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THE LOCATION OF THE GROUND FOCUS LINE PRODUCED BY A SUPERSONIC AIRCRAFT

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An aircraft flying at a speed higher than that of the propagation of sound produces a supersonic boom, and it has been observed that certain changes in the flight velocity or direction are accompanied by focussing the boom. In this paper procedural algorithms are presented which permit determination of the points on the Earth's surface at which this phenomenon occurs, for any manoeuvre (in section 2) and for accelerated rectilinear flight (in section 3).

1. Introduction

An aircraft flying at a supersonic velocity is the source of an acoustic disturbance called a *sonic boom* or *N wave* (Fig. 1).

This wave can be characterized by two parameters: the shock wave pressure jump Δp and the time Δt of its rising. It has been noted (e.g. WANNER [6]) that under certain conditions the value of Δp is very large (of

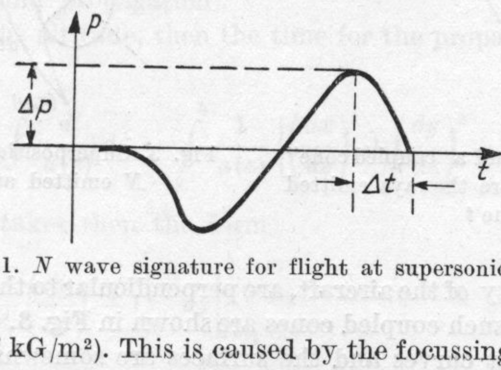

Fig. 1. N wave signature for flight at supersonic speed

the order of tens of kG/m²). This is caused by the focussing effected by a superposition of sonic booms. The rapid change of sign of Δp (from an excess pressure to a negative pressure — Fig. 1) is liable to cause damage to building-structures and living organisms (including the failure of internal organs).

The problem of localization of the focussing is of importance since although focussing is a hazard for the human environment, it is unavoidable if the benefits of supersonic aircraft are to be exploited. The primary cause of focussing is usually an aircraft manoeuvre, i.e. a change of velocity or flight direction. It is important that this flight manoeuvre (e.g. acceleration to supersonic velocity) should not lead to the focussing of waves on densely populated agglomerations or on recreation and rest areas.

In section 2 of this paper a procedural algorithm is proposed which permits the localization of the focussing region for any aircraft manoeuvre. The starting point for this procedure is the condition that at least two N waves will coincide in space and time. In other papers dealing with this problem (e.g. [2, 6, 7]) the focussing points are identified with the so-called *apex points* and thus tend to obscure the physical meaning of the focussing phenomenon. The algorithm developed for the case of rectilinear accelerated flight (section 3) is much simpler. It is based on a particular effect which occurs during this manoeuvre.

2. Determination of the focussing region in the general case

In the close proximity of the aircraft the sonic bang produces a Mach cone. Each point of the cone's surface can be considered to be the end of a ray emitted from the apex of the cone at an earlier moment (Fig. 2). The set of all rays emitted at any moment forms a "coupled cone" whose generating

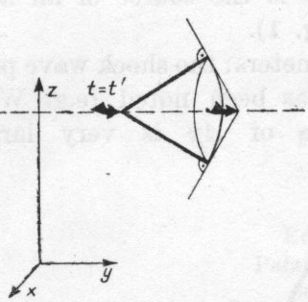

Fig. 2. The Mach cone and a "coupled cone" whose generating lines are the rays emitted at a time t

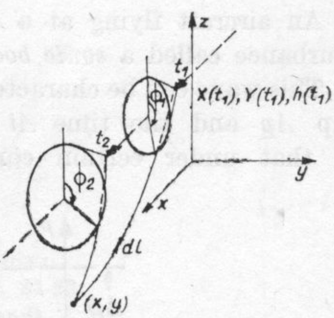

Fig. 3. Superposition (focussing) of waves N emitted at times t_1 and t_2

lines, in the proximity of the aircraft, are perpendicular to the generating lines of the Mach cone. Two such coupled cones are shown in Fig. 3. Owing to refraction, these rays are in fact curves and the surfaces are somewhat distorted.

