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DIRECTED BURGERS WAVES INTERACTING WITH MEAN FIELD THE MODIFIED BURGERS EQUATION AND ITS STATIONARY SOLUTIONS

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Projecting technique is used for a system of coupled nonlinear equations for interacting modes derivation. Three independent modes of the viscous flow: leftwards, rightwards propagating (acoustic) and the stationary one (heat) are specified by a set of orthogonal projectors. The final system of coupled equations is equivalent to the basic one with the accuracy up to the third-order nonlinear terms and its form allows to apply nonsingular perturbations method for an approximate solution in the case when one mode is dominant compared with other iterated modes. Caloric and thermal equations of state are taken in the general form. Thermoviscous media are treated, that leads to a system of coupled equations of the Burgers type. The modified Burgers equation for rightwards mode affected by other ones is obtained as well as its stationary solutions.

1. General remarks concerning the methods used for derivation of evolution equations in acoustics

Traditionally, the wide variety of acoustic problems relates to the progressive wave tracing. There are one-dimensional problems and quasi-plane ones such as focused/unfocused beams propagation generated by oscillating pistons. Exact solutions in the form of plane nonlinear travelling waves were obtained originally by Riemann. A common way for weak nonlinear flow investigation coming from the pioneering works of Korteweg and de Vries is based on the assumption that all field variables depend on τ (slowly varying time, that is responsible to the form of wave, $\tau = \mu t$, $\mu \ll 1$) and accompanying co-ordinate $x + ct$ (or $x - ct$) for leftwards (or rightwards) waves, respectively. This is the so-called high-frequency method of the nonlinear acoustics [1], suitable until shock formation. Then the procedure based on the use of small parameter μ is performed. In this manner, the most important evolution equations were derived. There are: in one-dimensional problems — the Burgers one for thermoviscous media and the Korteweg-de-Vries (KdV) equation for dispersive media, in quasi-plane problems: the Khokhlov-Zabolotskaya equation (KZ) and the Khokhlov-Zabolotskaya-Kuznetsov one (KZK) [2, 3]. It is clear that asymmetric choice of variables allows to get the evolution equations for one chosen directed mode with account of self-action only. Moreover, this approach seems to be incorrect since general initial disturbances cause all types of interacting modes: rightwards, leftwards and stationary ones. If even one mode is dominant and excited initially, a process of mutual interaction occurs and evolution equation should be corrected.

In the linear one-dimensional dynamics a problem of overall field separating to directed (leftwards and rightwards) and stationary modes is solved exactly with the projecting technique [4, 5]. These papers refer to flows in homogeneous, inhomogeneous (affected by gravity) and heterogeneous (bubbly liquid) media. Then relations between specific variables (velocity, pressure, density) of every mode are defined by the eigenvectors of the corresponding matrix. It seems more fruitful to separate the modes (i.e. to get evolution equations for every mode) on the level of the basic system of equations. This way we do not have to think about representation of every mode, only to act projectors on the overall field in order to separate the different modes. The similar ideas for radio waves have been used in [6].

For general nonlinear problem, the procedure also results in the application of new independent variables. Defining the relations between specific variables as in the linear problem, we pass to the coupled equations for these specific variables. The final system consists of equations with first-order derivatives with respect to time and is equivalent to the original one. One may use iteration procedure, going to a set of more and more exact equations.

The method applies to the three-dimensional flow investigation as well. The new two vorticity modes appear, according to the classification of linear motion [7]. The method leads to the definition of five projectors and five corresponding coupled nonlinear equations that are obviously more difficult for solution. This investigation demonstrates the algorithmic features of the method to get the final evolution equations as well as the way for approximate solution when one mode is dominant. The case of one-dimensional flow is taken as the most simple example for demonstration of the idea.

Orthogonal projectors are widely used in quantum mechanics but their application to fluid dynamics seems to be novel. In addition, projectors are suitable for problems with initial conditions as well as for boundary value problems.

We present:

Matrix projectors for a one-dimensional problem with viscosity and thermal conductivity;

Coupled nonlinear equations of Burgers type system in weak nonlinear regime;

The modified Burgers evolution equation for the right-propagating mode as a result of the iterated influence of other modes. Its stationary solutions are obtained and compared with those of the Burgers equation.

