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IMPEDANCE OF THE UNBAFFLED CYLINDRICAL PIPE OUTLET FOR THE PLANE WAVE INCIDENT AT THE OUTLET*

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The paper presents formulae for the impedance of the outlet of semi-infinite cylindrical wave-guide derived by considering the propagation of a plane wave and accounting for the generation of higher Bessel modes due to the diffraction at the opened end of the wave-guide. For this purpose expressions for the refraction and transformation coefficients of the basic mode were derived by solving exactly the wave equations with suitable boundary conditions using Wiener-Hopf factorization.

1. Introduction

In the practical applications of acoustics an important role is played by the phenomena occurring at the opened ends of wave-guides, e.g. of measuring pipes and acoustic horns. The first attempt to describe these phenomena was presented by Rayleigh [1] who had assumed uniform distribution of the velocity of vibrations at the outlet provided additionally with an infinitely rigid acoustic baffle. A further step towards the definition of the acoustic field inside the semi-infinite unbaffled cylindrical waveguide was made by Levine and Schwinger [2]. They assumed, however, that a basic mode plane wave propagates in the direction of the outlet and that because of diffraction at the opened end of the wave-guide only the plane wave with an amplitude described by the complex coefficient of reflection propagates. The impedance of the outlet calculated on the basis of the value of this coefficient is given, among others, by Żyszkowski ([3], p. 218). However, it is known, e.g. from the theory of the infinite cylindrical wave-quide (cf. [4]), that such assumptions are valid only when the diffraction parameter of the wave-guide, i.e. the product of the wave number and the pipe radius is smaller than the value of the zero-crossing of

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the Bessel function of the first order, equal to 3,8317... This model has thus a limited application to higher frequencies and larger diameters of the wave--guides.

In 1949 Wajnsztejn [6] developed an analytical theory of the acoustic field of a semi-infinite unbaffled cylindrical wave-guide utilizing the method of solution of a similar problem for electromagnetic waves [5]. Basing on his results the author of the present paper has calculated the impedance of the outlet of the unbaffled cylindrical wave-guide for the plane wave incident at the outlet with the aid of an exact solution of the wave equation for any value of the diffraction parameter. The result obtained can also be interpreted as an impedance of a circular sound source located at the bottom of a semi-infinite, rigid cylinder of the identical radius.

2. Solution of wave equation

Let us consider a cylindrical wave-guide with an infinite, thin and rigid wall and select a cylindrical coordinate system in which the Z-axis coincides with the symmetry axis of the wave-guide. The wave-guide wall Σ is given by the equation of the side-wall of the semi-infinite cylinder with a radius a:

$$\Sigma = \{(r, z): r = a, z \geqslant 0\}.$$

Let us assume further that the acoustic potential $\Phi(r, z)$ does not depend on angle φ and its dependence on time is described by the factor expressed in the form $\exp(-i\omega t)$. The wave equation for the potential has thus the following from:

$$\frac{1}{r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2 \Phi}{\partial z^2}+k^2\Phi = 0. \tag{1.1}$$

The assumption that the wave-guide is perfectly rigid leads to the boundary condition

$$\left.\frac{\partial \Phi}{\partial r}\right|_{\Sigma} = 0. \tag{1.2}$$

This means that the normal component of velocity vanishes at the wave-guide wall. The second boundary condition requires that the potential should be continuous at the surface extention in the negative direction of the Z-axis:

$$\lim_{r\to a_+} \Phi(r, z) = \lim_{r\to a_-} \Phi(r, z), \qquad z < 0. \tag{1.3}$$

The solution of the problem of acoustic field of the wave-guide consists in finding the function $\Phi(r, z)$ which satisfies equation (1.1) for the boundary conditions (1.2) and (1.3) and the Sommerfeld condition of radiation (cf. [4]).