Focussing of supersonic waves at the point (x, y) will take place when at least two waves reach this point simultaneously. Two rays are marked in Fig. 3 as paths along which the waves are propagated. If we denote the times of emission of the superimposing waves by t_1 and t_2, and the times of arrival at the point (x, y) by $t^{(1)}$ and $t^{(2)}$, then the following equation must be satisfied:

$$t^{(1)} = (t_2 - t_1) + t^{(2)}. \tag{1}$$

The length of the segment dl (Fig. 3) is

$$dl = \{(dx)^2 + (dy)^2 + (dz)^2\}^{1/2},$$

hence

$$dl = \left\{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right\}^{1/2}.$$

For the expressions dx/dz and dy/dz of the acoustic rays in an inhomogeneous medium (derived for example in [1]) we have

$$\begin{aligned} \frac{dx}{dz} &= \frac{[a(z)\cos\theta + W_x(z)]\cos\vartheta}{a(z)\{\cos^2\vartheta - \cos^2\theta\}^{1/2} + W_z(z)\cos\vartheta}, \\ \frac{dy}{dz} &= \frac{[a(z)\cos\theta\tan\vartheta + W_y(z)]\cos\vartheta}{a(z)\{\cos^2\vartheta - \cos^2\theta\}^{1/2} + W_z(z)\cos\vartheta}, \end{aligned} \tag{2}$$

where

$$\vartheta = \gamma + \arctan\left[\cot a\, \alpha \frac{\sin\Phi}{\cos\delta}\right],$$

$$\cos\theta = \cos\gamma[\cos\delta\sin\alpha - \sin\delta\cos\alpha\cos\Phi] - \sin\gamma\cos\alpha\sin\Phi,$$

with

$$\alpha = \arcsin\frac{a(h)}{V}.$$

It can be seen that dx/dz and dy/dz are functions of the flight parameters, i.e. the velocity V, the flight direction defined by the angles γ and δ, the angle Φ which determines the position of the rays in the "coupled cone" (Fig. 4), and the altitude z (W_x and W_y are components of the wind velocity, while a is the velocity of sound propagation).

If h is the flight altitude, then the time for the propagation of the N wave along any ray is

$$t = \int_0^h \frac{dl}{a(z)}dz = \int_0^h \frac{1}{a(z)}\left\{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right\}^{1/2} dz.$$

Condition (1) takes then the form

$$\left[\int_0^h \frac{1}{a(z)}\left\{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right\}^{1/2} dz\right]_{t=t_1} = t_2 - t_1 + \left[\int_0^h \frac{1}{a(z)}\left\{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right\}^{1/2} dz\right]_{t=t_2}. \tag{3}$$

The quantities V, γ and δ, i.e. the arguments of the functions dx/dz, dy/dz and h, which describe the flight, are, in general, functions of time.

At the point (x, y), where focussing occurs, the two or more intersecting rays will, in general, have different angles Φ (Fig. 3). (In the particular case

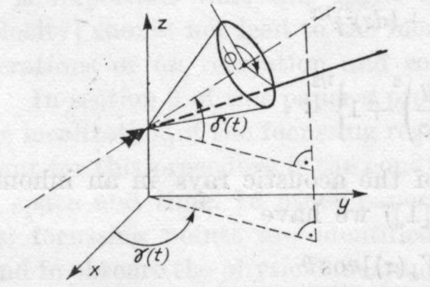

Fig. 4. Definition of the angles γ and δ describing the flight direction

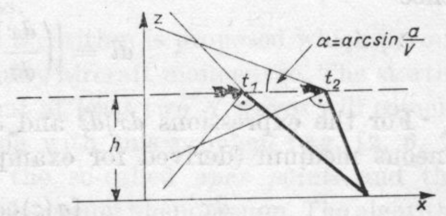

Fig. 5. Focussing of N waves beneath the flight path

of the focussing, occuring directly under the flight path, $\Phi_1 = \Phi_2$). Equation (3) is thus dependent on the angles Φ_1, Φ_2 and the times of emission t_1, t_2:

$$H_1(t_1, t_2, \Phi_1, \Phi_2) = 0. \tag{4}$$

The apparent form of this relation can only be determined when all quantities in the expressions dx/dz, dy/dz (formula (2)), and the altitude h are known functions, e.g. $V(t) = V_0 + mt$, $a(z) = a_0(1-\beta z)$, $h = h_0$ etc.