2. Basic equations. Projectors in a linear problem

We consider a one-dimensional gas flow in which processes of thermal conductivity and internal friction occur. A basic system thus represents hydrodynamic equations of Navier-Stokes (all terms representing a product of nonlinear and viscous terms are neglected):

$$\begin{aligned}(1+\varepsilon\rho)[\partial v/\partial t+\varepsilon v\partial v/\partial x]+\partial p/\partial x-\delta_1\partial^2 v/\partial x^2 &= 0,\\ (1+\varepsilon\rho)[\partial e/\partial t+\varepsilon v\partial e/\partial x]+(1+\varepsilon pc^2 p_0/\rho_0)\partial v/\partial x-\vartheta\partial^2 T/\partial x^2 &= 0,\\ \partial\rho/\partial t+\partial[(1+\varepsilon\rho)v]/\partial x &= 0,\end{aligned} \tag{2.1}$$

where ρ, p, e, T are non-dimensional perturbations of density, pressure, internal energy per unit mass and temperature, respectively, v is non-dimensional velocity; all variables relate to the dimensional ones, marked by asterisks, in the following way:

$$\rho_*=\rho_0+\varepsilon\rho_0\rho,\qquad p_*=p_0+\varepsilon c^2\rho_0 p,\qquad e_*=e_0+\varepsilon p_0 e/\rho_0,$$

$$T_*=T_0+\varepsilon p_0 T/(\rho_0 C_v),\qquad v_*=\varepsilon cv.$$

Background values are marked with zero, C_v is heat capacity per unit mass under constant volume, c is the sound velocity. Non-dimensional space co-ordinate x and time t were also introduced:

$$x_*=\lambda x,\qquad t_*=\lambda t/c,$$

(λ is the characteristic scale parameter of perturbation), as well as non-dimensional coefficients

$$\delta_1=\frac{\left(\frac{4}{3}\eta+\zeta\right)}{\rho_0 c\lambda},\qquad \vartheta=\frac{\chi}{\rho_0 c\lambda C_v}\,.$$

Coefficients of bulk and shear viscosity ζ and η and thermal conductivity χ are assumed to be constants.

In the second equation only linear terms relating to thermal conductivity (there are no linear viscous terms at all) are left. To treat a general case, let e and T have a form:

$$\begin{aligned}e&=E_1\frac{\rho_0}{p_0}c^2p+E_2\rho+\varepsilon E_3\left(\frac{\rho_0}{p_0}c^2\right)^2p^2+\varepsilon E_4\rho^2+\varepsilon E_5\frac{\rho_0}{p_0}c^2p\rho\\ &\quad+\varepsilon^2E_6\left(\frac{\rho_0}{p_0}c^2\right)^2p^2\rho+\varepsilon^2E_7\frac{\rho_0}{p_0}c^2p\rho^2+\varepsilon^2E_8\left(\frac{\rho_0}{p_0}c^2\right)^3p^3+\varepsilon^2E_8\rho^3+...,\\ T&=\Theta_1\frac{\rho_0}{p_0}c^2p+\Theta_2\rho+...,\end{aligned} \tag{2.2}$$

E_1, ..., Θ_1, ... are dimensionless coefficients. Thus system (2.1) – (2.2) is suitable for a wide variety of gases and liquids and we are not restricted with any special cases of internal energy and temperature dependence on pressure and density since we use caloric $e=e(p,\rho)$ and thermal $T=T(p,\rho)$ equations of state in the general form. When only two linear terms of temperature expansion (2.2) are preserved, the problem will be solved to the lowest order in the dissipation: cross terms proportional to Mach and Knudsen numbers product are usually neglected. Here we follow the common assumptions [1, 2, 8]. Expansion of internal energy is more extended according to our goals of accounting for modes interaction (what is described by cubic nonlinear terms) as well as to general interest for higher order nonlinear acoustic equations, see also [9, 10]. From (2.1) and (2.2) it follows that the sound speed depends on E_1, E_2 in the following way:

$$c=\sqrt{\frac{p_0(1-E_2)}{\rho_0 E_1}}\,.$$

The system (2.1) – (2.2) may be rewritten as follows:

$$\frac{\partial}{\partial t}\psi+L\psi=\widetilde{\psi}+\widetilde{\widetilde{\psi}}+O(\varepsilon^3,\varepsilon\beta). \tag{2.3}$$