Let us assume that the partial solution of this problem, depending additionally on a parameter w and modified by a function $F(w)$ is the solution obtained for the infinite wave-guide [5], [6]

$$\Phi(r,z,w) = i2\pi^2 v F(w) e^{iwz/a} \begin{cases} J_0\left(v\dfrac{r}{a}\right)[H_0^{(1)}(V)]', \\ [J_0(v)]' H_0^{(1)}\left(V\dfrac{r}{a}\right), \end{cases} \tag{1.4}$$

where

$$v = \sqrt{(ka)^2 - w^2}. \tag{1.5}$$

The upper product of cylindrical functions in braces refers to the interior of the cylinder described by Σ whereas the lower one to the outside of this cylinder, i.e. for $r > a$ [4].

The required potential $\Phi(r,z)$ is assumed to be a superposition of the above partial solutions [5], [6],

$$\Phi(r,z) = \int_C \Phi(r,z,w)\,dw, \tag{1.6}$$

where C is a contour which is selected so that the obtained solution satisfies the imposed boundary conditions. In particular, the boundary condition (1.2) now takes the form

$$\int_C e^{iwz/a} L(w) F(w)\,dw = 0, \quad z > 0, \tag{1.7}$$

where

$$L(w) = \frac{\pi V^2}{a} J_1(v) H_1^{(1)}(v). \tag{1.8}$$

By calculating then the potential step on the surface $r = a$,

$$\Phi(a_+, z, w) - \Phi(a_-, z, w) = 4\pi F(w) e^{iwz/a}, \tag{1.9}$$

it is possible to write the boundary condition (1.3) in the form

$$\int_C e^{iwz/a} F(w)\,dw = 0, \quad z < 0. \tag{1.10}$$

Finding the potential $\Phi(r,z)$ is thus reduced to the determination of such a function $F(w)$ and a contour that equations (1.7) and (1.10) are satisfied. The solution of these equations can be obtained by the Wiener-Hopf method, by factorizing analytically integrands $L(w)$ and $F(w)$ into factors $L_+(w)$ and $L_-(w)$ in the and lower half-plane of the complex variable w, respectively, as this permits to make use of the convolution theorem [7].

A further development of the factorization method, described extensively, among others, in papers [5] and [6], leads to the following expression for the acoustic potential of the wave-guide under consideration:

$$\Phi(r, z) = -A\left[e^{-ikz} + \sum_{n=0}^{\infty} R_n \frac{J_0\left(\mu_n \frac{r}{a}\right)}{J_0(\mu_n)} \cdot e^{i\gamma_n \frac{z}{a}}\right], \tag{1.11}$$

where R_n is the coefficient of transformation of the incident wave into the n-th wave mode with a wave number γ_n/a, with

$$\gamma_n = \sqrt{(ka)^2 - \mu_n^2}, \tag{1.12}$$

and μ_n is the n-th zero-crossing of the Besel function of the first order

$$J_1(\mu_n) = 0.$$

The first component in square brackets represents the plane wave which, according to the assumption, propagates in the direction of the wave-guide outlet and is transformed there into an infinite number of waves with a Bessel distribution which propagate in the opposite direction. Analyzing carefully the exponential expressions under the sum sign we see that for a fixed diffraction parameter only a certain number of components will represent the waves which can propagate along the wave-guide, since, if the condition

$$ka = \varkappa > \mu_n \tag{1.13}$$

is satisfied, the exponent of the exponential function will be an imaginary number. Starting, however, from a certain N such that

$$\mu_N < \varkappa < \mu_{N+1}, \tag{1.14}$$

the exponents will be negative real numbers and thus the corresponding components of the sum will represent a disturbance, attenuated exponentially with increasing coordinate Z. Since these disturbances are not the energy. carrying waves, they will be ignored in further considerations of impedance.

