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DETERMINATION OF THE LOCATION OF SONIC BOOM

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In the paper the results of theoretical considerations which permit to determine the location of sonic boom of shock wave produced during supersonic flight are given. It has been assumed that the earth's atmosphere is a stratified medium in which the velocity of sound propagation as well as wind velocity are magnitudes which depend only on the height. For general case (at arbitrary manoeuvring of aircraft) a procedure algorithm has been defined which permits to predict the boundary of audibility area. It has also been proved that for certain types of manoeuvring (including also rectilinear flight), defined by condition (9), this algorithm enables the determination of coordinates of audibility area (grazing points) (x_A, y_A) as well as (x_B, y_B) in the analytical form (10). In the case of still atmosphere the obtained results are simplified to a well-known relation (13) determining the width of audibility belt of a shock wave.

1. Introduction

A plane flying at a supersonic velocity speed is a source of a sound shock wave. This wave has bad effects upon the people, biological life, buildings as well as upon the equipment on the earth's surface. Having this in mind, experiments have for long been conducted aimed at establishing the method of the determination of the intensivity and audibility area of this wave. Both problems can be looked upon jointly and then the audibility area of the shock wave constitutes a set of singular points encountered in equations describing the shock strength. Relevant calculations are very complex.

In many practical problems (regional planning) the problem of the audibility area is very important, e.g. in the case where, one has to do with areas (hospitals, sanatoria etc.) to which even the slightest shock wave should not reach.

Many authors have been engaged in the problem of location of sonic boom at supersonic flights, among others Barlett and Friedmann [1], Hayes and Runyan [6], Lansing [9], Wanner [12] as well as Warren [13]. In said papers, as also in this one, theoretical deliberations are based on the perception that the boundary of the audibility area is a set of tangential points to earth's surface (Fig. 2). For a general case (arbitrary manoeuvring aircraft in presence of wind) the above-mentioned papers contain the algorithm which can be evaluated by use of quick operating computers. It has been shown in this paper that the satisfaction of a certain condition that limits somewhat the freedom of the manoeuvre (9) leads to simple relations (10) which determine the coordinates of tangential points (x_A, y_A) as well as (x_B, y_B) (Fig. 2).

2. Equations of acoustic rays

In direct proximity of a plane the shock wave forms the Mach's cone. It can be thought that each point of the cone's surface is the end of the ray emitted from the cone's apex at an earlier moment (Fig. 1). The set of all rays

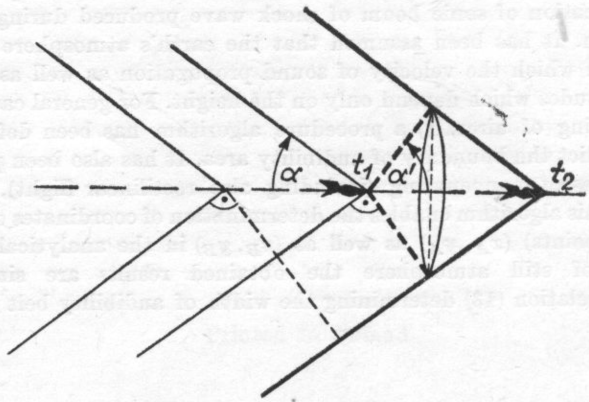

Fig. 1. Mach cone and coupled cone with obtuse angle α'

emitted at the moment t_1, the ends of which are on the surface of the cone at the moment t_2 (where $t_2 > t_1$) forms a coupled cone with obtuse angle

$$\alpha' = \frac{1}{2}\pi - \arcsin\frac{a(h)}{V},$$

whose generating lines perpendicular to Mach's cone. These generating lines are radii. Since the atmosphere is heterogeneous and anisotropic (presence of wind), these lines are curves. Some of them intersect the earth's surface and can then be heard at these points as shock waves.

The points of intersection (dashed lines on the plane x, y — Fig.2) are the axis of the influence zone of the shock wave at a certain moment. The purpose of our deliberations is to find the boundary of the audibility area not at one given moment, but over a length of time during which the influence zone moves forming a belt as shown in Fig. 2.

For the «coupled cone» there exist two radii tangent to the earth's surface φ_A and φ_B which limit the set of radii intersecting the earth's surface. In this manner the tangential points changing continuously in time form the boundary of the audibility area supersonic flight.

