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NONLINEAR DYNAMICS OF DIRECTED ACOUSTIC WAVES IN STRATIFIED AND HOMOGENEOUS LIQUIDS AND GASES WITH ARBITRARY EQUATION OF STATE

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A method of separating one-dimensional disturbances into components propagating upwards and downwards and the stationary one in a stratified medium was developed. The system of equations is split into three coupled nonlinear equations of interacting components. Weak nonlinear evolution formulae for the directed and stationary components of a medium with an arbitrary equation of state were obtained. The wave components treated by the numerical calculations keep their propagation direction, even for quite large initial amplitudes. The results of the numerical simulation are presented. The examples demonstrate a nonlinear evolution of the wave propagating downwards for both the models of the atmosphere: the exponentially stratified model and the homogeneous one.

1. The problem of separating the directed and stationary modes

The problem of separating gasdynamical field components comes from the adaptation and initialization problems in Geophysics [1 – 3]. The common way of separating is to share branches of the dispersion relation. In this way it is possible to separate, for example, acoustic, internal gravity and Rossby waves in the atmosphere. Every wave mode possesses a certain range of frequencies and corresponding group velocities [1, 2]. In the one-dimensional atmosphere model only acoustic wave modes can propagate.

Frequently, a particular direction of propagation is under consideration. This problem has many aspects, as for instance to separate the overall field into stationary and up- or down- moving components at any instant, or to choose initial conditions for one directed mode preferable generation and so on. The close problem is the transport of energy in one direction predominantly by a special selection of initial conditions.

The main aim of the linear problem is to pick up terms in the Fourier transformation corresponding to the signs of ω and k: the same signs for the down-moving component and opposite signs for the up-moving one (or left- and rightwards in a homogeneous medium, where the basic scheme looks quite simple). At first, we get connections between the independent variables (for example, p, ρ, v) in the (k, ω)-representation, and then we just obtain connections in the (r, t)-representation. For the stationary part, one should treat the components with $\omega = 0$. In this way, the overall wave motion is separated into components of a chosen direction or a stationary one.

The problem can be solved exactly in the case of linear dynamics: the basic system leads to three independent evolution equations for every mode. In the nonlinear problem, the modes obviously interact and, in general, one can't separate strictly the directed components from the stationary one. Thus, the method of the weak nonlinear problem arises consisting in acting projectors to the basic nonlinear system and getting coupled nonlinear evolution equations for the "nearly" directed and stationary modes. These equations may be simplified under certain conditions, for example, in the case of self-action only, if one mode was previously excited. In this case, we go on to the evolution equation for any gasdynamical variable of the selected mode taking into account connections between (p, ρ, v) for this mode obtained before. This is a general scheme.

2. Inhomogeneous gas and liquid affected by gravity

2.1. The dispersion relation and connection equations

The basic system of equations contains: the second Newton's law and the equations of mass and energy conservation in the differential form with gravitational force being taken into account:

$$\begin{aligned} \partial v/\partial t + v\partial v/\partial r + (1/\rho)\partial p/\partial r + g &= 0, \\ \partial\varepsilon/\partial t + v\partial\varepsilon/\partial r + (p/\rho)\partial v/\partial r &= 0, \\ \partial\rho/\partial t + \partial(\rho v)/\partial r &= 0. \end{aligned} \tag{2.1}$$

The above equations should be completed with the caloric equation of state $\varepsilon = \varepsilon(p, \rho)$, where r-coordinate (height over the Earth surface), t — time, ρ, p, ε, v — mass density, pressure, internal energy per mass unit and velocity, respectively, g — the gravity acceleration. The problem is to exclude ε from the system (2.1). Let a small change of the energy density for any stratified media has the general form:

$$\begin{aligned} \rho_0(\varepsilon - \varepsilon_0) = A(p - p_0) + B(\rho - \rho_0) + A1(p - p_0)^2/p_0 + B1(\rho - \rho_0)^2/\rho_0 \\ + D(p - p_0)(\rho - \rho_0)/\rho_0 \,. \end{aligned} \tag{2.2}$$