Here, $\psi=\begin{pmatrix} v\\ p\\ \rho\end{pmatrix}$ is the column of variables,

$$L=\begin{pmatrix} -\delta_1\partial^2/\partial x^2 & \partial/\partial x & 0\\ \partial/\partial x & -\delta_2^1\partial^2/\partial x^2 & -\delta_2^2\partial^2/\partial x^2\\ \partial/\partial x & 0 & 0\end{pmatrix}$$

is the linear matrix operator,

$$\delta_2^1=\frac{\Theta_1}{E_1}\vartheta,\qquad \delta_2^2=\frac{\Theta_2}{(1-E_2)}\vartheta,\qquad \beta=\delta_1+\delta_2^1+\delta_2^2,$$

and

$$\widetilde{\psi}=\varepsilon\begin{pmatrix} -v\dfrac{\partial}{\partial x}v+\rho\dfrac{\partial}{\partial x}p\\ -v\dfrac{\partial}{\partial x}p+\dfrac{\partial}{\partial x}v[N_1\cdot p+N_2\cdot\rho]\\ -v\dfrac{\partial}{\partial x}\rho-\rho\dfrac{\partial}{\partial x}v\end{pmatrix},\quad \widetilde{\widetilde{\psi}}=\varepsilon^2\begin{pmatrix} -\rho^2\dfrac{\partial}{\partial x}p\\ \dfrac{\partial v}{\partial x}[N_3\cdot p^2+N_4\cdot\rho^2+N_5\cdot p\rho]\\ 0\end{pmatrix}, \tag{2.4}$$

with the notations:

$$\begin{aligned} N_1 &= \frac{1}{E_1}\left(-1+2\frac{1-E_2}{E_1}E_3+E_5\right),\\ N_2 &= \frac{1}{1-E_2}\left(1+E_2+2E_4+\frac{1-E_2}{E_1}E_5\right),\\ N_3 &= \frac{(1-E_2)}{E_1^2}\left(\frac{-2E_3(1-E_5)-3E_8(1-E_2)}{E_1}+\frac{4E_3^2(1-E_2)}{E_1^2}-E_6\right),\\ N_4 &= \frac{1}{(1-E_2)E_1}\left(E_1(1-2E_4-3E_9)+E_5(1+E_2+2E_4)\right.\\ &\qquad\left.+\frac{(1-E_2)E_5^2}{E_1}-(1-E_2)E_7\right),\\ N_5 &= \frac{1}{E_1}\left(-1-E_5-2E_7+\frac{2(1+E_2+2E_4)E_3-E_5+E_5^2-2E_6+2E_2E_6}{E_1}\right.\\ &\qquad\left.+\frac{4E_3E_5(1-E_2)}{E_1^2}\right). \end{aligned} \tag{2.5}$$

For the linear version of the system (2.3) (with zero right-hand side), one can find a solution in the space of Fourier transforms and then return to the (x,t) space using the inverse Fourier transformation. We assume the plane waves $\sim \exp(i\omega t - ikx)$ with amplitudes V_k, P_k, R_k. The eigenvalues of the corresponding system of equations for the Fourier transforms:

$$\det\left|\begin{pmatrix} i\omega+\delta_1 k^2 & -ik & 0 \\ -ik & i\omega+\delta_2^1 k^2 & \delta_2^2 k^2 \\ -ik & 0 & i\omega \end{pmatrix}\right| = 0$$

are

$$i\omega_{1,2} = \pm ik - \frac{k^2}{2}\beta, \qquad i\omega_3 = k^2\delta_2^2.$$