The earth's atmosphere in the case under consideration can be looked upon as a stratified medium [6], that is, it can be assumed that the velocity of sound propagation $a(z)$ as well as component velocities of wind W_x, W_y, W_z

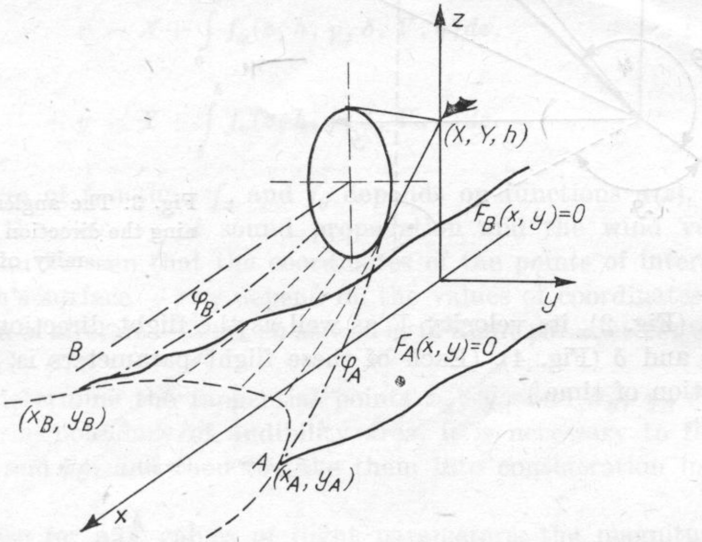

Fig. 2. Sets of the points of tangency of radii φ_A and φ_B with the earth's surface. Boundaries of audibility area: $F_A(x, y) = 0$, $F_B(x, y) = 0$

are functions of these heights. These parameters are decisive for the course of acoustic radii. If we assume that vertical component of wind $W_a = 0$, then the differential equation for radii takes the form [10]

$$\frac{dx}{dz} = -\cos\vartheta \frac{Ca^2 + W_x[1 - C(W_x + \tan\vartheta W_y)]}{a\sqrt{\cos^2\vartheta[1 - C(W_x + \tan\vartheta W_y)]^2 - C^2a^2}},$$
$$\frac{dy}{dz} = -\cos\vartheta \frac{Ca^2\tan\vartheta + W_y[1 - C(W_x + \tan\vartheta W_y)]}{a\sqrt{\cos^2\vartheta[1 - C(W_x + \tan\vartheta W_y)]^2 - C^2a^2}}, \tag{1}$$

where

$$C = \frac{\cos\theta}{a + \cos\theta(W_x + \tan\vartheta W_y)} \quad \text{for } z = h; \tag{2}$$

the angles ϑ and θ (Fig. 3) determine the direction of a radius in the proximity of the plane relative the coordinate system related to the earth.

The functions $W_x(z)$ and $W_y(z)$ can be determined with the aid of wind measurement at various heights. In a similar manner we determine the velocity of sound propagation $a(z)$.

The position of tangential points of radii φ_A and φ_B relative to earth's surface depends not only on the atmospheric conditions which were considered in equations (1) via the function a, W_x and W_y, but also on the aircraft position

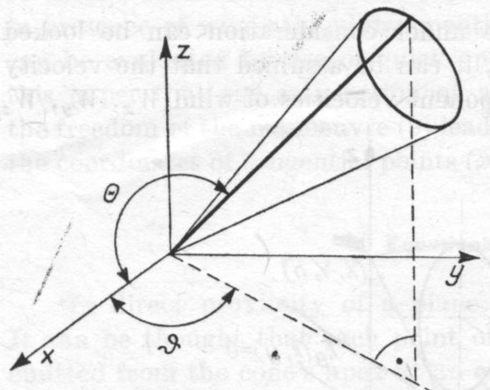

Fig. 3. The angles ϑ and θ determining the direction of ray in the proximity of aircraft

X, Y, h, (Fig. 2), its velocity V as well as the flight direction defined by the angles γ and δ (Fig. 4). (Each of these flight parameters is, in general case, the function of time.)