3. Reflection and transformations of impedance

It follows from (1.11) that the determination of the acoustic field inside the wave-guide is now reduced to the problem of explicit calculation of the coefficients of reflection and transformation of the incident wave. According to paper [6] these coefficients have the following form:

reflection coefficient

$$R_0 = \frac{L_+(\varkappa)}{L_-(\varkappa)}; \tag{2.1}$$

transformation coefficient

$$R_n = \frac{2\varkappa L_+(\varkappa)}{(\varkappa^2-\mu_n^2)L'_-(\gamma_n)}. \tag{2.2}$$

The factors $L_+(w)$ and $L_-(w)$ are defined as

$$L_+(w) = \frac{i\sqrt{a}}{\sqrt{\varkappa+w}}\psi_+(w), \tag{2.3}$$

$$L_-(w) = \frac{i\sqrt{a}}{\sqrt{\varkappa-w}}\psi_-(w), \tag{2.4}$$

where

$$\psi_+(w) = \sqrt{\pi(\varkappa+w)H_1^{(1)}(v)J_1(v)\prod_{i=1}^{N}\frac{\gamma_i+w}{\gamma_i-w}e^{S(w)/2}}, \tag{2.5}$$

$$\psi_-(w) = \sqrt{\pi(\varkappa-w)H_1^{(1)}(v)J_1(v)\prod_{i=1}^{N}\frac{\gamma_i-w}{\gamma_i+w}e^{-S(w)/2}}, \tag{2.6}$$

N is an index of the highest mode capable of propagating freely in the wave-guide (cf. (1.17)), $S(w)$ is the complex function

$$S(w) = X(w)+iY(w). \tag{2.7}$$

For real values of w that satisfy inequality $|w| \leqslant \varkappa$ the real and imaginary parts of the function $S(w)$ equal respectively to

$$X(w) = \frac{1}{\pi}\int_{-\varkappa}^{\varkappa}\frac{\Omega(v)\,dw'}{w'-w}, \tag{2.8}$$

$$\psi(w) = \frac{2w}{\pi} - \Omega(V) + \frac{1}{i}\lim_{M\to\infty}\left[\sum_{n=N+1}^{M}\ln\frac{\gamma_n+w}{\gamma_n-w} - \frac{1}{\pi}\int_{-\gamma_M}^{\gamma_M}\frac{\Omega(v')\,dw'}{w'-w}\right]. \tag{2.9}$$

$\Omega(v)$ is the argument of Hankel function of the first kind of the first order increased by $\pi/2$:

$$\Omega(v) = \operatorname{Arg}H_1^{(1)}(v)+\frac{\pi}{2} = \operatorname{arctg}\frac{N_1(v)}{I_1(v)}+\frac{\pi}{2}. \tag{2.10}$$

Substituting (2.3) and (2.4) into (2.1) and making use of (2.5) and (2.6) we get expression for the reflection coefficient of plane wave:

$$R_0 = -\prod_{i=1}^{N}\frac{\gamma_i+\varkappa}{\gamma_i-\varkappa}e^{S(\varkappa)}. \tag{2.11}$$

The calculation of the transformation coefficients R_n is a little more complicated, primarily because of the occurrence of the derivative of the function $L_-(w)$ in the denominator of formula (2.2). The derivative of the function $L_-(w)$ in (2.4) equals

$$L'_-(w) = \frac{i\sqrt{a}}{\sqrt{\varkappa - w}}\left[\frac{1}{2(\varkappa - w)} + \frac{\psi'_-(w)}{\psi_-(w)}\right]\psi_-(w), \tag{2.12}$$

where the second component in brackets can be calculated by means of the logarithmic derivative [6]:

$$\frac{\psi'_-(W)}{\psi_-(W)} = -\frac{1}{2(\varkappa - w)} + \sum_{i=1}^{\infty}\int_{\gamma_{i-1}}^{\gamma_i}\frac{d\Omega(v')}{dw'}\frac{dw'}{w'-w} + \sum_{i=1}^{\infty}\frac{1}{w-\gamma_i}. \tag{2.13}$$

At the point $w = \gamma_n$, in which the function $\psi_-(w)$ vanishes, its logarithmic derivative assumes an indefinite value. Hence, $L'_-(w)$ will exist in the sense of a limit. Direct calculation leads to the following expression:

$$L'_-(\gamma_n) = -i\frac{\sqrt{a}}{\mu_n}\sqrt{-i\prod_{\substack{i=1\\ i\neq n}}^{N}\frac{\gamma_i - \gamma_n}{\gamma_i + \gamma_n}}\cdot e^{-S(\gamma_n)/2}. \tag{2.14}$$