These relations serve as dispersion relations for the right- and left-propagating and stationary components. The eigenvectors in k-representation are:

$$\psi_{1,2} = \begin{pmatrix} \pm\left(1 \pm \dfrac{ik\beta}{2}\right) \\ 1 \pm ik(\delta_2^1+\delta_2^2) \\ 1 \end{pmatrix} R_{k1,2}, \qquad \psi_3 = \begin{pmatrix} -ik\delta_2^2 \\ 0 \\ 1 \end{pmatrix} R_{k3}\,.$$

Therefore, we obtain the relations for the specific variables appearing in the (x,t) space:

$$v_{1,2} = \pm\left(1 \mp \frac{\beta}{2}\frac{\partial}{\partial x}\right)\rho_{1,2}, \qquad p_{1,2} = \left(1 \mp (\delta_2^1+\delta_2^2)\frac{\partial}{\partial x}\right)\rho_{1,2}, \qquad v_3 = \delta_2^2\frac{\partial \rho_3}{\partial x}\,. \tag{2.6}$$

Following relations (2.6), matrix X should be determined:

$$\begin{pmatrix} v(r,t) \\ p'(r,t) \\ \rho'(r,t) \end{pmatrix} \equiv \begin{pmatrix} v_1+v_2+v_3 \\ p_1+p_2 \\ \rho_1+\rho_2+\rho_3 \end{pmatrix} = X \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix},$$

and also X^{-1}:

$$X^{-1} = \begin{pmatrix} \dfrac{1}{2} + \dfrac{\delta_2^1+\delta_2^2}{2}\dfrac{\partial}{\partial x} & \dfrac{1}{2} + \left(\dfrac{\beta}{4}+\dfrac{\delta_2^2}{2}\right)\dfrac{\partial}{\partial x} & -\dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} \\ -\dfrac{1}{2} + \dfrac{\delta_2^1+\delta_2^2}{2}\dfrac{\partial}{\partial x} & \dfrac{1}{2} - \left(\dfrac{\beta}{4}+\dfrac{\delta_2^2}{2}\right)\dfrac{\partial}{\partial x} & \dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} \\ -(\delta_2^1+\delta_2^2)\dfrac{\partial}{\partial x} & -1 & 1 \end{pmatrix}.$$

From this formula and Eq. (2.6) projectors follow immediately:

$$P_{1,2} = \begin{pmatrix} \dfrac{1}{2} \pm \left(\dfrac{\delta_2^1+\delta_2^2}{2} - \dfrac{\beta}{2}\right)\dfrac{\partial}{\partial x} & \pm\dfrac{1}{2} + \dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} & -\dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} \\ \pm\dfrac{1}{2} & \dfrac{1}{2} \pm \left(\dfrac{\beta}{2} - \dfrac{\delta_2^1+\delta_2^2}{2}\right)\dfrac{\partial}{\partial x} & \mp\dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} \\ \pm\dfrac{1}{2} + \dfrac{\delta_2^1+\delta_2^2}{2}\dfrac{\partial}{\partial x} & \dfrac{1}{2} \pm \left(\dfrac{\beta}{2} + \dfrac{\delta_2^1+\delta_2^2}{2}\right)\dfrac{\partial}{\partial x} & \mp\dfrac{\delta_2^2}{2}\dfrac{\partial}{\partial x} \end{pmatrix}, \tag{2.7}$$

$$P_3 = \begin{pmatrix} 0 & -\delta_2^2 \dfrac{\partial}{\partial x} & \delta_2^2 \dfrac{\partial}{\partial x} \\ 0 & 0 & 0 \\ -(\delta_2^1 + \delta_2^2)\dfrac{\partial}{\partial x} & -1 & 1 \end{pmatrix}. \tag{2.7}$$

[cont.]