The undisturbed values are marked by zero. Let also the undisturbed medium be exponentially density stratified:

$$\rho_0(r) = \rho_{00}\exp(-r/h), \qquad p_0(r) = p_{00}\exp(-r/h) = \rho_{00}gh\exp(-r/h),$$

where p_{00} and ρ_{00} are the values on the Earth surface, h is the scale height of the medium, $p_{00} = \rho_{00}gh$ is a consequence of the stationary equation $\partial p_0(r)/\partial r = -\rho_0(r)g$. The corresponding Fourier components for the perturbations in an exponentially stratified medium are written in the form:

$$\widetilde{v}'(k)\exp(r/2h + \alpha r)\exp(i(\omega t - kr)),$$

$$\widetilde{p}'(k)\exp(-r/2h+\alpha r)\exp(i(\omega t-kr)),$$
$$\widetilde{\rho}'(k)\exp(-r/2h+\alpha r)\exp(i(\omega t-kr)).$$

Here and later the disturbed values are primed. The way of the dispersion relation obtaining from the linear analogue of (2.1) is quite clear [4−6]; the final expressions are:

$$\begin{aligned}\omega^2 &= k^2(gh-B)/A+g^2(A+1)^2/(4A(gh-B)) \quad \text{or} \quad \omega=0,\\ \alpha &= -(Agh+B)/(2h(gh-B)).\end{aligned} \tag{2.3}$$

Both signs of the frequency roots and the zero solution yield the ability of three independent types of the linear wave motion. We propose to distinguish three independent modes: the up-propagating one, the down-propagating one, and the stationary one in the overall wave motion by the complete set of orthogonal projection operators. This is convenient from the physical point of view because the projection procedure may be performed by certain calculations at any moment of the evolution.

The main steps of the derivation of equations waves directed vertically are as follows from [5−7]. First of all, taking into account the dispersion relation (2.3), one can obtain equations connecting the Fourier-components of the perturbed variables

$$\begin{aligned}\widetilde{p}'(k) &= \frac{-i\rho_{00}\left[g(ik+1/2h-\alpha)-\omega^2\right]}{\omega(ik+1/2h-\alpha)}\widetilde{v}'(k),\\ \widetilde{\rho}'(k) &= \frac{-i\rho_{00}(ik+1/2h-\alpha)}{\omega}\widetilde{v}'(k).\end{aligned} \tag{2.4}$$

Now, keeping in the Fourier-transformation of each gasdynamic variable the terms corresponding to the same (the upwards directed mode) or to different (downwards directed one) sign of ω and k or $\omega=0$ (the stationary one), one can get the connection equations for each couple of variables in the (r,t)-representation.

Let

$$\widetilde{\widetilde{v}}'(k,t)=\widetilde{v}'(k)\exp\left[i\sqrt{\frac{gh-B}{A}k^2+\frac{g^2(A+1)^2}{4A(gh-B)}}\,t\right].$$

Then

$$v'(r,t)\exp(-r/2h-\alpha r)=\int\limits_{-\infty}^{\infty}\widetilde{\widetilde{v}}'(k,t)\exp(-ikr)dk+\text{c.c.}$$

is a common Fourier-transformation, which leads to the overall field disturbance that is presented as the sum of the upwards and downwards components, while the stationary component of the velocity equals obviously to zero. The expressions for the rightwards wave perturbations are as follows:

$$v_+(r,t) \equiv v'_{\text{up}}(r,t)\exp(-r/2h-\alpha r)=\int\limits_{-\infty}^{\infty}\widetilde{\widetilde{v}}'(k,t)\theta(k)\exp(-ikr)\,dk+\text{c.c.},$$

$$p_+(r,t) \equiv p'_{\text{up}}(r,t)\exp(r/2h-\alpha r)=\int\limits_{-\infty}^{\infty}\widetilde{\widetilde{v}}'(k,t)F1(k)\theta(k)\exp(-ikr)\,dk+\text{c.c.},$$

$$\rho_+(r,t) \equiv \rho'_{\mathrm{up}}(r,t)\exp(r/2h-\alpha r) = \int_{-\infty}^{\infty} \widetilde{\widetilde{v}}'(k,t)F2(k)\theta(k)\exp(-ikr)\,dk + \text{c.c.}, \quad (2.5)$$

$$F1(k) = \frac{-i\rho_{00}\left[g(ik+1/2h-\alpha) - \left(k^2(gh-B)/A + g^2(A+1)^2/\left[4A(gh-B)\right]\right)\right]}{(ik+1/2h-\alpha)\sqrt{k^2(gh-B)/A + g^2(A+1)^2/\left[4A(gh-B)\right]}},$$

$$F2(k) = \frac{-i\rho_{00}(ik+1/2h-\alpha)}{\sqrt{k^2(gh-B)/A + g^2(A+1)^2/\left[4A(gh-B)\right]}},$$

where

$$\theta(k) = \begin{cases} 1, & k>0, \\ 0, & k \le 0. \end{cases}$$

Therefore, only the components corresponding to different signs of ω and k are left. The dispersion relation (2.3) and the connections (2.4) have been taken into account.

