b\sin\theta_0}{\sqrt{2\pi}(\nu-k\lambda\cos\theta_0)}\left[1 - e^{-i(\nu-k\lambda\cos\theta_0)l}\right],
\end{aligned} \tag{4.8}$$

where

$$\gamma = \left(k^2\lambda^2 - \nu^2\right)^{1/2}, \qquad N(\nu) = 1 + \frac{k(\alpha-\beta)}{\gamma},$$

$$\begin{aligned}
J_1(\nu,0,\xi) &= \frac{1}{2}\left[\,\overline{\phi}_1(\nu,0^+,\xi) - \overline{\phi}_1(\nu,0^-,\xi)\right], \\
b &= -\frac{a_1}{4i}\sqrt{\frac{2}{\pi k\lambda r_0}}\, e^{(k\lambda r_0-\pi/4)}, \\
r_0 &= \sqrt{x_0^2+y_0^2}\,, \qquad 0<\theta_0\le\pi/2,
\end{aligned}$$

and $\overline{\phi}_+$ is regular for $\operatorname{Im}\nu > -\operatorname{Im}k\lambda$, $\overline{\phi}_-$ is regular for $\operatorname{Im}\nu < \operatorname{Im}k\lambda$, $\overline{\phi}_1$ is an integral function and is therefore analytic in $-\operatorname{Im}k\lambda < \operatorname{Im}\nu < \operatorname{Im}k\lambda$ and ν is a Fourier transform parameter on x.

5. Solution of the Wiener-Hopf equation

The unknown diffracted wavefield have been determined using the procedure discussed by Noble [16, pp. 196–202]. Several steps in the procedure are given in Appendix and the final result is given by

$$\chi(x,y,z) = \frac{isgn(y)}{8\pi^2\sqrt{rr_0}}\int_{-\infty}^{\infty}\frac{[g_2(-k\lambda\cos\theta)+g_3(-k\lambda\cos\theta)]}{\sqrt{1-\zeta^2}} \cdot e^{ik\lambda(r+r_0)+ik\zeta(z-z_0)}d\zeta, \tag{5.1}$$

where

$$g_2(-k\lambda\cos\theta) = \left[\frac{f_1(-k\lambda\cos\theta)}{\cos\theta+\cos\theta_0} + k\lambda f_2(-k\lambda\cos\theta)\right], \tag{5.2}$$

$$g_3(-k\lambda\cos\theta) = -g_1(-k\lambda\cos\theta)\left[\begin{array}{c}(\alpha-\beta)\left\{\begin{array}{c}\dfrac{1}{\cos\theta_1-\lambda\cos\theta}\\+\dfrac{e^{ik\lambda l\cos\theta}}{\cos\theta_1+\lambda\cos\theta}\end{array}\right\}+f_3(-k\lambda\cos\theta)\\-\dfrac{iS_+(k\cos\theta_1)}{k}\left\{\begin{array}{c}\dfrac{S_+(-k\lambda\cos\theta)}{\cos\theta_1-\lambda\cos\theta}\\+\dfrac{e^{ik\lambda l\cos\theta}S_+(k\lambda\cos\theta)}{\cos\theta_1+\lambda\cos\theta}\end{array}\right\}\end{array}\right]. \quad (5.3)$$

The integrals in Eq. (5.1) can be solved asymptotically using the method of steepest descent. Introducing $r+r_0 = r_{12}\sin\theta_{12}$, $z-z_0 = r_{12}\cos\theta_{12}$, $\zeta = \cos(\theta_{12}+iq)$, $(-\infty < q < \infty$, $0 < \theta_{12} < \pi)$, the far field is given by

$$\chi = \chi_A + \chi_w, \quad (5.4)$$

where χ_A denotes that part of χ that arises when there is no wake and χ_w the part that arises when there is a wake. They are explicitly given by

$$\chi_A = \frac{isgn(y)g_2(-k\cos\theta\sin\theta_{12})}{4\pi\sqrt{2\pi krr_0r_{12}}}e^{i(kr_{12}-\pi/4)}, \quad (5.5)$$

$$\chi_w = \frac{isgn(y)g_3(-k\cos\theta\sin\theta_{12})}{4\pi\sqrt{2\pi krr_0r_{12}}}e^{i(kr_{12}-\pi/4)}, \quad (5.6)$$

where $g_2(-k\cos\theta\sin\theta_{12})$ and $g_3(-k\cos\theta\sin\theta_{12})$ are given by Eqs. (5.2) and (5.3) respectively.

6. Concluding remarks

A new canonical diffraction problem of spherical wave (emanating due to a point source) by a penetrable finite plane has been solved in the presence of a wake. The problem studied takes into account the material properties and thickness of the finite plane. I address this problem using an analytical approach based on the Wiener-Hopf method. A key attribute of such an approach is that it is not fundamentally numerical in nature and thus allows additional insight into the mathematical and physical structure of the diffracted field.