The obtained operators (2.7) have general properties of orthogonal projectors: $P_1 + P_2 + P_3 = \widetilde{I}$; $P_1P_2 = P_1P_3 = ... = \widetilde{0}$, where $\widetilde{I}$ and $\widetilde{0}$ are unit and zero matrices, $P_1 = P_1P_1$, ... In order to obtain the rightwards progressive column of specific components at any instant, for example, it is sufficient to act P_1 on the total field:

$$P_1 \begin{pmatrix} v(x,t) \\ p(x,t) \\ \rho(x,t) \end{pmatrix} = \begin{pmatrix} v_1(x,t) \\ p_1(x,t) \\ \rho_1(x,t) \end{pmatrix}$$

or, defining rightwards mode as $\psi_1 : P_1\psi = \psi_1$, just the same for other inputs: $P_2\psi = \psi_2$, $P_3\psi = \psi_3$. Projectors P_1, P_2. P_3 do commute both with L and $\partial/\partial t$, so one can act them on the basic system directly.

Note that the cross nonlinear-viscous terms ($\sim \beta\varepsilon$) are neglected in the system (2.3) though cubic nonlinear ones of order of ε^2 are preserved. That implies the relation between small viscous and nonlinear parameters as follows: $\beta \sim \varepsilon^2$. Cross nonlinear-viscous terms may be placed in the right-hand side of (2.3) and would influence the final evolution equations. Here, in order to demonstrate the method in the possibly simple way, we relate to the case $\beta \sim \varepsilon^2$.

3. Nonlinear coupled equations of the Burgers type

We underline once more that the system (2.1) – (2.2) and following formulae for operators are suitable for fluids treated by the general caloric and thermal equations of state. Further, the case of ideal gas is considered with coefficients:

$$E_1 = E_4 = E_7 = \Theta_1 = \frac{1}{\gamma - 1}, E_2 = E_5 = E_9 = \Theta_2 = -\frac{1}{\gamma - 1}, E_3 = E_6 = E_8 = 0,$$

γ is specific heats ratio $\gamma = C_p/C_v$, sound velocity is $c = \sqrt{\gamma(p_0/\rho_0)}$, $\delta_2 = (1/C_v - 1/C_p)C_v\vartheta$, $\delta_2^1 = \delta_2\gamma/(\gamma - 1)$, $\delta_2^2 = -\delta_2/(\gamma - 1)$. System (2.1) – (2.2) is transformed to:

$$\frac{\partial}{\partial t}\psi + L\psi = \widetilde{\psi} + \widetilde{\widetilde{\psi}} + O(\varepsilon^3, \varepsilon\beta) \tag{3.1}$$

with the right-hand columns:

$$\widetilde{\psi} = \varepsilon \begin{pmatrix} -v\dfrac{\partial}{\partial x}v + \rho\dfrac{\partial}{\partial x}p \\ -v\dfrac{\partial}{\partial x}p - \gamma p\dfrac{\partial v}{\partial x} \\ -v\dfrac{\partial}{\partial x}\rho - \rho\dfrac{\partial}{\partial x}v \end{pmatrix}, \qquad \widetilde{\widetilde{\psi}} = \varepsilon^2 \begin{pmatrix} -\rho^2\dfrac{\partial p}{\partial x} \\ 0 \\ 0 \end{pmatrix}.$$

Then we denote nearly rightwards, leftwards and stationary modes of nonlinear problem by ψ_1, ψ_2 and ψ_3 as was as in the linear problem and assume that relations (2.6) also take place. Thus the defined modes are strictly progressive and stationary in the linear limit and form a system of coupled nonlinear equation when projectors act on both sides of (3.1). We get finally the system:

$$\frac{\partial \rho_n}{\partial t} + c_n \frac{\partial \rho_n}{\partial x} + \frac{\varepsilon}{2} \sum_{i,m=1}^{3} Y^n_{i,m} \rho_i \frac{\partial}{\partial x} \rho_m + \frac{\varepsilon^2}{2} \sum_{i,m=1}^{3} T^n_{i,m} \rho_i \rho_m \frac{\partial}{\partial x} (\rho_1 + \rho_2) + \frac{1}{2} \sum_{i=1}^{3} B^n_i \frac{\partial^2}{\partial x^2} \rho_i + O(\varepsilon^3, \varepsilon\beta) = 0;$$