Just the same formulae apply to the leftwards component with $\theta(-k)$ instead of $\theta(k)$. At last, an inverse connection should be used:

$$\widetilde{\widetilde{v}}'(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v'(r,t)\exp(-r/2h-\alpha r)\exp(ikr)\,dk + \text{c.c.}.$$

After substitution this equation to (2.5), the final connections in the (r,t)-representation are obtained:

$$p_+ = L1 v_+\,, \quad \rho_+ = L2 v_+\,, \quad p_- = -L1 v_-\,, \quad \rho_- = -L2 v_-\,, \quad \rho_{\mathrm{stat}} = L3 p_{\mathrm{stat}}\,. \qquad (2.6)$$

For the stationary component a mass density or pressure should be designed since the velocity component equals to zero:

$$\widetilde{\rho}' = \frac{(ik+1/2h-\alpha)}{g}\widetilde{p}'(k).$$

The integrodifferential operators $L1$, $L2$, $L3$ have the form:

$$L3 = (1-2\alpha h - 2h\partial/\partial r)/(2gh),$$

$$L1 = \rho_{00}/(\pi\sqrt{(gh-B)/A})\int_{-\infty}^{\infty} dr'\left\{-\frac{gh-B}{A}F_{AB}(r-r')\partial/\partial r' + \frac{g(A-1)}{2A}F_{AB}(r'-r)\right\},$$

$$L2 = \rho_{00}/(\pi\sqrt{(gh-B)/A})\int_{-\infty}^{\infty} dr' F_{AB}(r-r')\left\{-\partial/\partial r' + \frac{g(A+1)}{2(gh-B)}\right\},$$

$$F_{AB}(r) = \int_0^{\infty} \frac{\sin(kr)dk}{\sqrt{k^2 + \left[\frac{g(A+1)}{2(gh-B)}\right]^2}} = \frac{\pi}{2}\left[I_0\left(r\frac{g(A+1)}{2(gh-B)}\right) - L_0\left(r\frac{g(A+1)}{2(gh-B)}\right)\right],$$

where I_0, L_0 — the modified Bessel function of zero order and the Struve function, respectively.

Then, using (2.6), from the first equation of (2.1) we also get the linear evolution equations for the upwards and downwards directed waves:

$$\partial v_\pm/\partial t \pm \sqrt{\frac{gh-B}{\pi^2 A}} \int\limits_{-\infty}^{\infty} \left\{ v_{\pm r'r'} - \frac{g^2(A+1)^2}{4(gh-B)^2} v_\pm \right\} F_{AB}(r-r')\, dr' = 0. \tag{2.7}$$

The evolution equations could be obtained for $p_\pm$ and $\rho_\pm$ from the second and third equations (2.1) and the relations (2.6); they look just the same. The acoustic field separation is mathematically unique and may be proceeded at any instant of the evolution.

2.2. Projectors

It seems more convenient to use variables $v'_s = v'(r,t)\exp(-r/2h - \alpha r)$, $p'_s = p'(r,t)$ $\exp(r/2h - \alpha r)$, and $\rho'_s = \rho'(r,t)\exp(r/2h - \alpha r)$ instead of v', p', and ρ'. Following the relations (2.6), a matrix $\mathbf{M}$ should be determined:

$$\begin{pmatrix} v'_s \\ p'_s \\ \rho'_s \end{pmatrix} \equiv \begin{pmatrix} v_+ + v_- \\ p_+ + p_- + p_{\mathrm{stat}} \\ \rho_+ + \rho_- + \rho_{\mathrm{stat}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} v_+ \\ v_- \\ \rho_{\mathrm{stat}} \end{pmatrix},$$

and also

$$\mathbf{M}^{-1} : \mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ L1 & -L1 & L3 \\ L2 & -L2 & 1 \end{pmatrix}, \qquad \mathbf{M}^{-1} = \begin{pmatrix} 1/2 & l1 & l2 \\ 1/2 & -l1 & -l2 \\ 0 & l3 & l4 \end{pmatrix},$$

from which the projectors follow immediately.