It is also of interest to note from Eqs. (A.8) and (A.9) that ϕ^{sep} consists of two parts each representing the diffracted field produced by the two edges at $x = 0$ and $x = -l$ respectively, as though the other edges were absent while ϕ^{int} gives the interaction of one edge upon the other. Furthermore, the transmitted sound level, for wood or plastic barriers is almost proportional to kh. Thus most of the transmitted noise is the low frequency sound. The high frequency sound is diffracted into the shadow of the barrier via the edges.

The present work with no wake also has applications in electromagnetism when considering diffraction by a dielectric finite plane. For this we introduce $n = \sqrt{\epsilon_1\sigma_{1/\epsilon\sigma}}$, $N = K_1\epsilon/k\epsilon_1\sin\theta_0$, (for $\chi_t = H_z$ magnetic vector parallel to z-axis), $N = K_1\sigma/k\sigma_1\sin\theta_0$, (for

$\chi_t = E_z$ electric vector parallel to z-axis) where σ, ϵ and σ_1 and ϵ_1 are the permeability and permittivity of the media $|y| > h$ and $|y| < h$, respectively.

Several physically interesting features of Eq. (5.7) are further noted. First, it is once again found that the imposition of the Kutta-Joukowski condition and associated wake has the effect of producing a stronger diffracted field away from the wake than that in the absence of the Kutta-Joukowski condition when the source is near the edge. In the neighbourhood of the wake an intense sound is created; it is much stronger than the scattered field away from the wake and does not decay downstream. This is true whether or not the source is near the edge. Second, the results for no wake situation can be obtained by taking $a = 0$. Third, the field corresponds to a rigid barrier if we put $\alpha = \beta = 0$. Fourth, the results for an absorbing finite plane in presence of a wake can be obtained by taking $\beta = 0$ and $\alpha = \rho_0 c/z_a$ (ρ_0 is the density of the undisturbed stream and z_a is the acoustic impedance of the surface). Thus, the consideration of the penetrable finite plane with wake represent a more generalized model in the theory of diffraction and quite a few interesting situations can be obtained as a special case by choosing suitable parameters.

Appendix

For the solution of the Wiener-Hopf functional equation (4.8), we make the following factorizations:

$$\gamma = (k\lambda + \nu)^{1/2}(k\lambda - \nu)^{1/2} = K_+(\nu)K_-(\nu), \tag{A.1}$$

$$N(\nu) = N_+(\nu)N_-(\nu), \tag{A.2}$$

where $N_+(\nu)$ and $K_+(\nu)$ are regular for $\operatorname{Im}\nu > -\operatorname{Im}k\,\lambda$ and $N_-(\nu)$ and $K_-(\nu)$ are regular for $\operatorname{Im}\nu < -\operatorname{Im}k\lambda$. The factorization (A.2) is obtained by employing the method of Noble [16, p. 164] and is given by

$$N_\pm(\nu) = 1 - \frac{i(\alpha-\beta)}{\pi}\left[(\nu/k)^2 - \lambda^2\right]^{-1/2}\cos^{-1}(\pm\nu/k\lambda). \tag{A.3}$$

Thus, substitution of Eqs. (A.1) and (A.2) in Eq. (4.8) yields

$$\begin{aligned}\frac{d\overline{\phi}_+(\nu,0,\xi)}{dy} &+ \frac{d\overline{\phi}_-(\nu,0,\xi)}{dy}e^{-i\nu l} + S_+(\nu)S_-(\nu)J_1(\nu,0,\xi)\\ &+\frac{i\tilde{a}S_+(\nu)S_-(\nu)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu} - \frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right] - \frac{\tilde{a}k(\alpha-\beta)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu} - \frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]\\ &= \frac{-k\lambda b\sin\theta_0}{\sqrt{2\pi}(\nu - k\lambda\cos\theta_0)}\left[1 - e^{-i(\nu-k\lambda\cos\theta_0)l}\right].\end{aligned} \tag{A.4}$$

In Eq. (A.4), $S_+(\nu)[= K_+(\nu)N_+(\nu)]$ is regular for $\operatorname{Im}\nu > -\operatorname{Im}k\lambda$ and $S_-(\nu)$ $[= K_-(\nu)N_-(\nu)]$ is regular for $\operatorname{Im}\nu < -\operatorname{Im}k\lambda$. The unknown functions $\dfrac{d\overline{\phi}_+(\nu,0,\xi)}{dy}$ and $\dfrac{d\overline{\phi}_-(\nu,0,\xi)}{dy}$ in Eq. (A.4) have been determined using the procedure discussed by Noble