$n = 1, 2, 3$, $c_{1,2} = \pm 1$, $c_3 = 0$. The coefficients are determined in tables (3.2):

$$\begin{array}{cccc} Y^1_{i,m} & m/1 & 2 & 3 \\ i/1 & \gamma+1 & -\gamma-1 & 0 \\ 2 & \gamma-3 & -\gamma-1 & 0 \\ 3 & -1 & -1 & 0 \end{array}, \qquad \begin{array}{cccc} Y^2_{i,m} & m/1 & 2 & 3 \\ i/1 & -\gamma-1 & \gamma-3 & 0 \\ 2 & -\gamma-1 & \gamma+1 & 0 \\ 3 & 1 & 1 & 0 \end{array},$$

$$\begin{array}{cccc} Y^3_{i,m} & m/1 & 2 & 3 \\ i/1 & -2\gamma+2 & 2\gamma-2 & 2 \\ 2 & -2\gamma+2 & 2\gamma-2 & -2 \\ 3 & 2 & -2 & 0 \end{array}, \qquad \begin{array}{cccc} B^n_i & n/1 & 2 & 3 \\ i/1 & -\beta & 0 & 0 \\ 2 & 0 & -\beta & 0 \\ 3 & 0 & 0 & 2\delta_2^2 \end{array}, \qquad (3.2)$$

$T^1_{i,m} = 1$, $T^2_{i,m} = -1$, $T^3_{i,m} = 0$ for all i, m.

In the case of self-action of, say, rightwards mode, the well-known Burgers equation for the rightwards density component follows from (3.2):

$$\frac{\partial \rho_1}{\partial t} + c_1 \frac{\partial \rho_1}{\partial t} + \varepsilon \frac{\gamma+1}{2} \rho_1 \frac{\partial}{\partial x} \rho_1 - \frac{\beta}{2} \frac{\partial^2}{\partial x^2} \rho_1 + O(\varepsilon^2, \varepsilon\beta) = 0.$$

4. The modi ed Burgers equation for rightwards mode and its stationary solutions

The ordinary method of the perturbation theory is obviously unsuitable to solve (3.2). We use a generalized non-singular perturbation method similar to that from [11]. The zero approximation $\rho_n^{(0)}(x,t)$ satisfies an equation (case of the self-action):

$$\frac{\partial \rho_n^{(0)}}{\partial t} + c_n \frac{\partial \rho_n^{(0)}}{\partial x} + \frac{\varepsilon}{2} Y^n_{n,n} \rho_n^{(0)} \frac{\partial \rho_n^{(0)}}{\partial x} + \frac{\varepsilon^2}{2} T^n_{n,n} \rho_n^{(0)2} \frac{\partial \rho_n^{(0)}}{\partial x} + \frac{B^n_n}{2} \frac{\partial^2}{\partial x^2} \rho_n^{(0)} = 0,$$

with an initial condition as follows: $\rho_n^{(0)}(x,0) = \phi_n(x)$.

Then the approximate solution of (3.2), which accounts for the interaction effects of the first order, is:

$$\rho_n^{(1)}(x,t) = \rho_n^{(0)}(r,t) - \frac{\varepsilon}{2} \int_0^t \sum_{m,\, k \neq n} Y^n_{m,k} \rho_m^{(0)} \frac{\partial}{\partial x} \rho_k^{(0)} \Big|_{x - c_n(t-\tau), \tau} d\tau. \qquad (4.1)$$

Suppose that only one mode of number i is excited initially, $\phi_n(x) = 0$, $n \neq i$. Evolution of all other modes is described by (4.1):

$$\rho_n^{(1)}(x,t) = \frac{\varepsilon Y_{i,i}^n}{4(c_i - c_n)} \left[\left(\rho_i^{(0)}(x,t) \right)^2 - (\varphi_i(x - c_n t))^2 \right].$$