If we assume that at $t = 0$ the aircraft is at the point $x = 0$, $y = 0$, $z = h$, then by integration of differential equations (2) we obtain the coordinates of the point at which the ray emitted at a time t_i $(i = 1, 2)$ meets the earth's surface:

$$x_i = X(t_i) + \Big[\int_0^h f_1(V, \gamma, \delta, \Phi, z)\,dz\Big]_{t=t_i}, \tag{5}$$

$$y_i = Y(t_i) + \Big[\int_0^h f_2(V, \gamma, \delta, \Phi, z)\,dz\Big]_{t=t_i}. \tag{6}$$

$X(t_i)$, $Y(t_i)$, $h(t_i)$ determine the position of the aircraft at the time of emission of the i-th N wave.

Using the definition of angles γ and δ (Fig. 4), we have

$$X(t_i) = \int_0^{t_i} V\cos\gamma\cos\delta\,dt, \qquad Y(t_i) = \int_0^{t_i} V\sin\gamma\cos\delta\,dt. \tag{7}$$

If the conditions $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$ are introduced into (5) and (6), which denote that the focussing will take place only when the two N waves emitted at the moments t_1 and t_2 will meet at the same point on the earth's surface (Fig. 3), we obtain equations which, like the condition for the synchronization of the sonic booms (equation (4)), can be rewritten in the form

$$H_2(t_1, t_2, \Phi_1, \Phi_2) = 0, \quad H_3(t_1, t_2, \Phi_1, \Phi_2) = 0.$$

From these equations, eliminating Φ_1 and Φ_2, we obtain

$$t_2 = f(t_1). \tag{8}$$

This equality shows that at the point (x, y) there will occur a superposition of the waves emitted at times t_1 and $f(t_1)$. It is obvious that different manoeuvres (i.e. variations of the flight with time) and different atmospheric conditions, described by $a(z)$, $W_x(z)$ and $W_y(z)$, will give different forms for the function f. If the three above-mentioned equations do not take the form of equation (8), the focussing will not take place.

Let us assume that the functional relation (8) exists for the interval (t_1', t_2''). For any time t_1 within this interval, a time t_2 can be found which corresponds to the time of emission of the N wave which coincides at the same point with the wave emitted at time t_1. For these times from equations (5) and (6) we obtain a set of coordinates $\{x, y\}$ at which focussing occurs.

3. The location of the ground focus line produced by a supersonically accelerating aircraft (for the particular case of linear travel)

It has already been stated that manoeuvres involving a change of flight direction can result in the phenomenon of focussing which can be of hazard. Thus, irrespective of such considerations as comfort and safety during the flight, planned flight paths are straight lines. During such a flight only one manoeuvre is performed which may cause focussing – the initial acceleration. The uniform retardation in the last stages of the flight does not cause focussing.

The procedure, proposed here, for the localization of the focus of the sonic boom differs from the algorithm previously proposed. This procedure is based on the phenomenon which in the most simple manner can be explained on the assumptions that the medium, in which the N wave is propagating, is homogeneous (i.e. no refraction occurs), and the rays are thus straight lines.

If the aircraft is flying at a constant altitude h along the x-axis at a varying speed $V(t)$, then the condition for the synchronization (1) of the rays beneath the flight path takes the form

$$\frac{h}{a}\left[\frac{V}{\{V^2-a^2\}^{1/2}}\right]_{t=t_1} = t_2 - t_1 + \frac{h}{a}\left[\frac{V}{\{V^2-a^2\}^{1/2}}\right]_{t=t_2}. \tag{9}$$

From the condition for the intersection at the same point of two rays along which the N waves propagate at two different times, i.e. from $x(t_1) =$

$= x(t_2)$, we obtain

$$\int_{t_1}^{t_2} V(t)\,dt = \left[\frac{h_a}{\{V^2-a^2\}^{1/2}}\right]_{t=t_1} - \left[\frac{h_a}{\{V^2-a^2\}^{1/2}}\right]_{t=t_2}. \tag{10}$$

Assuming that the aircraft is flying with uniform acceleration $V = a+mt$, and at time $t = 0$, the angle aperture of the Mach cone is $\alpha = \frac{1}{2}\pi$, since

$$\lim_{t\to 0} \arcsin\frac{a}{a+mt} = \frac{1}{2}\pi.$$

The intersection of the x-axis with the N wave propagating along the rays (Fig. 6) lies at infinity. With increasing velocity this point approaches the coordinate origin and later moves away from it.