On the other hand, the factor $L_+(\varkappa)$ in the nominator of (2.2) can be written as

$$L_+(\varkappa) = i\sqrt{a}\sqrt{-i\prod_{i=1}^{N}\frac{\gamma_i + \varkappa}{\gamma_i - \varkappa}}\; e^{S(\varkappa)/2}, \tag{2.15}$$

where use was made of the asymptotic formulae for the special functions at small values of the argument [9]:

$$J_k(v) = \frac{1}{\Gamma(k+1)}\left(\frac{v}{2}\right)^k, \qquad N_k(v) = -\frac{\Gamma(k)}{\pi}\left(\frac{2}{v}\right)^k. \tag{2.16}$$

Finally, we can write the expression for the transformation coefficient of plane wave into the n-th wave mode

$$R_n = -\frac{2\varkappa}{\mu_n}\sqrt{\prod_{\substack{i=0\\ i\neq n}}^{N}\frac{\gamma_i + \gamma_n}{\gamma_i - \gamma_n}\prod_{i=1}^{N}\frac{\gamma_i + \varkappa}{\gamma_i - \varkappa}}\; e^{[S(\gamma_n)+S(\varkappa)]/2}. \tag{2.17}$$

Effective calculations of the values of coefficients as functions of diffraction parameter are only possible by numerical methods, since the integrals in the definition of the function $S(w)$ cannot be expressed by analytic functions.

The graphs in Figs. 1 and 2 represent respectively the moduli and phases of the reflection coefficient R_0 and the transformation coefficients R_n of plane wave for all the allowed wave modes because of their values within the range [0,20] except for R_6 appearing as late as for $\varkappa = 19.62$. Numerical computations have been made starting from the point $\varkappa = 0$ with a step 0.1.

In the calculations use has been made of the generally accepted definition of modulus and phase of wave reflection coefficient