Now we present the new formulae taking into account the energy density decomposition (2.2):

$$\mathbf{P}_\pm = \begin{pmatrix} 1/2 & \pm l1 & \pm l2 \\ \pm L1/2 & L1l1 & L1l2 \\ \pm L2/2 & L2l1 & L2l2 \end{pmatrix}, \qquad \mathbf{P}_{\mathrm{stat}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l3 & l4 \\ 0 & L3l3 & L3l4 \end{pmatrix}, \tag{2.8}$$

where

$$\begin{aligned}
K_- &= \int\limits_{-\infty}^{r} dr' \exp\left[(r'-r)\frac{g(A+1)}{2(gh-B)}\right], \\
K_+ &= \int\limits_{r}^{\infty} dr' \exp\left[-(r'-r)\frac{g(A+1)}{2(gh-B)}\right], \\
l1 &= 1/\left(2\rho_{00}\pi\sqrt{(gh-B)/A}\right) \times \int\limits_{-\infty}^{\infty} dr' F_{AB}(r-r') \left\{-\partial/\partial r' + \frac{g(A+1)}{2(gh-B)}\right\},
\end{aligned}$$

$$\begin{aligned}
l2 &= -g/\left(2\rho_{00}\pi\sqrt{(gh-B)/A}\right) \times \int\limits_{-\infty}^{\infty} dr' F_{AB}(r-r'),\\
l3 &= \frac{Ag}{gh-B}K_-\,,\\
l4 &= \frac{g}{A+1}K_+ - \frac{gA}{A+1}K_-\,,\\
L1l1 &= \frac{1}{2} - \frac{Ag}{2(gh-B)}K_-\,,\\
L2l1 &= \frac{A}{2(gh-B)} - \frac{A(A+1)g}{2(gh-B)^2}K_-\,,\\
L1l2 &= -\frac{g}{2(A+1)}K_+ + \frac{gA}{2(A+1)}K_-\,,\\
L2l2 &= \frac{Ag}{2(gh-B)}K_-\,,\\
L3l3 &= -\frac{A}{gh-B} + \frac{Ag(A+1)}{2(gh-B)^2}K_-\,,\\
L3l4 &= 1 - \frac{Ag}{(gh-B)}K_-\,;
\end{aligned}$$

$L1$, $L2$, $L3$ have been determined earlier. For the inverse matrix $\mathbf{M}^{-1}$ and the projector element derivation, the equation

$$\int\limits_{-\infty}^{\infty} F_{AB}(r'-r)\left(\frac{\partial^2}{\partial r^2} - \left[\frac{g(A+1)}{2(gh-B)}\right]^2\right) F_{AB}(r-r'')\,dr = \pi^2\delta(r'-r'')$$

was used.

The operators (2.8) obtained have the ordinary properties of projectors:

$$P_- + P_+ + P_{\text{stat}} = \widetilde{\mathbf{I}}, \qquad P_-P_+ = P_-P_{\text{stat}} = ... = \widetilde{\mathbf{0}}, \qquad P_- = P_-P_-\,, ...,$$

where $\widetilde{\mathbf{I}}$ and $\widetilde{\mathbf{0}}$ are unit and zero matrices. For example, in order to obtain the upwards propagating component at any instant, it is sufficient to apply P_+ on the total field:

$$P_+\begin{pmatrix} v'(r,t)\exp(-r/2h-\alpha r)\\ p'(r,t)\exp(r/2h-\alpha r)\\ \rho'(r,t)\exp(r/2h-\alpha r)\end{pmatrix} \equiv P_+\begin{pmatrix} v'_s(r,t)\\ p'_s(r,t)\\ \rho'_s(r,t)\end{pmatrix} = \begin{pmatrix} v_+(r,t)\\ p_+(r,t)\\ \rho_+(r,t)\end{pmatrix}.$$

One can see that the operators depend only on the coefficients A, B. It follows from the linearized caloric equation of state. This is quite clear because the operators are obtained in fact from linear gasdynamic equations. Let us stress, that the stratified medium is strongly dispersive, but the projectors are got explicitly in the dispersive linear problem. So, one can calculate every disturbance evolution without any restriction of its extent.