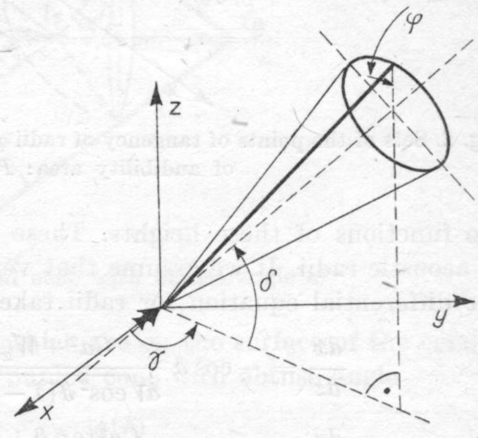

Fig. 4. The angles γ and δ determining the direction of flight

The dependence of the audibility area boundary on the flight parameters is obtained by the dependencies between the angles ϑ and θ (Fig. 3) as well as on the obtuse angle of the coupled cone α', the direction of its axis defined by the angles γ, δ and the position of the radius on the cone side. From trigonometric calculations we get

$$\vartheta = \gamma + \arctan\left\{\cot\alpha \frac{\sin\varphi}{\cos\delta}\right\}, \tag{3}$$

$$\cos\theta = \cos\gamma[\cos\delta\sin\alpha - \sin\delta\cos\alpha\cos\varphi] - \sin\gamma\cos\alpha\sin\varphi,$$

where

$$\alpha = \arcsin \frac{a(h)}{V}.$$

If we denote correspondingly the right members of equations (1) by f_x and f_y, then after substituting the relations (2) and (3) and integrating both members with respect to z, these equations take the form

$$x = X + \int_0^h f_x(z, h, \gamma, \delta, V, \varphi)\,dz,$$
$$y = Y + \int_0^h f_y(z, h, \gamma, \delta, V, \varphi)\,dz. \tag{1'}$$

(The apparent form of functions f_x and f_y depends on functions $a(z)$, $W_x(z)$, $W_y(z)$ describing the velocity of sound propagation and the wind velocity, respectively). It can be seen that the coordinates of the points of intersection of radii with earth's surface $-x, y$ depend on the values of coordinates determining the position of aircraft $-X, Y, h$ as well as of flight parameters: velocity $-V$ and direction $-\gamma, \delta$.

In order to determine the tangential points x_A, y_A and x_B, y_B (Fig. 2) which constitute the boundary of audibility area, it is necessary to find the value of angles φ_A and φ_B, and then to take them into consideration in equations (1′).

In general case for any values of flight parameters, the magnitudes φ_A and φ_B cannot be described by an analytical formula, but it is only possible to point to a procedure which enables to find numerical values φ_A and φ_B (this case will be referred to as «general case»). It requires the use of computers.

The problem is considerably simplified when we make an assumption limiting the intervals of variability of flight parameters. Then the magnitudes φ_A and φ_B, thus also the coordinates of the tangential points, can be described by means of simple dependencies (the case will be referred to as «particular case»).

3. Determination of audibility area — general case

The tangency of a ray with the earth's surface means that the relations

$$\frac{dz}{dx} = 0, \qquad \frac{dz}{dy} = 0 \quad \text{for } z = 0$$

are satisfied.

From equations (1) it can be seen that this implies the equality

$$\cos\vartheta\left[1 - C\left(W_x(0) + \tan\vartheta\, W_y(0)\right)\right] = Ca_0. \tag{4}$$

Taking into account (2) we have

$$\cos\vartheta = \frac{a_0}{a(h)}\cos\theta\left[1-\sin\vartheta\,\frac{W_x(h)-W_x(0)}{a_0}-\cos\vartheta\,\frac{W_y(h)-W_y(0)}{a_0}\right], \quad (5)$$

where a_0, $W_x(0)$, $W_y(0)$; $a(h)$, $W_x(h)$, $W_y(h)$ denote velocities of sound propagation and component wind velocities at the earth's surface $z = 0$ and at the height $z = h$, respectively. The angles ϑ and θ are described by equations (3).

Expression (5) is a transcedental equation with respect to φ whose solution are the values of angles φ_A and φ_B for which the rays are tangent to the earth's surface. This solution can be obtained with the aid of a computer. By substituting φ_A and φ_B in equations (1′) we get the coordinates of tangential points $x_A(t)$, $y_A(t)$ as well as $x_B(t)$, $y_B(t)$ (Fig. 2) as a time function. By the elimination of the parameter t we get the boundary of the audibility area of sound shock wave in the form of equations $F_A(x, y) = 0$ and $F(x, y) = 0$.

4. Determination of audibility area — particular case

The above-presented algorithm of the determination of the audibility area is rather complex. It can be seen that if the inequality

$$C\,|W_x+\tan\vartheta\cdot W_y| \ll 1 \quad \text{for } 0 \leqslant z \leqslant h \quad (6)$$

is satisfied, then this algorithm leads to comparatively simple analytical expressions. (The physical interpretation of this inequality will be given.)