$$R_n = -|R_n|\, e^{i\Theta n}. \tag{2.18}$$

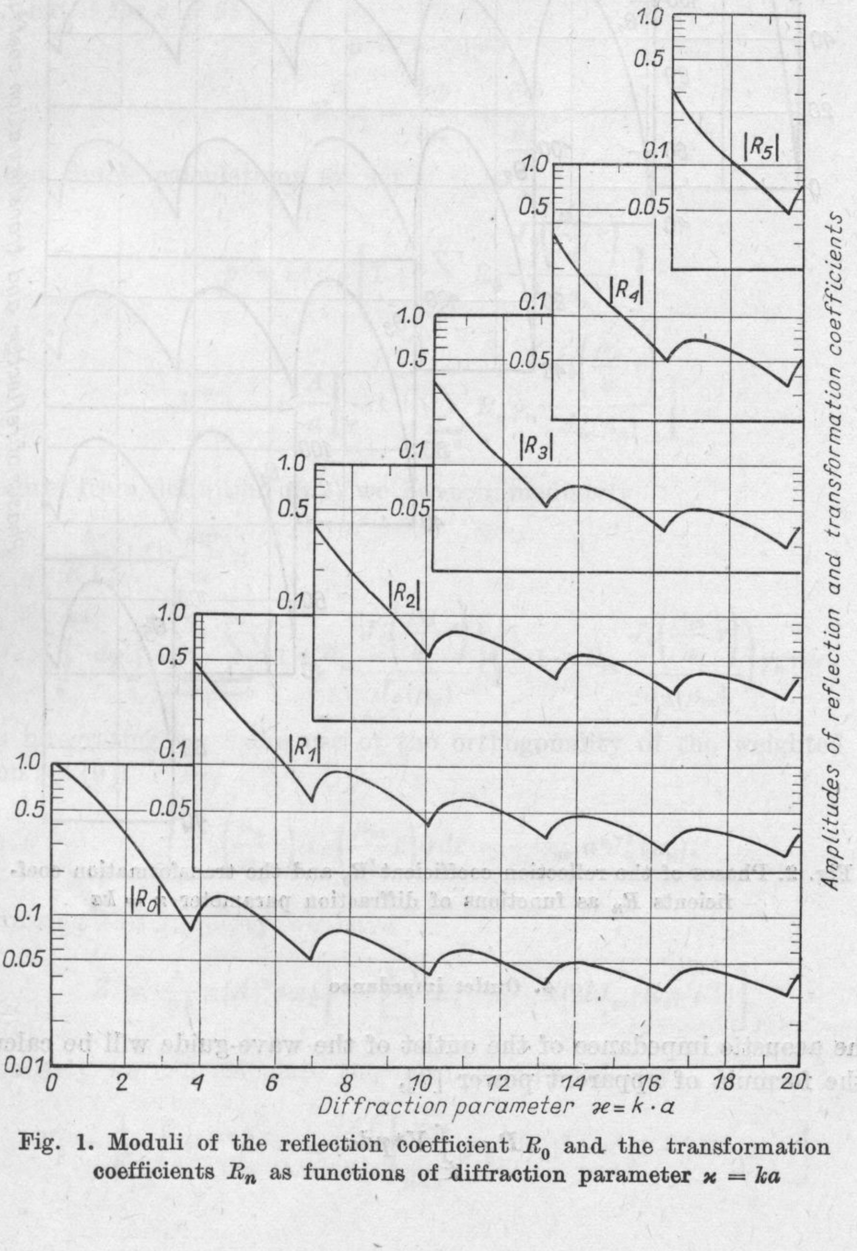

Fig. 1. Moduli of the reflection coefficient R_0 and the transformation coefficients R_n as functions of diffraction parameter $\varkappa = ka$