2.3. Nonlinear evolution of directed waves

By letting P_-, P_+ or $P_{\rm stat}$ act on the complete system of nonlinear equations (2.1) we obtain new evolution equations which account for the interaction between modes which is a generalization of (2.7). It will be useful to write on the system (2.1), separated into the linear left-hand side and the nonlinear right-hand side involving the second-order terms, only in (v_s', p_s', ρ_s')-variables:

$$\begin{pmatrix} \dfrac{\partial v_s'}{\partial t} + \dfrac{(\partial/\partial r + \alpha - 1/2h)p_s'}{\rho_{00}} + \dfrac{g\rho_s'}{\rho_{00}} \\ \dfrac{\partial p_s'}{\partial t} + \rho_{00}\dfrac{(p_{00}/\rho_{00} - B)}{A}\dfrac{\partial v_s'}{\partial r} + \dfrac{\rho_{00}Bv_s'}{Ah} \\ \dfrac{\partial \rho_s'}{\partial t} + \rho_{00}(\partial/\partial r + \alpha - 1/2h)v_s' \end{pmatrix} = \exp(r(\alpha + 1/2h))$$

$$\times \begin{pmatrix} -v_s'\left(\dfrac{\partial}{\partial r} + \alpha + \dfrac{1}{2h}\right)v_s' + \dfrac{\rho_s'}{\rho_{00}^2}\left(\dfrac{\partial}{\partial r} + \alpha - \dfrac{1}{2h}\right)p_s' \\ v_s'\left[\left(-\dfrac{\partial}{\partial r} + \dfrac{1}{2h} - \alpha + \dfrac{\dfrac{2A1B}{Agh} - A - D}{Ah}\right)p_s' + \dfrac{\dfrac{DB}{A} - B - 2B1}{Ah}\rho_s'\right] \\ -\dfrac{p_s'\left(\dfrac{2A1c^2}{gh} + D - 1\right) + \rho_s'(B + 2B1 + c^2D + gh)}{A}\left(\dfrac{\partial}{\partial r} + \dfrac{1}{2h} + \alpha\right)v_s' \\ -v_s'\left(\dfrac{\partial}{\partial r} + \alpha\right)\rho_s' - \rho_s'\left(\dfrac{\partial}{\partial r} + \alpha\right)v_s' \end{pmatrix}. \tag{2.9}$$

Then, the projector (P_- or P_+) acting on both sides of (2.9) with the non-zero right side results in the non-zero nonlinear right-hand side of the evolution equation (2.7) for the corresponding directed mode. The evolution nonlinear equations for the upwards and downwards velocity are as follows:

$$\partial v_\pm/\partial t \pm \sqrt{\frac{gh - B}{\pi^2 A}} \int_{-\infty}^{\infty} \left\{ v_{\pm r'r'} - \frac{g^2(A+1)^2}{4(gh-B)^2} v_\pm \right\} F_{AB}(r - r')\, dr$$
$$= (l/2 \quad \pm l1 \quad \pm l2)\exp(r(\alpha + 1/2h))$$

$$\times\begin{pmatrix} -v_s'\left(\dfrac{\partial}{\partial r}+\alpha+\dfrac{1}{2h}\right)v_s'+\dfrac{\rho_s'}{\rho_{00}^2}\left(\dfrac{\partial}{\partial r}+\alpha-\dfrac{1}{2h}\right)p_s' \\ v_s'\left[\left(\dfrac{\partial}{\partial r}+\dfrac{1}{2h}-\alpha+\dfrac{\dfrac{2A1B}{Agh}-A-D}{Ah}\right)p_s'+\dfrac{\dfrac{DB}{A}-B-2B1}{Ah}\rho_s'\right] \\ -\dfrac{p_s'\left(\dfrac{2A1c^2}{gh}+D-1\right)+\rho_s'(B+2B1+c^2D+gh)}{A}\left(\dfrac{\partial}{\partial r}+\dfrac{1}{2h}+\alpha\right)v_s' \\ -v_s'\left(\dfrac{\partial}{\partial r}+\alpha\right)\rho_s'-\rho_s'\left(\dfrac{\partial}{\partial r}+\alpha\right)v_s' \end{pmatrix}. \tag{2.10}$$

Similar equations can be derived for the pressure and density. The nonlinear right-hand side can be further simplified, for example, by ignoring the upwards propagating and the stationary inputs of the nonlinear evolution of the "downwards directed" disturbances [7, 8]. That corresponds to the case of self-interaction only. The numeric calculations have been performed and proved that upwards and stationary modes caused by "downwards" initial disturbances are of a considerably smaller amplitude than that of the initial one. Thus, it is reasonable to use only the self-acting right-hand side for the approximate investigation of the concrete excited initially mode evolution.