The iterated influence is apparently of the order of ε^2. The main term that does influence the i-mode is the first one, since it has the same velocity. We would say that the second input "goes away" without leaving any sufficient trace, but the influence of the first one is resonant and stored over time. In view of (4.1), the new evolution equation for the initially excited mode has a form

$$\frac{\partial \rho_i^{(1)}}{\partial t} + c_i \frac{\partial \rho_i^{(1)}}{\partial x} + \frac{\varepsilon}{2} \left(Y_{i,i}^i \rho_i^{(1)} + \varepsilon \left[T_{i,i}^i + A_i \right] \rho_i^{(1)^2} \right) \frac{\partial \rho_i^{(1)}}{\partial x} + \frac{B_i^i}{2} \frac{\partial^2 \rho_i^{(1)}}{\partial x^2} = 0, \qquad (4.2)$$

where

$$A_i = \frac{1}{2} \sum_{m \neq i} \left(Y_{m,i}^i / 2 + Y_{i,i}^m \right) Y_{i,i}^m / (c_i - c_m).$$

To illustrate and compare (4.2) with the Burgers equation in the well-known form, let us consider the rightwards mode and pass to non-dimensional variables $z = Y_{1,1}^1 \omega v_0 x / c^2$, $\tau = (t - x/c)\omega$, $V = v/v_0$ (t, x, c are dimensional ones, ω, v_0 are mean characteristic parameters of initial disturbance as well as λ we have already introduced). Then, taking into account relation (2.6) between the rightwards mode velocity and density, finally one gets from (4.2) the modified Burgers equation in agreement with the high-frequency method of nonlinear acoustics approach [1, 2]:

$$\frac{\partial V}{\partial z} - V \frac{\partial V}{\partial \tau} - \alpha M V^2 \frac{\partial V}{\partial \tau} - \Gamma \frac{\partial^2 V}{\partial \tau^2} = 0, \qquad (4.3)$$

$$\alpha = (A_1 + T_{1,1}^1)/Y_{1,1}^1 = \alpha_1 + \alpha_2, \qquad \Gamma = \frac{\omega(\zeta + 4\eta/3 + \chi(1/C_V - 1/C_P))}{(\gamma + 1) c \rho_0 v_0},$$

$M = v_0/c$ is the Mach number, $\alpha_1 = -(\gamma + 1)/8 - 1/(\gamma + 1)$ is the term caused by the iterated influence of other modes and $\alpha_2 = 1/(\gamma + 1)$ corresponds to cubic term for self-action of the rightwards mode. So an absolute value of α_1 for any ideal gas is lager than that of α_2. Since they have opposite signs, partial compensation occurs. The sign of final expression is negative, which is rather unexpected result in view of positive sign α_2. The famous Burgers equation looks just the same with zero cubic nonlinear term and serves as zero approximation for the rightwards mode (case of self-action only). Equation (4.3) takes into account the iterated influence of the other modes generated by the dominant one.

The Burgers equation and its solutions have been completely studied: there is the well-known Hopf-Cole transformation that leads it to linear one [12]. A stationary solution in the assumed co-ordinates which satisfies the conditions

$$\lim_{\tau \to \infty} V = V_0, \qquad \lim_{\tau \to \infty} \frac{dV}{d\tau} = 0 \qquad (4.4)$$

can be found from the equation

$$\frac{\tau-\tau_0}{\Gamma} = -V_0 \ln\left|\frac{V-V_0}{V+V_0}\right|. \tag{4.5}$$

Here, τ_0 is integration constant, $V_0 = 1$, that is just a scale multiplier which we may choose without any restrictions. The stationary solution of (4.3) satisfying (4.4) with $V_0 = 1$ is

$$\frac{\tau-\tau_0}{\Gamma} = -\frac{1}{2K_3}\ln\frac{(V-1)^2}{K_1(V-1)^2+K_2(V-1)+K_3}+Q,$$

$$Q = \begin{cases} \dfrac{K_2}{2K_3\sqrt{K_2^2-4K_1K_3}}\ln\left|\dfrac{2K_1(V-1)+K_2-\sqrt{K_2^2-4K_1K_3}}{2K_1(V-1)+K_2+\sqrt{K_2^2-4K_1K_3}}\right|, & K_2^2-4K_1K_3>0,\\ -\dfrac{K_2}{K_3(2K_1(V-1)+K_2)}, & K_2^2-4K_1K_3=0,\\ \dfrac{K_2}{K_3\sqrt{4K_1K_3-K_2^2}}\operatorname{arctg}\dfrac{2K_1(V-1)+K_2}{\sqrt{4K_1K_3-K_2^2}}, & K_2^2-4K_1K_3<0; \end{cases}$$

$$K_1 = \alpha M/3, \quad K_2 = \alpha M + 1/2, \quad K_3 = \alpha M + 1.$$

A solution is presented as a superposition of logarithmic functions if $M < 3/(2|\alpha|)$. For $\gamma = 1.4$ that means that $M < 5$ and is obviously correct. Solutions of (4.3) depend also on the Mach number M. Figure 1 presents stationary solution $V(\tau-\tau_0)$ for