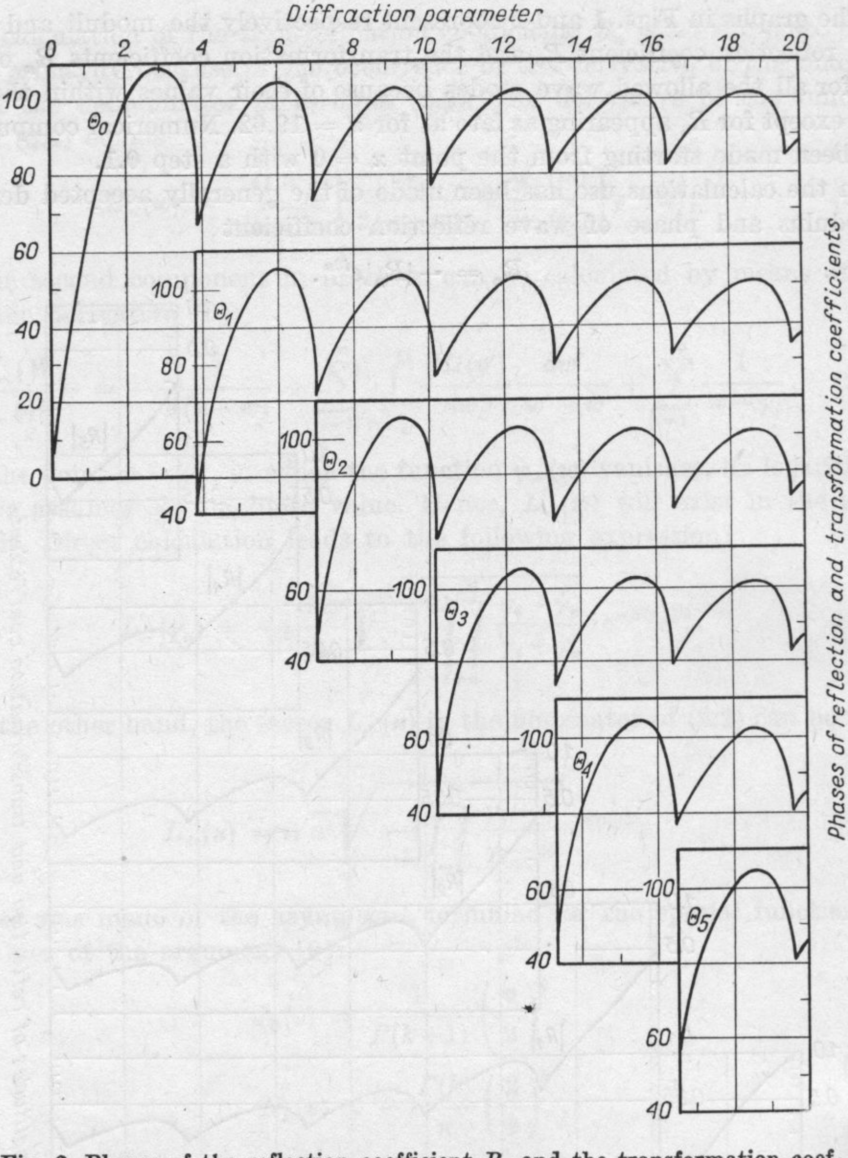

Fig. 2. Phases of the reflection coefficient R_0 and the transformation coefficients R_n as functions of diffraction parameter $\varkappa = ka$

4. Outlet impedance

The acoustic impedance of the outlet of the wave-guide will be calculated from the formula of apparent power [8],

$$P = \int_K V^* p d\sigma, \tag{3.1}$$

where k is the surface of the outlet, V — normal component of the velocity of vibrations, p — acoustic pressure.

The required impedance is related to the apparent power by the formula

$$Z = \frac{1}{KV_0^2}P, \tag{3.2}$$

where V_0 is the mean square of the velocity at the outlet. Knowing the acoustic potential, we can calculate the acoustic pressure and normal velocity at the outlet, that is for $z = \theta$:

$$p = -i\omega\varrho\Phi, \tag{3.3}$$

$$V = -\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial z}. \tag{3.4}$$

After simple calculations we get

$$p = iA\omega\varrho\left[1+\sum_{n=0}^{\infty} R_n \frac{J_0\left(\frac{\mu_n}{a}r\right)}{J_0(\mu_n)},\right] \tag{3.5}$$

$$V = -i\,\frac{A}{a}\left[-1+\sum_{n=0}^{\infty} R_n\gamma_n \frac{J_0\left(\frac{\mu_n}{a}r\right)}{J_0(\mu_n)}\right]. \tag{3.6}$$

Hence, from definition (3.2) we have immediately

$$Z = -\frac{1}{KV_0^2}|A|^2\frac{\omega\varrho}{a}\times$$

$$\times\int_0^{2\pi} d\varphi\int_0^a \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(1+R_n\frac{J_0\left(\frac{\mu_n}{a}r\right)}{J_0(\mu_n)}\right)\left(-1+R_m\frac{J_0\left(\frac{\mu_m}{a}r\right)}{J_0(\mu_m)}\right)\gamma_m r dr. \tag{3.7}$$

In integrating we make use of the orthogonality of the weighted Bessel function set [9]:

$$\int_0^a I_0\left(\frac{\mu_n}{a}r\right)I_0\left(\frac{\mu_m}{a}r\right)rdr = \frac{1}{2}\delta_{nm}a^2J_0^2(\mu_n). \tag{3.8}$$

Utilizing this property we have

$$Z = \frac{1}{V_0^2}\pi|A|^2a\omega\varrho\left[-\sum_{n=0}^{\infty}|R_n|^2\gamma_n^* + \varkappa\left(2iJ_m(R_0)+1\right)\right]. \tag{3.9}$$

Similarly we can calculate the mean square velocity

$$V_0^2 = \frac{1}{K}\int_K VV^*d\sigma = \pi\frac{|A|^2}{a^2}\left[\sum_{n=0}^{\infty}|R_n|^2|\gamma_n|^2 + \varkappa^2\left(1-2\,\mathrm{Re}(R_0)\right)\right]. \tag{3.10}$$

Acoustical impedance at the outlet is thus equal to

$$Z = \omega \varrho a \frac{\sum_{n=0}^{\infty} -\gamma_n^* |R_n|^2 + \varkappa\big(2i\,\mathrm{Im}(R_0)+1\big)}{\sum_{n=0}^{\infty} |R_n|^2 |\gamma_n|^2 + \varkappa\big(1-2\,\mathrm{Re}(R_0)\big)}. \tag{3.11}$$

We now separate the real and imaginary part of the impedance.

