ARCHIVES OF ACOUSTICS
4, 1, 57-64 (1979)

PROPAGATION OF ELASTIC WAVE IN SOLID LAYER-LIQUID SYSTEM*

ANNA GRABOWSKA

The Institute of Fundamental Technological Research (00-049 Warszawa, Świętokrzyska 21)

The paper presents the solution of the problem of a flat wave propagating without attenuation along a solid layer placed between a semispace filled with liquid and the vacuum.

The wave equation of scalar and vector potentials of displacement has been solved for this case. A characteristic equation accounting for boundary conditions has been derived. This equation has been solved numerically and it has been shown that in these conditions the wave can propagate at a velocity slightly smaller than the wave velocity in the liquid.

The distributions of stress, acoustic pressure, and displacement of the propagating wave have been determined numerically for a layer of a thickness $a = 0.075$ cm and $a = 0.010$ cm, contacting with water on one side, and for a frequency of 3×10^6 Hz. The type of wave is close to a surface wave.

1. Introduction

The problem of the propagation of an elastic wave in the solid layer-liquid system originated in the course of ultrasonic investigations of tumours. Using a probe containing a piezoelectric transducer and generating ultrasonic waves, a needle is driven through the patient's skin toward the tumour. The punctured organ is observed by an ultrasonic visualization system. The needle passes through a hole in the centre of the piezoelectric transducer which has the form of a plate.

In the course of these investigations it has been observed that the wave propagating along the needle placed already inside the body is accompanied by a wave which after reaching the end of the needle is reflected backwards and returns giving an image of the needle end on the oscilloscope screen. This effect makes it possible to locate precisely the puncture and to sample the tumour tissue instead of cutting the whole organ apart.

---

* The paper was written under problem MR.I.24.

This work is aimed at investigation of the effects accompanying the propagation of the above wave along the needle surrounded by the body tissue. We shall reduce this problem to the consideration of a wave propagating in a flat solid layer-liquid system. In addition we shall assume that the considered wave is a running, continuous sinusoidal wave.

The solution of this problem consists in a description of an elastic wave propagating in an infinite isotropic homogeneous solid layer in contact with an mimobile and infinitely deep liquid on one side and with the vacuum on the other.

2. Basic equations

The coordinate system is chosen as follows (cf. Fig. 1). The x-axis coincides with the upper edge of the layer and is parallel to the direction of the wave propagation. The z-axis is directed vertically upwards. The layer thickness

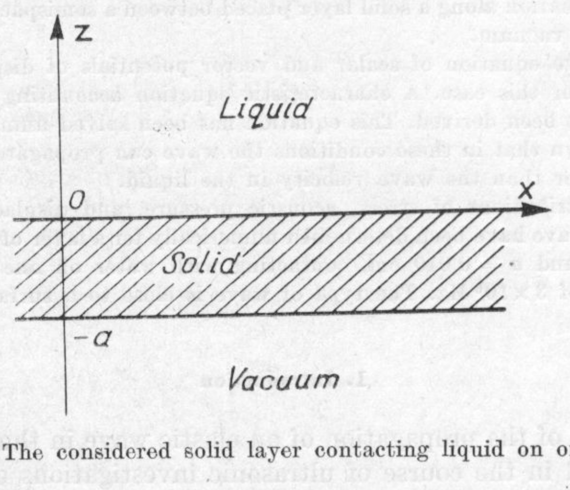

Fig. 1. The considered solid layer contacting liquid on one side

is a; the densities ϱ_s, λ and μ are the Lamé constants and the velocities of longitudinal and transverse waves in the material are c_d and c_t, respectively. The liquid denisty is ϱ_c and the longitudinal wave velocity in the liquid is c_0.

The displacement vector $\boldsymbol{u}$ can be presented in the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} \tag{1}$$

with conditions curl $\boldsymbol{v} = 0$, div $\boldsymbol{w} = 0$.

It follows from the vector analysis that such representation of a vector field is always possible. This is a representation of a vector in the form of a sum of the gradient of a scalar potential φ and the curl of a vector potential $\boldsymbol{\psi}\,(\psi_x, \psi_y, \psi_z)$:

$$\boldsymbol{u} = [u, v, w] = \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\psi}. \tag{2}$$

In view of a two-dimensional character of the problem, the vector potential $\boldsymbol{\psi}$ contains only one component ψ_y.