[16, p. 196–202] and are given by

$$\begin{aligned}\frac{d\overline{\phi}_+(\nu,0,\xi)}{dy} &= \frac{-k\lambda b\sin\theta_0}{\sqrt{2\pi}}\left[S_+(\nu)G_1(\nu)+T(\nu)S_+(\nu)C_1\right]\\ &\quad+\frac{\tilde{a}}{2\sqrt{2\pi}}\left[\frac{k(\alpha-\beta)}{(\nu+\mu)}-\frac{iS_+(\mu)S_+(\nu)}{(\nu+\mu)}+\frac{T(\nu)S_+(\nu)}{(k+\mu)}C_3\right], \qquad (A.5)\end{aligned}$$

$$\begin{aligned}\frac{d\overline{\phi}_-(\nu,0,\xi)}{dy} &= \frac{-k\lambda b\sin\theta_0}{\sqrt{2\pi}}\left[S_-(\nu)G_2(-\nu)+T(-\nu)S_-(\nu)C_2\right]\\ &\quad+\frac{\tilde{a}}{2\sqrt{2\pi}}\left[\frac{k(\alpha-\beta)}{(\mu-\nu)}-\frac{iS_+(\mu)S_-(\nu)}{(\mu-\nu)}+\frac{T(-\nu)S_-(\nu)}{(k+\mu)}C_3\right]. \qquad (A.6)\end{aligned}$$

Using above expressions and following [8], the diffracted wavefield ϕ is given by

$$\phi(x,y,\zeta)=\phi^{\text{sep}}(x,y,\zeta)+\phi^{\text{int}}(x,y,\zeta), \qquad (A.7)$$

where

$$\begin{aligned}\phi^{\text{sep}}(x,y,\zeta) &= \frac{isgn(y)f_1(-k\lambda\cos\theta)}{4\pi k\lambda(\cos\theta+\cos\theta_0)\sqrt{rr_0}}e^{ik\lambda(r+r_0)-ik\zeta z_0}\\ &\quad+\frac{asgn(y)}{2\sqrt{2\pi k\lambda r}}\left[\begin{array}{c}(\alpha-\beta)e^{i\pi/4}\left\{\begin{array}{c}\dfrac{1}{\cos\theta_1-\lambda\cos\theta}\\ +\dfrac{e^{ik\lambda l\cos\theta}}{\cos\theta_1+\lambda\cos\theta}\end{array}\right\}\\ +\dfrac{e^{-i\pi/4}S_+(k\cos\theta_1)}{k}\left\{\begin{array}{c}\dfrac{S_+(-k\lambda\cos\theta)}{\cos\theta_1-\lambda\cos\theta}\\ +\dfrac{e^{ik\lambda l\cos\theta}S_+(k\lambda\cos\theta)}{\cos\theta_1+\lambda\cos\theta}\end{array}\right\}\end{array}\right]e^{ik\lambda r}, \qquad (A.8)\end{aligned}$$

$$\begin{aligned}\phi^{\text{int}}(x,y,\zeta) &= \frac{isgn(y)f_2(-k\lambda\cos\theta)}{4\pi\sqrt{rr_0}}e^{ik\lambda(r+r_0)-ik\zeta z_0}\\ &\quad+\frac{\widetilde{a}e^{i(k\lambda r+\pi/4)}}{2\sqrt{2\pi k\lambda r}}sgn(y)f_3(-k\lambda\cos\theta), \qquad (A.9)\end{aligned}$$

$$f_1(-k\lambda\cos\theta) = -\sin\theta_0\left[\frac{S_+(-k\lambda\cos\theta)}{S_+(k\lambda\cos\theta_0)}-\frac{S_+(k\lambda\cos\theta)e^{ik\lambda l(\cos\theta+\cos\theta_0)}}{S_+(-k\lambda\cos\theta_0)}\right], \qquad (A.10)$$

$$f_2(-k\lambda\cos\theta) = \sin\theta_0\left[\begin{array}{c}S_+(-k\lambda\cos\theta)R_1(-k\lambda\cos\theta)e^{ik\lambda l\cos\theta_0}\\ -S_+(k\lambda\cos\theta)R_2(k\lambda\cos\theta)e^{ik\lambda l\cos\theta}\\ -S_+(-k\lambda\cos\theta)T(-k\lambda\cos\theta)C_1\\ -S_+(k\lambda\cos\theta)T(k\lambda\cos\theta)C_2e^{ik\lambda l\cos\theta}\end{array}\right], \qquad (A.11)$$

$$f_3(-k\lambda\cos\theta) = \frac{C_3}{k+k\cos\theta_1}\left[\begin{array}{c}S_+(-k\lambda\cos\theta)T(-k\lambda\cos\theta)\\ +S_+(k\lambda\cos\theta)T(k\lambda\cos\theta)e^{ik\lambda l\cos\theta}\end{array}\right], \qquad (A.12)$$