Inequality (6) simplifies expression (2) to the form

$$C = \frac{\cos\theta}{a(h)}.$$

In view of (3) we get

$$C = \frac{1}{a(h)}\{\cos\gamma\,(\cos\delta\sin\alpha-\sin\delta\cos\alpha\cos\varphi)-\sin\gamma\cos\alpha\cos\varphi\}. \quad (7)$$

Equation (4) and inequality (6) imply that $\cos\vartheta = Ca_0$. From (3) and (7) we obtain the following equation with respect to φ:

$$\cos\left\{\gamma+\arctan\frac{\sin\varphi\cot\alpha}{\cos\delta}\right\} = \frac{a_0}{a(h)}\left[\cos\gamma(\cos\delta\sin\alpha-\sin\delta\cos\alpha\cos\varphi)-\sin\gamma\cos\alpha\sin\varphi\right]$$

The identical relation is obtained from (5) under the assumption that $W_x = 0$, $W_y = 0$. This means that in this case the value of the angle φ for which the radii are tangent to the earth's surface do not depend on the γ. Hence it can be concluded that said equations can be reduced to the form

$$\cos\left\{\gamma+\arctan\frac{\sin\varphi\cot\alpha}{\cos\delta}\right\}=\cos\{\gamma+\psi\},$$

where

$$\cos\psi=\pm\frac{a_0}{a(h)}(\cos\delta\sin\alpha-\sin\delta\cos\alpha\cos\varphi),$$

$$\sin\psi=\pm\frac{a_0}{a(h)}\cos\alpha\sin\varphi;$$

from the identity $\cos^2\psi+\sin^2\psi=1$ we get

$$\cos\varphi=-\tan\delta\tan\alpha-\frac{\sqrt{1-a^2(h)/a_0^2}}{\cos\alpha\cos\delta}.$$

Substituting $\cos\varphi$ as well as $\sin\varphi=\pm\sqrt{1-\cos^2\varphi}$ into (7) we get the constant values C_A and C_B, from the relation $\cos\vartheta=Ca_0$ — the constants ϑ_A and ϑ_B. Next, to abbreviate the notation, we shall use the symbols $C_{A,B}$ and this will mean that we have to do either with the constant C_A or with the constant C_B (the same applies to the constants ϑ_A and ϑ_B as well as to the coordinates of the points of tangency x_A, x_B, y_A, y_B).

Equations (1′) can be re-written in the following form:

$$x_{A,B}=X+\int_0^h\frac{C_{A,B}a^2+W_x}{a\sqrt{1-a^2/a_0^2}}dz,$$

$$y_{A,B}=Y+\int_0^h\frac{\pm(a^2/a_0)\sqrt{1-C_{A,B}^2a_0^2}+W_y}{a\sqrt{1-a^2/a_0^2}}dz. \tag{8}$$

Let us return to inequality (6) which is the starting point for the method in question. If we substitute (2) in this inequality, we get

$$\cos\theta\left\{\frac{W_x}{a}+\left(\frac{W_x}{a}\right)_h+\tan\vartheta\left[\frac{W_y}{a}+\left(\frac{W_y}{a}\right)_h\right]\right\}\ll 1.$$

Since the wind velocity is, in general considerably smaller than the velocity of sound propagation, this inequality will be satisfied if $|\vartheta|<\frac{1}{2}\pi$, which means that the following relation for the parameters γ, δ, V

$$\left|\gamma+\arctan\left\{\frac{\sqrt{V^2-a^2(h)}}{a(h)\cos\delta}\right\}\right|<\frac{1}{2}\pi \tag{9}$$

holds.

This inequality is satisfied, when the aircraft flight is very similar to the rectilinear flight, i.e. when turns in the horizontal plane — the angle γ, as well as in the vertical plane — the angle δ (Fig. 4), are not too acute. This means that in the case of ascent (upward flight) for which we have $\delta \to \frac{1}{2}\pi$ or of turn $\gamma \to \frac{1}{2}\pi$ we cannot use equations (8). On the other hand, if we assume that we have to do wit passenger aircraft, then from the viewpoint of the flying comfort it is necessary to dispense with such manoeuvres. Then, relations derived in this chapter can be utilized for planned flight channels or can be used as one of the prerequisites for country's regional planning under the assumption that flight channels are determined beforehand.