Eq. (2) leads to the following expressions for the components of displacement **u**, described in the layer by potentials $\varphi_s(x, z, t)$, $\psi_y(x, z, t)$ and in the liquid by $\varphi_0(x, z, t)$:
– in the layer

$$u_s = \frac{\partial \varphi_s}{\partial x} - \frac{\partial \psi_y}{\partial z}, \quad w_s = \frac{\partial \varphi_s}{\partial z} - \frac{\partial \psi_y}{\partial x}; \tag{3a}$$

– in the liquid

$$u_0 = \frac{\partial \varphi_0}{\partial x}, \quad w_0 = \frac{\partial \varphi_0}{\partial z}, \tag{3b}$$

where u_s, w_s are components of displacement vector, parallel to the x-axis, w_s, w_0 – components of displacement vector normal to the x-axis.

Potentials φ_s, ψ_y, φ_0, satisfy the following wave equation:

$$\nabla^2 \varphi_s = \frac{1}{c_d^2} \frac{\partial^2 \varphi_s}{\partial t^2} \quad \text{in the layer},$$

$$\nabla^2 \psi_y = \frac{1}{c_t^2} \frac{\partial^2 \psi_y}{\partial t^2} \quad \text{in the layer} \tag{4}$$

$$\nabla^2 \varphi_0 = \frac{1}{c_0^2} \frac{\partial^2 \varphi_0}{\partial t^2} \quad \text{in the liquid}.$$

Eq. (4) is satisfied by any periodic function. By separating the variables with respect to eqs. (4) we obtain the following solution:

$$\varphi_s(x, z, t) = [A_1 \cos k_d z + A_2 \sin k_d z] e^{-jkx} e^{j\omega t},$$

$$\psi_y(x, z, t) = [B_1 \cos k_t z + B_2 \sin k_t z] e^{-jkx} e^{j\omega t}, \tag{5}$$

$$\varphi_0(x, z, t) = E e^{-jk_0 z} e^{-jkx} e^{j\omega t},$$

where

$$k_d^2 = \frac{\omega^2}{c_d^2} - k^2, \quad k_t^2 = \frac{\omega^2}{c_t^2} - k^2, \quad k_0^2 = \frac{\omega^2}{c_0^2} - k^2 \tag{5a}$$

and $c = \omega/k$. The potentials φ_s, ψ_y, φ_0 in (5) describe waves propagating along the x-axis with a phase velocity c and a wavelength λ related to the wave number by relation $k = 2\pi/\lambda$. The frequency f is given by relation $f = \omega/2\pi$.

The normal stress τ_{zz} and shear stress τ_{zx} in the solid can be expressed by potentials φ_s and ψ_y and by elastic constants as follows:

$$\tau_{zz} = \lambda \nabla^2 \varphi_s + 2\beta \left(\frac{\partial^2 \varphi_s}{\partial z^2} + \frac{\partial^2 \psi_y}{\partial x \partial z} \right),$$

$$\tau_{zx} = \eta \left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_y}{\partial z^2} + 2 \frac{\partial^2 \varphi_s}{\partial x \partial z} \right). \tag{6}$$

The acoustic pressure in liquid is given by relation

$$p = -\varrho_c \frac{\partial^2 \varphi_0}{\partial t^2}. \tag{7}$$

3. Boundary conditions and characteristic equation

The solution of the problem should satisfy appropriate boundary conditions posed by the requirement of continuity of stresses and displacements perpendicular to the surface of the layer. These conditions are as follows:

$$\begin{aligned} &\tau_{zz} = -p \ (z=0), \quad \tau_{zx} = 0 \ (z=0), \quad w_s = w_0 \ (z=0), \\ &\tau_{zz} = 0 \ (z=-a), \quad \tau_{zx} = 0 \ (z=-a). \end{aligned} \tag{8}$$

In agreement with the convention used in acoustics [2] it has been assumed that positive tensile stress corresponds to negative pressures. Hence the negative sign appears at acoustic pressure in the first boundary condition (8).