$$\begin{aligned}
S_+(\nu) &= \sqrt{k\lambda+\nu}N_+(\nu), S_-(\nu) = e^{i\pi/2}\sqrt{\nu-k\lambda}N_-(\nu),\\
N_\pm(\nu) &= 1-\frac{i(\alpha-\beta)}{\pi}\left[(\nu/k)^2-\lambda^2\right]^{-1/2}\cos^{-1}(\pm\nu/k\lambda),\\
C_1 &= \frac{S_+(k\lambda)}{[1-T^2(k\lambda)S_+^2(k\lambda)]}\left(G_2(k\lambda)+G_1(k\lambda)T(k\lambda)S_+(k\lambda)\right),\\
C_2 &= \frac{S_+(k\lambda)}{[1-T^2(k\lambda)S_+^2(k\lambda)]}\left(G_1(k\lambda)+G_2(k\lambda)T(k\lambda)S_+(k\lambda)\right),\\
C_3 &= -iS_+(\mu)\frac{S_+(k\lambda)}{[1-T^2(k\lambda)S_+^2(k\lambda)]}\left(T(k\lambda)S_+(k\lambda)-e^{i\mu l}\right),\\
G_1(\nu) &= \frac{1}{\nu-k\lambda\cos\theta_0}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(k\lambda\cos\theta_0)}\right]-R_1(\nu)e^{ik\lambda l\cos\theta_0},\\
G_2(\nu) &= \frac{1}{\nu+k\lambda\cos\theta_0}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(-k\lambda\cos\theta_0)}\right]e^{ik\lambda l\cos\theta_0}-R_2(\nu),\\
R_{1,2}(\nu) &= \frac{E_{-1}\left[W_{-1}\left\{-i(k\lambda+k\lambda\cos\theta_0)l\right\}-W_{-1}\left\{-i(k\lambda+\nu)l\right\}\right]}{2\pi i(\nu\mp k\lambda\cos\theta_0)},\\
T(\nu) &= \frac{1}{2\pi i}E_{-1}W_{-1}\left[-i(k\lambda+\nu)l\right], \qquad E_{-1}=2\sqrt{l}e^{ik\lambda l-3i\pi/4},\\
W_{-1}(m) &= \Gamma(1/2)e^{m/2}(m)^{-3/4}W_{-1/4,-1/4}(m)
\end{aligned}$$

($m=-i(k\lambda+\nu)l$ and $W_{i,j}$ is a Whittaker function).

In the limit $r\to 0$, Eq. (A.7) shows that

$$\phi(x,y,\zeta)\approx 2\sqrt{r}sgn(y)\left[\begin{array}{c}-\dfrac{e^{ik\lambda r_0-ik\zeta z_0}\left\{f_1(-k\lambda\cos\theta)+\frac{k\lambda}{2}f_2(-k\lambda\cos\theta)\right\}}{4\pi\sqrt{r_0}}+\dfrac{i\widetilde{a}e^{i\pi/4}k\lambda}{2\sqrt{2\pi k\lambda}}\\ \times\left[\begin{array}{c}(\alpha-\beta)\left(1+e^{ik\lambda l\cos\theta}\right)+\dfrac{f_3(-k\lambda\cos\theta)}{2}\\ -\dfrac{iS_+(k\cos\theta_1)}{k}\left(\begin{array}{c}S_+(-k\lambda\cos\theta)\\ +S_+(k\lambda\cos\theta)e^{ik\lambda l\cos\theta}\end{array}\right)\end{array}\right]\end{array}\right],$$

where we have neglected the terms which are constant and $O(r)$. Therefore, the velocity will remain bounded at the edge if and only if the coefficient of $\sqrt{r}$ vanishes. Hence the Kutta-Joukowski condition requires that

$$\widetilde{a}=\frac{e^{ik\lambda r_0-ik\zeta z_0-3i\pi/4}}{\sqrt{2\pi k\lambda r_0}}g_1(-k\lambda\cos\theta), \tag{A.13}$$

where

$$\begin{aligned}
g_1(-k\lambda\cos\theta) &= \left\{f_1(-k\lambda\cos\theta)+\frac{k\lambda}{2}f_2(-k\lambda\cos\theta)\right\}\\
&\times\left[\begin{array}{c}(\alpha-\beta)\left(1+e^{ik\lambda l\cos\theta}\right)+\dfrac{f_3(-k\lambda\cos\theta)}{2}-\dfrac{iS_+(k\cos\theta_1)}{k}\\ \times\left(S_+(-k\lambda\cos\theta)+S_+(k\lambda\cos\theta)e^{ik\lambda l\cos\theta}\right)\end{array}\right]^{-1}.
\end{aligned}$$

Using Eq. (A.13) in Eq. (A.7) and then taking inverse Fourier transform over the variable z we get Eq. (5.1).

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