But if the disturbance includes some modes of nearly equal amplitudes, one should take into account all terms of the right-hand nonlinear side. In this way the reciprocal influence of the generated modes on the initial one can be studied as well. Generally, the nonlinear evolution equations are coupled, because the nonlinear vector includes all inputs of the modes. There are three coupled nonlinear equations which could be written in any basic variables: ρ_+, ρ_-, ρ_{stat}, for example, p_+, p_-, ρ_{stat} or v_+, v_-, ρ_{stat}.

3. Homogeneous media

3.1. Dispersion relation and connection equations

In the case of a homogeneous medium (liquid or gas), the formulae become quite simple. At first, the background pressure and density are constants now independent on r. The corresponding Fourier-components of the perturbations are presented as $\widetilde{v}'(k)\exp(i(\omega t-kr))$, $\widetilde{p}'(k)\exp(i(\omega t-kr))$, and $\widetilde{\rho}'(k)\exp(i(\omega t-kr))$; the background pressure and density are equal to constants: $p_0=p_{00}$, $\rho_0=\rho_{00}$. Similarly, $\rho_\pm=\pm L2v_\pm$ and $p_\pm=\pm L1v_\pm$. The dispersion relation has the form: $\omega^2=k^2((p_0/\rho_0)-B)/A$ or $\omega=0$. There is no dispersion now, the group and phase velocities are equal to $c=(p_0/\rho_0-B)/A)^{1/2}$ respectively. Expressions for the corresponding operators can be got directly from the previous formulae by going to the limits: $h\to\infty$, $g\to 0$, $gh\to p_0/\rho_0$, $\alpha\to 0$ which leads to $L1=\rho_0 c$, $L2=\rho_0/c$ and $l1=1/2\rho_0 c$, $l2=0$. A new connection of the stationary components takes place: $p_{\text{stat}}=L\rho_{\text{stat}}=0$, thus $L=0$. The operator chosen in this way is inverse to $L3$. In the case of the homogeneous medium, it seems natural to consider the directed waves as propagating leftwards and rightwards.

Therefore, there exist only two pressure components in a homogeneous medium, the leftwards component and rightwards one, in contrast to the stratified medium. The three density components exist in both cases. One could also get the primitive linear evolutionary equations for the directed waves:

$$\partial v_\pm/\partial t \pm c\partial v_\pm/\partial r = 0.$$

Finally, $v_\pm$, $p_\pm$, $\rho_\pm$, ρ_{stat} are corresponding parts of the usual variables.

3.2. Projectors and nonlinear evolution of directed waves

Their elements are not integrodifferential operators, but numbers:

$$P_\pm = \begin{pmatrix} 1/2 & \pm 1/(2L1) & 0 \\ \pm L1/2 & 1/2 & 0 \\ \pm L2/2 & L2/(2L1) & 0 \end{pmatrix}, \qquad P_{\text{stat}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -L2/L1 & 1 \end{pmatrix}.$$

The common properties of orthogonal operators are retained, too. Let us put in this case an analogue of (2.9):

$$\begin{pmatrix} \dfrac{\partial v'}{\partial t} + \dfrac{\partial p'/\partial r}{\rho_0} \\ \dfrac{\partial p'}{\partial t} + \rho_0 c^2 \dfrac{\partial v'}{\partial r} \\ \dfrac{\partial \rho'}{\partial t} + \rho_0 \dfrac{\partial v'}{\partial r} \end{pmatrix}$$

$$= \begin{pmatrix} -v'\dfrac{\partial}{\partial r}v' + \dfrac{\rho'}{\rho_0^2}\dfrac{\partial}{\partial r}p' \\ -v'\dfrac{\partial}{\partial r}p' + \dfrac{\partial v'}{\partial r}\,\dfrac{\left[p'(-1+D+2A1c^2\rho_0/p_0) + \rho'(p_0/\rho_0 + Dc^2 + 2B1 + B)\right]}{A} \\ -v'\dfrac{\partial}{\partial r}\rho' - \rho'\dfrac{\partial}{\partial r}v' \end{pmatrix}.$$