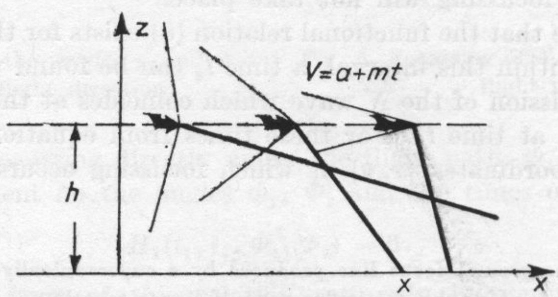

Fig. 6. The "motion" of the point of intersection of a ray with the x-axis during accelerated flight (x is the point of minimum range)

Let us denote by x the point of intersection nearest to the origin of the coordinate system and call it *the point of minimum range*.

A similar phenomenon will be observed along any straight line parallel to the x-axis (i.e. for all y), even when the rays are curved, as in the case of the earth's (refractive) atmosphere. The points of minimum range are also focussing points for the sonic boom.

If at $t = 0$ the aircraft is passing the point $x = 0$, then the point of intersection of the N wave (emitted at time t) with the x-axis has the coordinate

$$x(t) = \int_0^t (a+mt)\,dt + \frac{ha}{\{(a+mt)^2-a^2\}^{1/2}}.$$

If the acceleration is not too high and the inequality

$$\frac{mt}{a} \ll 1 \tag{11}$$

is satisfied, then from $dx(t)/dt = 0$ we obtain

$$t = \frac{h^{2/3}}{2(am)^{1/3}}, \tag{12}$$

which determines the time of emission of the N wave reaching the point of minimum range x.

Under condition (11), equations (9) and (10) take the form

$$a(t_2-t_1) = h\left(\frac{1+\dfrac{mt_1}{a}}{\sqrt{2\dfrac{mt_1}{a}}} - \frac{1+\dfrac{mt_2}{a}}{\sqrt{2\dfrac{mt_2}{a}}}\right),$$

$$a(t_2-t_1)+\frac{1}{2}m(t_2^2-t_1^2) = h\left(\frac{1}{\sqrt{2\dfrac{mt_1}{a}}} - \frac{1}{\sqrt{2\dfrac{mt_2}{a}}}\right). \tag{13}$$

The first of these equations relates to the times of emission $\{t_1, t_2\}$ of the sonic booms which reach the earth's surface at the same time, but not necessarily at the same place. The second equation relates to pairs $\{t_1, t_2\}$ which correspond to the times of emission of the N waves reaching the same point, but not necessarily at the same time. In order to find the points of focussing it is necessary to find the pairs $\{t_1, t_2\}$ which simultaneously satisfy the two equations.

If $mh/a^2 \ll 1$, then from equations (13) we obtain

$$t_2 = \frac{h}{\sqrt{2\,amt_1}} \quad \text{and} \quad t_1 = t_2 = \frac{h^{2/3}}{2\,(am)^{1/3}}.$$

Thus the focussing takes place at the point of minimum range, as if the ray during its travel along the x-axis stopped at this point (Fig. 8). This results in the superposition of the N waves.

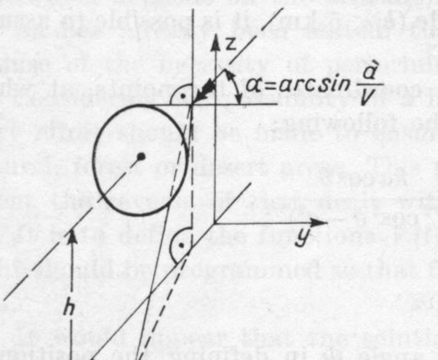

Fig. 7. The definition of the angle ϑ describing the position of a ray within the coupled cone

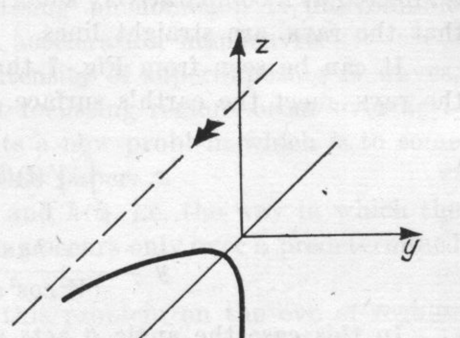

Fig. 8. The curve along which focussing occurs (equation (15))

Since the points of minimum range for accelerated rectilinear flying occur over the whole area of sonic boom audibility, and not only beneath the flight path as is shown in Fig. 6, it is obvious that the points of focussing lie on a curve which is symmetrical relative to the flight path (Fig. 8).