If N is the highest index of the wave for which n is a real number, then we have equalities

$$\gamma_n = \begin{cases} \gamma_n^*, & \text{when } n \leqslant N, \\ -\gamma_n^*, & \text{when } n > N, \end{cases}$$

and this leads to the following expressions for the real and imaginary parts of the impedance referred to the a specific impedance of environment:

$$\mathrm{Re}(Z) = \varkappa \frac{-\sum_{n=0}^{N} \gamma_n |R_n|^2 + \varkappa}{\sum_{n=0}^{\infty} |R_n|^2 |\gamma_n|^2 + \varkappa^2\big(1-2\,\mathrm{Re}(R_0)\big)}, \tag{3.12}$$

$$\mathrm{Im}(Z) = \varkappa \frac{\sum_{n=N+1}^{\infty} |\gamma_n|\,|R_n|^2 + 2\varkappa\,\mathrm{Im}(R_0)}{\sum_{n=0}^{\infty} |R_n|^2 |\gamma_n|^2 + \varkappa^2\big(1-2\,\mathrm{Re}(R_0)\big)}. \tag{3.13}$$

According to the remark concluding section 2, we can neglect the components of sums with an index $n > N$. Thus we finally get

$$\mathrm{Re}(Z) = \varkappa \frac{\varkappa - \sum_{n=0}^{N} \gamma_n |R_n|^2}{\sum_{n=0}^{N} |R_n|^2 |\gamma_n|^2 + \varkappa^2\big(1-2\,\mathrm{Re}(R_0)\big)}, \tag{3.14}$$

$$\mathrm{Im}(Z) = \frac{2\varkappa^2\,\mathrm{Im}(R_0)}{\sum_{n=0}^{N} |R_n|^2 |\gamma_n|^2 + \varkappa^2\big(1-2\,\mathrm{Re}(R_0)\big)}. \tag{3.15}$$

Putting $N = 0$ in (3.14) and (3.15), we shall confine ourselves to the case considered in [1], where it has been assumed that only the plane wave is reflected from the outlet and the wave modes of higher orders have been neglected.

Then these formulae take the well-known form of the expression for impedance

$$Z_0 = \frac{1+R_0}{1-R_0}. \tag{3.16}$$

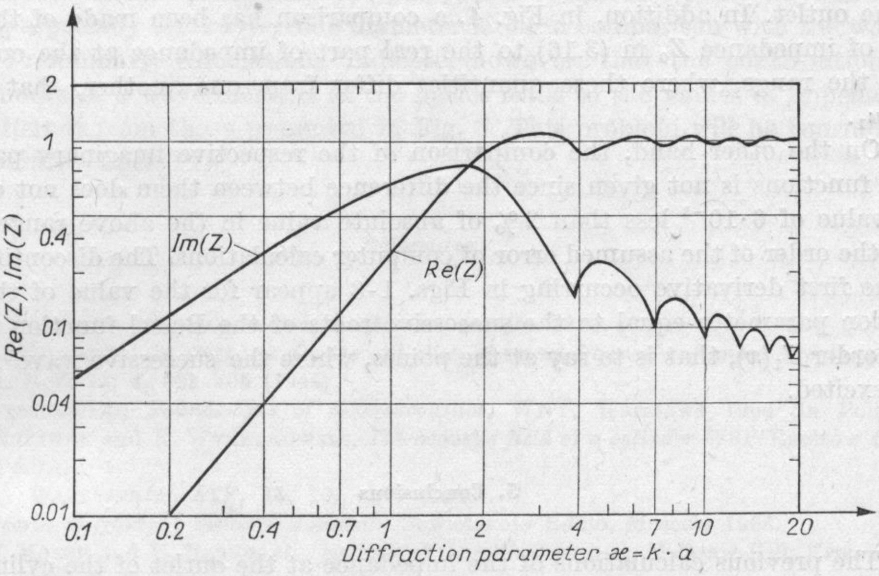

Fig. 3. The diagrams of the real and imaginary parts of the impedance at the outlet of semi-infinitive cylindrical wave-guide without baffle