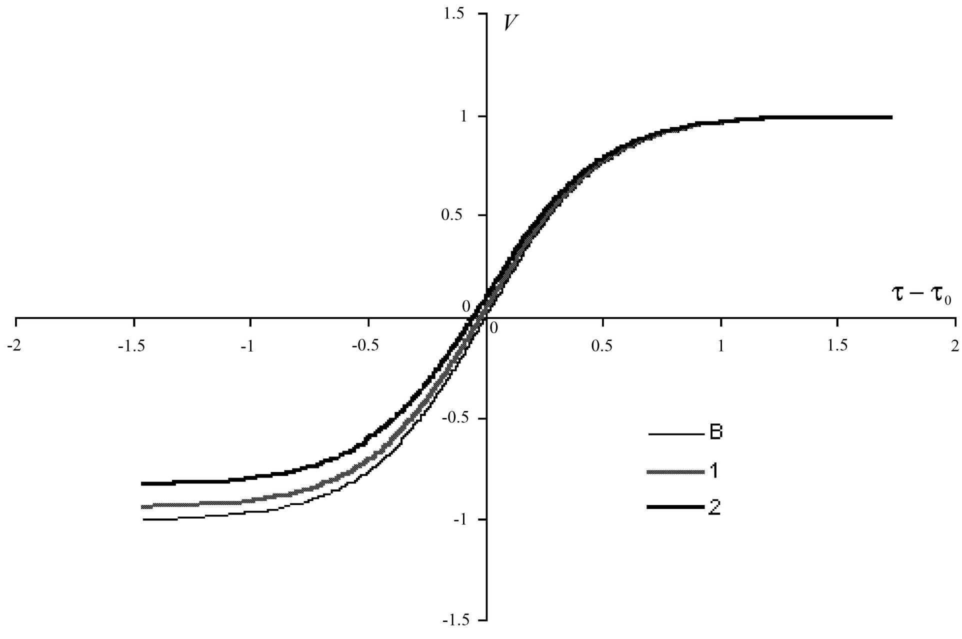

Fig. 1. Stationary solutions of the Burgers (B) and the modified Burgers equations (1 for $M = 0.3$, 2 for $M = 1$).

the Burgers and the modified Burgers equations, $\Gamma = 0.25$. For the Burgers equation $\lim\limits_{\tau\to-\infty} V = -1$, limit for the modified Burgers equation depends on M and equals to $(-K_2 + \sqrt{K_2^2 - 4K_1K_3})/2K_1 + 1$, that corresponds to -0.94 when $M = 0.3$ (the first curve presented at the figure) and to -0.83 when $M = 1$ (the second one). If only self-action third-order nonlinear terms are taken into account, $\lim\limits_{\tau\to-\infty} V < -1$ though real mutual interaction yields $\lim\limits_{\tau\to-\infty} V > -1$. Naturally, stationary solutions of the modified Burgers equation are not symmetric in contrast to the Burgers ones.

5. Conclusion

The projectors are written on in the general form depending on the equations of state of fluid for flows with viscosity and thermal conductivity. The projection operators are used for the fluid nonlinear dynamics investigation. In this manner, a system of coupled nonlinear evolution equations for the rightwards, leftwards and stationary modes is obtained that is equivalent to the basic one. The final form of the system allows to use nonsingular perturbation method to improve all evolution equations. The modified Burgers equation that describes mutual interaction of the dominant mode and other modes generated by this first one, is derived. Stationary solutions depending on the Mach number are studied and compared with the well-known stationary solution of the Burgers equation.

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