Putting relations (5) into boundary conditions (8) and making use of (3a), (3b), (6) and (7) we obtain a set of five homogeneous equations with unknown coefficients A_1, A_2, B_1, B_2, E. The solution to this set requires the determinant W formed of the coefficients of this set to vanish. This determinant has the form

$$W = \begin{vmatrix} \omega^2\varrho_s - 2\mu k^2 & 0 & 0 & 2\mu jkk_t & -\varrho_c\omega^2 \\ 0 & 2jkk_d & k^2-k_t^2 & 0 & 0 \\ 0 & k_d & -jk & 0 & jk_0 \\ (\omega^2\varrho_s - 2\mu k^2)\cos k_d a & (2\mu k^2 - \omega^2\varrho_s)\sin k_d a & 2\mu jkk_t \sin k_t a & 2\mu jkk_t \cos k_t a & 0 \\ 2jkk_d \sin k_d a & 2jkk_d \cos k_d a & (k^2-k_t^2)\cos k_t a & (k_t - k^2)\sin k_t a & 0 \end{vmatrix}. \tag{9}$$

The characteristic equation is the very condition of vanishing of this determinant and has the form

$$\begin{aligned} &(\omega^2\varrho_s - 2\mu k^2)^2 (k^2-k_t^2)^2 k_0 \sin k_t a \cdot \sin k_d a + \\ &\quad + 4\varrho_c \omega^2 \mu j (k^2+k_t^2)^2 k^2 k_d^2 k_t \cos k_t a \cdot \sin k_d a + \\ &\quad + 16\mu^2 k^4 k_d^2 k_t^2 k_0 \sin k_d a \cdot \sin k_t a + \\ &\quad - (\omega^2\varrho_s - 2\mu k^2)\varrho_c \omega^2 k_d (k^4 - k_t^4) j \sin k_t a \cdot \cos k_d a - \\ &\quad - 8(\omega^2\varrho_s - 2\mu k^2)\mu (k^2-k_t^2) k^2 k_d k_t k_0 + \\ &\quad + 8(\omega^2\varrho_s - 2\mu k^2)\mu (k^2-k_t^2) k^2 k_d k_t k_0 \cos k_d a \cdot \cos k_t a = 0. \end{aligned} \tag{10}$$

If we put $\varrho_c = 0$ in characteristic equation (10) we obtain the characteristic equation for the layer in the vacuum as given by Ewinge and Jardetzky [1].

The characteristic equation (10) determines the relation between the phase velocity c of the wave and the wave number k. The relation between the phase velocity c and the angular frequency ω can be obtained after putting the relation $k = \omega/c$ in (10).

The solutions of (10) can be real or complex. The real values of k satisfying (10) correspond to wave propagation along the x-axis without attenuation. The complex values of k correspond to propagation with attenuation.

The characteristic equation (10) has been solved numerically for the following parameters.

(a) The layer is made of steel:

$$\varrho_s = 7.7 \text{ g/cm}^3, \quad \lambda = 1.07 \times 10^{12} \text{ g/cm}\cdot\text{s}^2,$$
$$\mu = 8.03 \times 10^{11} \text{ g/cm}\cdot\text{s}^2, \quad c_d = 5.9 \times 10^5 \text{ cm/s},$$
$$c_t = 3.23 \times 10^5 \text{ cm/s}.$$

The layer thickness was $a = 0.075$ cm in the first case and $a = 0.010$ cm in the second case. These thickness are equal to the thickness of the needle used for puncturing the body tissue.

(b) Liquid – water:

$$\varrho_c = 1 \text{ g/cm}^3, \quad c_0 = 1.48 \times 10^5 \text{ cm/s}.$$

(c) Frequency:

$$f = 3\text{MHz}.$$

Only the real values of k were taken into account when solving (10) since they correspond to the waves which are not attenuated in the x-direction. Under such an assumptions it has been obtained: for $a = 0.075$ cm, propagation constant $k = 127.4$ cm^{-1}, wave numbers, $k_d = j\cdot 123$ cm^{-1}, $k_t = j\cdot 113$ cm^{-1}, $k_0 = -j\cdot 3.12$ cm^{-1}. Then the phase velocity of the surface wave is

$$c = \omega/k = 1.47840 \times 10^5 \text{ cm/s},$$

i.e., it is slightly below the assumed velocity of wave in water. As it follows from (5a), k_0 is imaginary. Then, by virtue of (5) the first exponent of potential $\varphi_0(x, z, t)$ will be real. Thus the wave considered decays in liquid with the increase of depth z.