On the basis of results presented in [4] it can be assumed that the velocity of sound propagation is the linear function of altitude

$$a = a_0(1-\beta z) \quad \text{for } z < 12\,\text{km},$$

while the components of wind velocity can take the form of multinominals

$$W_x = \sum_{k=0}^{n} a_k^{(x)} z^k, \quad W_y = \sum_{k=0}^{n} a_k^{(y)} z^k.$$

Substituting these magnitudes into equations (8) we get

$$x_{A,B} = X + C_{A,B} a_0 \sqrt{\frac{h(2-\beta h)}{\beta}} + \frac{1}{a_0\beta} \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{a_k^{(x)}}{\beta^k} \binom{k}{m} (-1)^m \int_1^{1-\beta h} \frac{\xi^m d\xi}{\sqrt{1-\xi^2}},$$

$$y_{A,B} = Y \pm \sqrt{1-C_{A,B}^2 a_0^2} \sqrt{\frac{h(2-\beta h)}{\beta}} + \frac{1}{a_0\beta} \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{a_k^{(x)}}{\beta^k} \binom{k}{m} (-1)^m \int_1^{1-\beta h} \frac{\xi^m d\xi}{\sqrt{1-\xi^2}}. \tag{10}$$

If the weather is windless ($W_x = 0$, $W_y = 0$), then the coordinates of points which constitute the boundary of the audibility area of supersonic flight $x_A y_A$, $x_B y_B$, considering the mode of notation $x_{A,B}$, $y_{A,B}$, $C_{A,B}$ as explained above, take the form

$$x_{A,B} = X + C_{A,B} a_0 \sqrt{\frac{h(2-\beta h)}{\beta}}, \quad y_{A,B} = Y \pm \sqrt{1-C_{A,B}^2 a_0^2} \sqrt{\frac{h(2-\beta h)}{\beta}}, \tag{11}$$

where X, Y, h denote the coordinates that determine the position of aircraft (Fig. 2), $C_{A,B}$ — the constant defined by (2), a_0 — the velocity of sound propagation at the earth's surface, β — the constant determining the changes of velocity of the sound propagation in the atmosphere.

5. Rectilinear flight

During the rectilinear flight on the constant height h, i.e. with $\delta = 0$, it is a good practice to choose the coordinate system so that $\gamma = 0$. In this case, in agreement with (7), we have $C = 1/V$.

Furthermore, if we assume that $W_x = 0$ and $W_y = 0$, then from equations (11) we get

$$x_{A,B} = \int V(t)\,dt + \frac{a_0}{V(t)}\sqrt{\frac{h(2-\beta h)}{\beta}}, \quad y_{A,B} = \pm\sqrt{1-\frac{a_0^2}{V^2(t)}}\sqrt{\frac{h(2-\beta h)}{\beta}}. \tag{12}$$

Hence, it can be seen that the audibility area of a supersonic flight is a belt with the axis of symmetry X (Fig. 2).

For the constant air speed we obtain the well-known formula for the width of this belt:

$$d = 2\sqrt{1-\frac{a_0^2}{V_0^2}}\sqrt{\frac{h(2-\beta h)}{\beta}}. \tag{13}$$

In paper [11] the results of numerical calculations of d are given.

From the above formulae it results that $d = 0$ if $V(t) = a_0$. This means that the supersonic flight is inaudible when air speed is smaller than the velocity of sound propagation at the earth's surface.

Let us assume that the initial flight stage is effected at a speed $V = a_0 + m_1 t$. Substituting $V(t)$ into (12) and eliminating t we get the equation that corresponds to the broken curves shown in Fig. 5:

$$x = \frac{a_0^2}{m_1}\left[\frac{1}{\sqrt{1-y^2/x^2}}-1\right] + \frac{a_0^2}{2m_1}\left[\frac{1}{\sqrt{1-y^2/x^2}}-1\right]^2 + \sqrt{x_0^2+y^2},$$

$$x_0 = \sqrt{\frac{h(2-\beta h)}{\beta}}.$$

It can be proved that the course of those curves depends on the satisfaction of the relation

$$M_1 \gtrless \frac{a_0^2}{x_0}.$$

Since the boundary of the audibility area of the flight is a set of the tangential points of radii to the earth's surface, which simultaneously confine the beam of rays that intersect the earth's surface. This implies that with accelerations $m_1 < a_0^2/x_0$ the shock wave will two times reach the shaded area in Fig. 5. This phenomenon of «double boom», caused by supersonic flight, is observed in practice [2, 5, 12].