The evolution nonlinear equation for the upwards and downwards velocity is:

$$\partial v_\pm/\partial t \pm c\partial v_\pm/\partial r = \begin{pmatrix} \dfrac{1}{2} & \pm\dfrac{1}{2\rho_0 c} & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} -v'\dfrac{\partial}{\partial r}v' + \dfrac{\rho'}{\rho_0^2}\dfrac{\partial}{\partial r}p' \\ -v'\dfrac{\partial}{\partial r}p' + \dfrac{\partial v'}{\partial r}\,\dfrac{\left[p'(-1+D+2A1c^2\rho_0/p_0) + \rho'(p_0/\rho_0 + Dc^2 + 2B1 + B)\right]}{A} \\ -v'\dfrac{\partial}{\partial r}\rho' - \rho'\dfrac{\partial}{\partial r}v' \end{pmatrix}.$$

4. Applications of the theory

4.1. Application of the theory to gas and liquid dynamics

Since the equation of state is taken in the general form depending on the coefficients A, B, $A1$, $B1$, D of (2.2), the theory may be applied to both gases and liquids with an arbitrary caloric equation of state. Besides of these coefficients, the background values of pressure and mass density are needed. The right-hand nonlinear vector may be completed by viscous and dispersive terms of the order of nonlinear ones. Since the viscous term depends on temperature, a thermal equation of state (that decomposition looks like (2.2)) $T(p,\rho)$ should be involved in the basic system.

4.2. Homogeneous and exponencially stratified atmosphere

For an ideal atmospheric gas the problem becomes quite simple. In this case, $\varepsilon = p/(\rho(\gamma-1))$, $A = 1/(\gamma-1)$, $B = -gh/(\gamma-1)$, $A1 = 0$, $B1 = -B$, $D = -A$, $\alpha = 0$, and the same can be applied in the stratified and homogeneous cases putting p_0/ρ_0 instead of gh. Here $\gamma = C_p/C_v$ is the ratio of specific heats.

Then, the evolution equations, by accounting only self-interaction, is:

$$\partial v_\pm/\partial t \pm c\partial v_\pm/\partial r + \frac{\gamma+1}{2} v_\pm \partial v_\pm/\partial r = 0. \tag{4.1}$$

The evolution equation for the stratified atmosphere has a somewhat complicated form which can be obtained directly from (2.10) and the connecting equations (2.6).

The same can be obtained by the well-known method of slowly changing variables [9, 10]. An analytical solution of (4.1) is well-known, too. The proposed method, in contrast to the method of slowly changing variable, where only one directed mode is traced, allows to get coupled nonlinear equations. The upwards and downwards propagation of some types of initial disturbances are discussed in [5, 6, 8]. In this case, $\alpha = 0$, so that there is now an additional amplitude growth (or decrease). Both upwards and downwards-propagating modes retain their properties even for essentially large initial perturbation amplitudes, such as the velocity up to 150 m/s. The initial conditions were constructed using the linear equations and the numerical calculations were carried out by means of a nonlinear Lagrange finite-difference scheme [4–6].

4.3. Illustrations

Numerical simulations of equations (2.10) and (4.1) were carried out. Some peculiarities of the wave propagation in a stratified atmosphere caused by dispersion properties of medium and exponential dependence of nonlinear input on r are expected. We adopted the values $h = 1.033 \cdot 10^3$ m, $g = 9.8066\,\mathrm{m/s^2}$, and $\gamma = 1.4$, $A = 2.5$, $B = -2.533 \cdot 10^5$ J/kg.

Figure 1 presents the nonlinear evolution of a saw initial disturbance of velocity in the stratified and homogeneous atmosphere models. The influence of dispersion is clear: it widens the disturbance. For initial disturbances enough narrow, even additional