The distributions of normal and shear stresses, of the acoustic pressure, and of the components of displacement vector of the propagating wave have been calculated numerically for the above example with the aid of eqs. (6), (7), (3a), (3b) (for $x = t = 0$). These equations, after some elementary transformations, assume the form:

$$\tau_{zz} = \lambda[C_1 \cos k_d a - C_2 \sin k_d a](-k^2 - k_d^2) +$$
$$+ 2\mu[-k_d^2(C_1 \cos k_d a - C_2 \sin k_d a) + (-D_1 \sin k_t a - D_2 \cos k_t a) jkk_t], \quad (11)$$

$$\tau_{zx} = \mu[(D_1 \cos k_t a - D_2 \sin k_t a)(k_t^2 - k^2) + 2jkk_d(-C_1 \sin k_d a - C_2 \cos k_d a)], \quad (12)$$

$$p = -\omega^2 \varrho_c E e^{-jk_0 a},$$
$$w_s = (C_1 \sin k_d a + C_2 \cos k_d a) k_d - (D_1 \cos k_t a - D_2 \sin k_t a) jk,$$
$$u_s = -(C_1 \cos k_d a - C_2 \sin k_d a) jk + (D_1 \sin k_t a + D_2 \cos k_t a) k_t, \tag{13}$$
$$w_0 = -jk_0 E e^{-jk_0 a}, \quad u_0 = -jkE e^{-jk_0 a}.$$

The plots of the stresses and displacements are presented in Figs. 2 and 3.

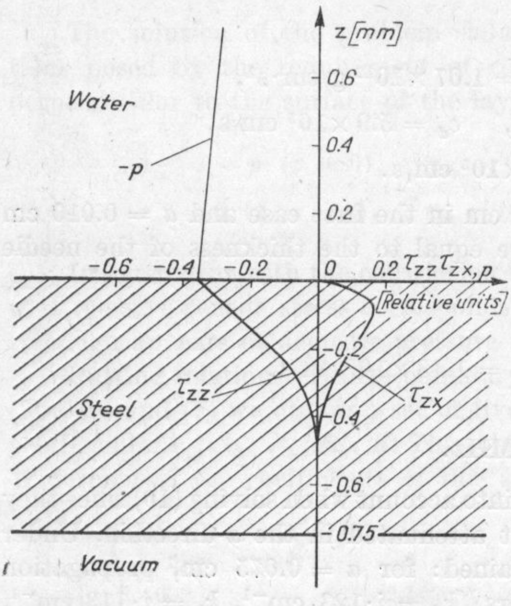

Fig. 2. Distribution of normal stress τ_{zz} and shear stress τ_{zx} in the steel layer and the distribution of acoustic pressure p in water; $f = 3 \times 10^6$ Hz, $a = 0.075$ cm

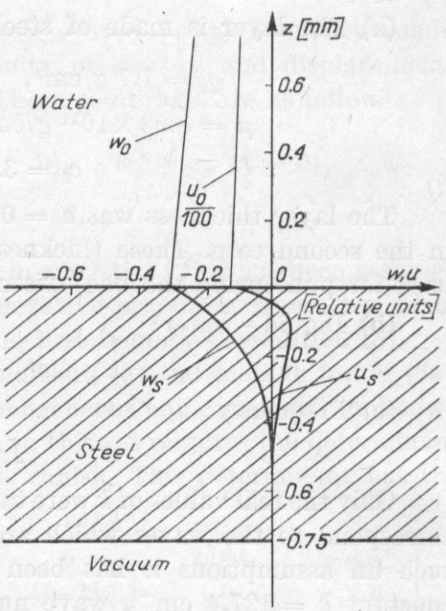

Fig. 3. Distribution of the components of displacement vector in the steel layer and in water for $a = 0.075$ cm

For $a = 0.010$ cm the propagation constant $k = 132$ cm^{-1}, wave numbers $k_d = j \cdot 128$ cm^{-1}, $k_t = j \cdot 118$ cm^{-1}, $k_0 = -j \cdot 34.7$ cm^{-1}. The velocity of surface wave is $c = 1.43 \cdot 10^5$ cm/s. The distributions of acoustic pressure, stresses, and displacements are presented in Figs. 4 and 5.