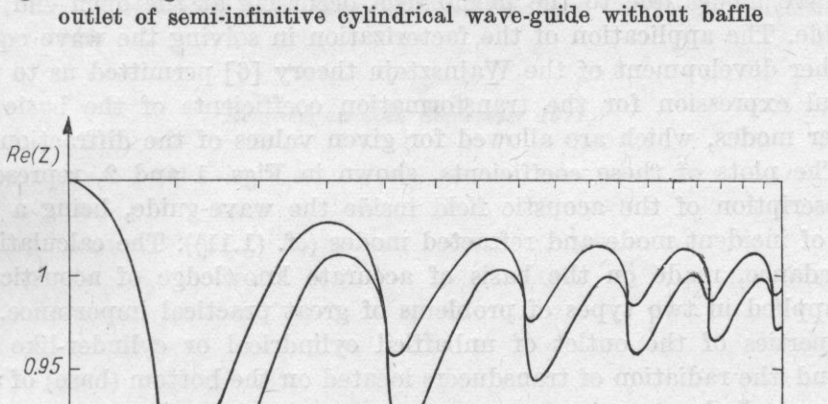

Fig. 4. The diagrams of the real part of the impedance at the outlet of the semi-infinitive cylindrical wave-guide in the case where the higher moduli of reflected waves are taken into account and in the case where it is disregarded

Fig. 3 shows the graph of the real and imaginary parts of the impedance at the outlet. In addition, in Fig. 4, a comparison has been made of the real part of impedance Z_0 in (3.16) to the real part of impedance at the outlet Z over the range, where these quantities differ from one another, that is for $\varkappa > \mu_1$.

On the other hand, the comparison of the respective imaginary parts of both functions is not given since the difference between them does not exceed the value of $6 \cdot 10^{-3}$ less than 2% of absolute value in the above range. This is of the order of the assumed error of computer calculations. The discontinuities of the first derivative occurring in Figs. 1-3 appear for the value of the diffraction parameter equal to the successive roots of the Bessel function of the first order $J_1(x)$, that is to say at the points, where the successive wave modes are excited.

5. Conclusions

The previous calculations of the impedance at the outlet of the cylindrical wave-guide, excited by the basic mode, did not account for the appearance of higher wave modes due to the phenomena occurring at the open end of the wave-guide. The application of the factorization in solving the wave equation and further development of the Wajnsztejn theory [6] permitted us to obtain the useful expression for the transformation coefficients of the basic mode into other modes, which are allowed for given values of the diffraction parameter. The plots of these coefficients, shown in Figs. 1 and 2, represent an exact description of the acoustic field inside the wave-guide, being a superposition of incident mode and refracted modes (cf. (1.11)). The calculations of the impedance, made on the basis of accurate knowledge of acoustic field, can be applied in two types of problems of great practical importance. They are: properties of the outlet of unbaffled cylindrical or cylinder-like wave-guides and the radiation of transducers located on the bottom (base) of a relatively long cylinder.

Commenting on the results obtained, it can be concluded that accounting for the higher modes resulting from the diffraction at the opened end of the wave-guide has only little influence on the magnitude of impedance if a plane wave is an incident at the outlet, although the values of the transformation coefficients of this wave into others attain considerable values even for the highest permissible modes. This means that in these problems for which only the magnitude of impedance is interesting, the assumption of the non-excitation of these modes is a quite good approximation, which is the more accurate the greater is the diffraction parameter of the wave-guide. However, one should keep in mind that the condition of the applicability of this approximation is the propagation of a "pure" basic mode in the direction of the outlet.

It is known that practically the generation of an ideal plane wave is very difficult, especially for wave-guide diameter large in comparison with the wave-length. Preliminary calculations indicate, however, that the contribution of higher modes in a wave incident at the outlet leads to the values of impedance quite different from those presented in Fig. 3. This problem will be considered in a separate paper.

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