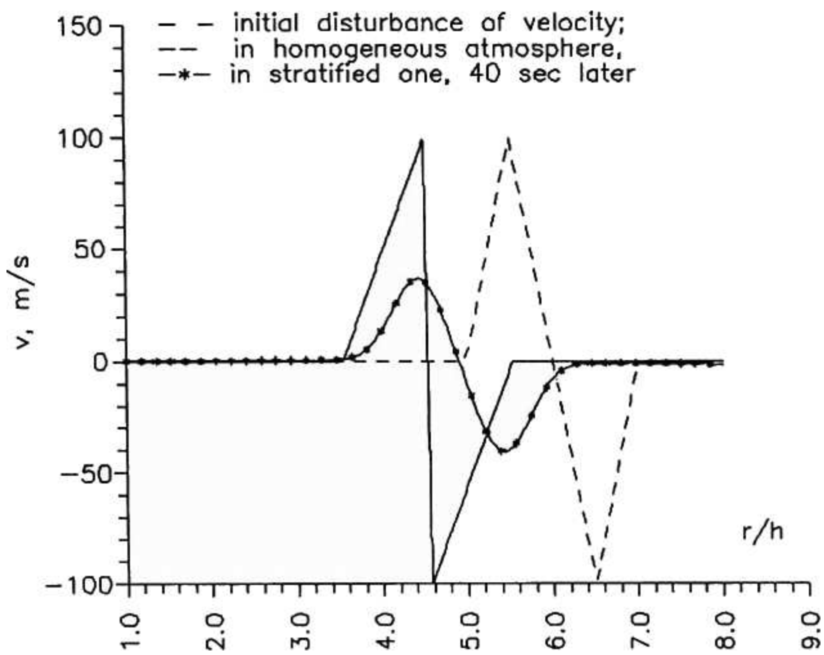

Fig. 1. Evolution of the initial disturbance of the velocity for the downwards directed wave in a stratified and a homogeneous gas.

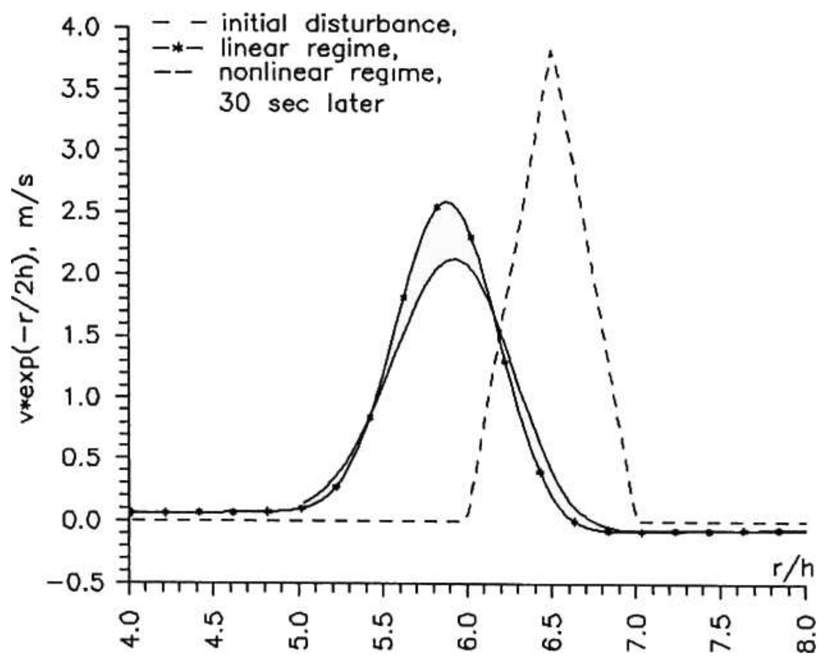

Fig. 2. Evolution of the initial disturbance of the velocity for the downwards directed wave in the linear and weak nonlinear regimes.

"hills" following the propagating disturbance appear. Evolution (4.1) of the disturbance in a homogeneous atmosphere gas is described by the known expressions [9, 10]. The amplitude of the wave in a stratified atmosphere seems to be much smaller then in a homogeneous one. The formulae for the directed wave were obtained for the variable $v \cdot \exp(-r/2h - \alpha r)$ instead of v in the homogeneous case (for an ideal gas $\alpha = 0$). The figure plotted for this variable shows that the propagating mode is more close to the initial one.

Figure 2 presents the evolution of the downwards directed wave in the stratified atmosphere with the account of the nonlinearity and without it. The initial amplitude of the velocity is about 100 m/s. Weak nonlinearity influences the wave shape in the same way as in the homogeneous atmosphere: points of larger amplitude move with a relatively smaller speed (for positive disturbances). This is in agreement with (4.1).

5. Conclusions

The projectors are written in a general form depending on the equation of state only both for the stratified and homogeneous models of a gas or liquid. The possibilities of the method are wide: nonlinear self-action or mutual interaction of the directed waves, generation of a mean field by the directed wave and so on. Generally, the coupled evolution equations obtained are suitable for the boundary regime problems as well as for the problems with initial conditions.

Thus, the projectors serve as an universal tool for the investigation of nonlinear evolution both in homogeneous and stratified gases and liquids. Moreover, the formulae for the projectors may be improved by applying of the perturbation theory.

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