4. Conclusions

The paper is concerned with the problem of propagation of an elastic, running wave along an infinite solid layer placed between the semispace filled with liquid and the vacuum.

By solving wave equations (4) with boundary conditions (8) the characteristic equation (10) has been obtained. Unlike in the case of the layer in the vacuum [3], this equation cannot be decomposed into two terms describing the symmetric and antisymmetric modes.

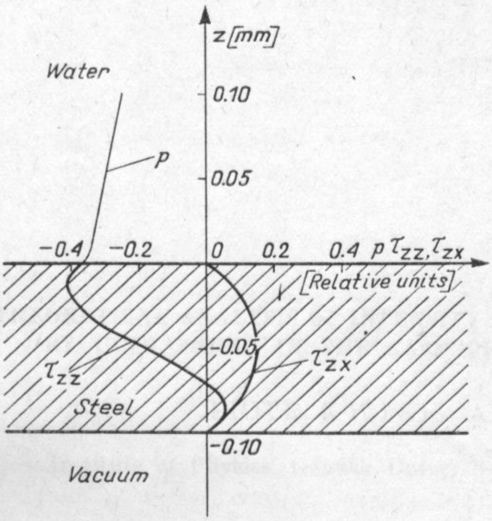

Fig. 4. Distribution of stresses τ_{zz} and τ_{zx} in the steel layer and the distribution of acoustic pressure p in water; $f = 3 \times 10^6$ Hz, $a = 0.010$ cm

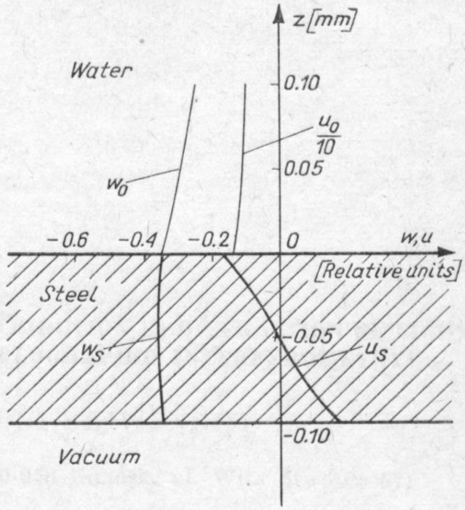

Fig. 5. Distribution of components w and u of displacement vector in the steel plate and in water for $a = 0.010$ cm

The characteristic equation (10) determines the phase velocity c of the wave as a function of the angular frequency ω. The phase velocity c may be real or complex and in a general case depends on the frequency.

In the case considered in this paper it has been assumed that the wave in question should be unattenuated in the direction of its propagation. For steel layers of thicknesses of 0.075 cm and 0.01 cm, equal to the thicknesses of needles used for puncturing the body tissue, and for the assumed wave velocity in water $c_0 = 1.48 \times 10^5$ cm/s it has been found that for a frequency of 3×10^6 Hz the velocities are $c = 1.47840 \times 10^5$ cm/s and $c = 1.42810 \times 10^5$ cm/s for the thick and the thin layer, respectively. These velocities are only slightly lower than c_0. The wave decays exponentially with the increase of the penetration depth in water. The penetration depth of this wave in water is much larger than its penetration depth in the layer. The wave is conducted without attenuation along the solid-liquid interface; its character resembles that of a surface wave.

References

[1] W. Ewing, W. Jardetzky, *Elastic waves in layered media*, McGraw-Hill Book Company, INC, 1957, 281-288, 293-299.

[2] I. Malecki, *Teoria fal i układów akustycznych (The theory of waves and acoustic systems)*, PWN, Warszawa, 1964, 79-82.

[3] M. Redwood, *Mechanical wave-guides*, Pergamon Press, Oxford 1960, 119-121.

Received on 15th November 1977