If the transition to subsonic speed is effected with uniformly retarded motion $V(t) = V^* - m_2 t$, then the boundary of the audibility area is a curve (continuous line in Fig. 5):

$$x = x_0 + \frac{V^*}{m_2}\left[V^* - \frac{a_0}{\sqrt{1-y^2/x_0^2}}\right] - \frac{1}{2m_2}\left[V^* - \frac{a_0}{\sqrt{1-y^2/x^2}}\right]^2 + x_0\sqrt{1-\frac{y^2}{x_0^2}}.$$

6. Conclusions

The boundary of the audibility area of the shock wave caused by supersonic flight can – in general case (when the aircraft makes arbitrary manoeuvres) – be determined by means of the algorithm described in Section 3.

If the flight is similar to rectilinear one, then this area can be found with the aid of relations derived in Section 4.

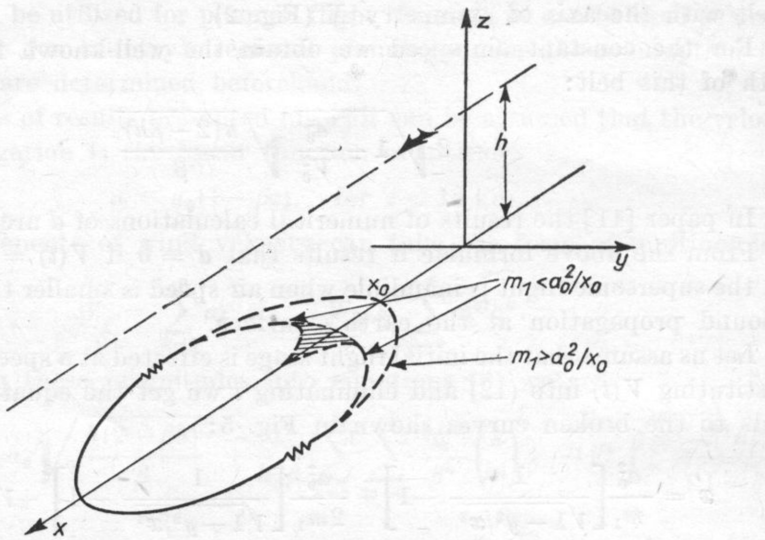

Fig. 5. The boundary of audibility area during the flight at a speed $V = a_0 + m_1 t$ (broken line) and a speed $V = V^* - m_2 t$ (continuous line).
Shaded area – the place of the occurrence of «double boom»

The reason and the mechanism of the so-called «double boom» are explained and the area of its occurrence is determined (Section 5).

The results presented can be helpful in regional planning to scale of macroregion and of the country.

References

[1] C. Barlett, M. Friedmann, *A method for calculating the effect of aircraft manoeuvres on sonic boom*, Jour. of Aircraft, 2, 5 (1965).

[2] A. Biestek, *Analysis and methods of the determination and propagation of sonic boom*, doctor's thesis [in Polish], Department of Mathem., Phys., Chem. of A. Mickiewicz University, Poznań 1973.

[3] J. Edwards, *Lateral extents of sonic boom carpets in real atmospheres*, Roya Aircraft Establishment, Tech. Mem. Aero., 1445, 1972.

[4] G. Groves, *The structure of the atmosphere up to 150 kilometre*, Contemp. Phys. 14, 1 (1973).

[5] G. Haglund, E. Kane, *Flight test measurements and analysis of sonic boom phenomena near the shock wave extremity*, NASA CR-216, 1973.

[6] W. HAYES, H. RUNYAN, *Sonic-boom propagation through a stratified atmosphere*, JASA, 51, 2 (1972).

[7] D. HILTON, H. HENDERSON, *Measurements of sonic-boom overpressures from Apollo space vehicles*, JASA, **56**, 2 (1974).

[8] D. MAGLIERI, *Experiments on the effects of atmospheric refraction and airplane accelerations on sonic-boom groundpressure patterns*, NASA TN-3520,1966.

[9] D. LANSING, *Application of acoustic theory to prediction of sonic boom ground patterns from manoeuvering aircraft*, NASA TN D-1860, 1964.

[10] R. MAKAREWICZ, *Differential equations of rays in moving non-homogeneous medium* [in Polish], Archiwum Akustyki, **11**, 1 (1976).

[11] R. MAKAREWICZ, *Ergonomic analysis of manoeuvers leading to inaudibility of shock wave* [in Polish], Materials from the conference «Ergonomy in Aviation», Warszawa 1975.

[12] I. WANNER, *Bang Sonique. Théorie et expérimentation du phénomène de focalisation*, Association Technique Maritime et Aerotique, Session 1969.

[13] C. WARREN, *The propagation of sonic bangs in a non-homogeneous still atmosphere*, 4-th Congress Aeronautical Sciences, London 1965.

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