The identity of the points of minimum range with the points of focussing largely facilitates their determination.

During the rectilinear flight along the x-axis, $\gamma = 0$ and $\delta = 0$ (in Fig. 4). This simplifies equation (5) to the form

$$x_i = \int_0^{t_i} V(t)\,dt + \Big[\int_0^h f_1(V, \Phi, z)\,dz\Big]_{t=t_i},$$

$$y_i = \Big[\int_0^{t_i} f_2(V, \Phi, z)\,dz\Big]_{t=t_i}.$$

Eliminating the parameter Φ, we have

$$x(t_i) = \int_0^{t_i} V(t)\,dt + \Big[\int_0^h f_1(V, y, z)\,dz\Big]_{t=t_i}. \tag{14}$$

The times of emission of the N waves, which reach the points of minimum range, are obtained from the equation

$$\frac{dx(t_i)}{dt_i} = 0,$$

whence we may obtain the function in the form $t_i = h(y)$. Substituting this relation into equation (14) we obtain a set of focal points in the form (Fig. 8)

$$x = f(y).$$

Let us illustrate this procedure with an example. If the rectilinear flying is effected at a comparatively small altitude ($h < 5$ km), it is possible to assume that the rays are straight lines.

It can be seen from Fig. 7 that the coordinates of the points, at which the rays meet the earth's surface, are the following:

$$x = \int_0^t V(t)\,dt + \frac{ha\cos\vartheta}{(V^2\cos^2\vartheta - a^2)^{1/2}},$$

$$y = \frac{ha\sin\vartheta}{(V^2\cos^2\vartheta - a^2)^{1/2}}.$$

In this case the angle ϑ acts as the angle Φ in defining the position of the ray.

If the motion is uniformly accelerated, then

$$\int_0^t V(t)\,dt = \frac{V^2 - a^2}{2m}.$$

In this case it is preferable to use the velocity V rather than the time t.

We may further obtain

$$x(t) = \frac{V^2-a^2}{2m} + \frac{a\sqrt{h^2+y^2}}{(V^2-a^2)^{1/2}},$$

corresponding to equation (14).

Differentiating $x(t)$ with respect to t we obtain

$$1-a\{h^2+y^2\}^{1/2}\{V^2-a^2\}^{-3/2}m = 0.$$

The value V, which satisfies this equation, corresponds to the time of emission of the N wave of the shortest range. Eliminating V from the last two equations, we obtain the equations of the loci along which the focussing occurs:

$$y^2 - \frac{8}{27}\frac{m}{a^2}x^3 + h^2 = 0. \tag{15}$$

WANNER [7] has obtained the same result by another method.

4. Conclusions

Due to changing flight parameters (a manoeuvre), the phenomenon of focussing may occur during flights at supersonic speeds.

On the basis of the algorithms, proposed in sections 2 and 3, the location of the focus may be determined if data describing the flight are known. Relevant data are the velocity $V(t)$, the angles γ and δ that characterize the direction, the altitude $h(t)$ and the velocity of sound propagation in the atmosphere $a(z)$ (which depends on the altitude).

As has already been stated, the focussing phenomenon is unavoidable because of the necessity of performing an acceleration manoeuvre.

Considering the possibility of a high intensity of superimposing N waves, every effort should be made to ensure that focussing regions occur over agricultural, forest or desert areas. This presents a new problem which is to some extent the reverse of that dealt with in this paper.

It is to define the functions $V(t)$, $\delta(t)$ and $h(t)$, i.e. the way in which the flight should be programmed so that focussing occurs only over a predetermined area.

It would appear that the solution of this problem on the eve of regular supersonic flights is of paramount importance in the protection of the human environment.

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