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THE BLURRED CUT-OFF FREQUENCY OF ACOUSTIC HORNS

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On the basis of discussion of the Webster equation the conditions and possibilities of a blurred cut-off frequency for horns of arbitrary geometry were analyzed. The transmission properties of horns in the blurred region were discussed. Subsequently the blurred cut-off frequency was considered for hyperbolic horns of annular cross-section.

1. Introduction

The notion of the cut-off frequency of a horn occurs in the theory of acoustic horns in a discussion of the so-called Webster equation [7, 8]. This equation describes the wave motion in the horn with the simplifying assumptions that the propagating wave is plane, harmonic, of infinitely small amplitude, and propagates without energy losses. In the case when the geometrical axis of the horn coincides with the axis of the abscissa, the Webster equation written in the reduced form is the following [8]

$$\frac{d^2F}{da^2}+\left(\mu^2-\frac{1}{\varrho}\frac{d^2\varrho}{da^2}\right)F = 0, \tag{1}$$

where F is the wave function [1] related to the sound pressure p and the area of cross section of the horn S by the formula

$$p = \frac{F}{\sqrt{S}}, \tag{2}$$

a is the dimensionless abscissa determined by the formula

$$a = \frac{x}{x_0}, \tag{3}$$

with x_0 being the coefficient of divergence of the walls of the horn; μ is dimensionless frequency

$$\mu = \frac{f}{f_0}, \tag{4}$$

where f is the absolute frequency and f_0 is a constant with the dimensions of frequency defined by the relation

$$f_0 = \frac{c}{2\pi x_0}, \tag{5}$$

with c being adiabatic velocity of the acoustic wave; ϱ is the dimensionless radius of the cross-section of the horn defined by the formula

$$\varrho = \sqrt{\frac{S}{S_0}}, \tag{6}$$

with S_0 being the area of the horn inlet.

The form of the solution of the wave equation (1) depends on the sign of the expression in the brackets by the function F. If this expression is larger than zero, which occurs when $\mu^2 < \varrho^{-1} d^2\varrho/d\alpha^2$, then the function F becomes periodic. However, in the case where the term in the brackets is less than zero, F is an aperiodic function. This occurs when $\mu^2 < \varrho^{-1}d^2\varrho/d\alpha^2$. The function F is connected with the acoustic pressure by formula (2) and describes the wave motion in the horn; it should, therefore, be a periodic function. It should be stated, therefore, that for the frequencies which satisfy the condition $\mu^2 > \varrho^{-1} d^2\varrho/d\alpha^2$ the wave motion occurs in the horn, while for the frequencies for which $\mu^2 < \varrho^{-1} d^2\varrho/d\alpha^2$ this motion does not occur and the horn does not guide acoustic waves. The boundary between these two frequency ranges can be obtained by equating the above mentioned expression, which occurs in formula (1), to zero. Then one obtains the cut-off frequency of the horn, μ_{gr}, which was mentioned in the introduction and below which the wave motion in the horn will decay

$$\mu_{gr} = \sqrt{\frac{1}{\varrho}\frac{d^2\varrho}{d\alpha^2}}. \tag{7}$$

It can be seen from formulae (6) and (7) that the cut-off frequency μ_{gr} depends on the geometry of the horn. For the horns discussed so far in the literature the expression under the square root sign in formula (7) took a constant value, which implied a constant value for μ_{gr}. For example, in the case most frequently described and used practically, Salmon's horn, this expression is equal to unity [5, 6, 8],

$$\frac{1}{\varrho}\frac{d^2\varrho}{d\alpha^2} = 1. \tag{8}$$

Thus the dimensionless cut-off frequency μ_{gr} is also equal to unity, and from (4) it can be seen that in this particular case the absolute cut-off frequency is equal to the constant f_0.

Generally, however, the expression under the square root sign in formula (7) must be a function of the position on the axis of the horn. This function will be subsequently denoted by $V_{(a)}$ in this paper,

$$V_{(a)} = \frac{1}{\varrho}\frac{d^2\varrho}{da^2}. \tag{9}$$

In this case the frequency μ_{gr} also becomes a function of position and extends over a certain frequency range for a horn of a given length. This is the so called blurred cut-off frequency. Analysis of this phenomenon is the subject of the present paper.

2. Analysis of the transmission properties of a horn in the blurred cut-off frequency region

Let us assume that $V_{(a)}$ is a continuous function. Moreover, we shall assume that the function is monotonic* and consider the case where $V_{(a)}$ decreases. This case is illustrated by Fig. 1 for a horn of length a_l, whose inlet was placed at the origin of the coordinate system.

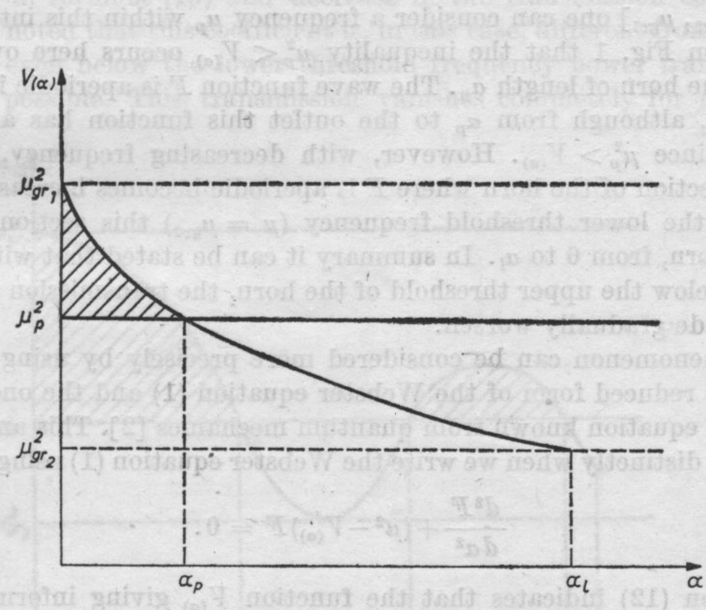

Fig. 1. The monotonically decreasing function $V_{(a)}$ as a "geometric barrier" of the horn for waves at frequencies $\mu < \mu_{gr1}$

* assumption of monotonicity of the function $V_{(a)}$ corresponds to horns used in practice.

Fig. 1 shows that for frequencies higher than μ_{gr_1} over the whole length of the horn $\mu^2 > V_{(a)}$. According to what has been said in section 1, this signifies that wave motion occurs in the horn for $\mu > \mu_{gr1}$. Below the frequency μ_{gr2} over the whole length of the horn the relation $\mu^2 < V_{(a)}$ is satisfied, which is a lack of periodicity in the wave function F and a decay of wave motion. However, at frequencies in the interval from μ_{gr2} to μ_{gr1} the function F is periodic only over some sections of the horn: this is the interval of the blurred cut-off frequency of the horn. Accordingly, it is proposed that the quantities μ_{gr2} and μ_{gr1}, which are the limits of this interval, should be called the lower and upper threshold frequencies of the horn. In the present case μ_{gr1} and μ_{gr2} can be determined from formula (7) by insertion into it of the dimensionless value of the abscissa of the inlet ($a = 0$) and the outlet ($a = a_1$) of the horn

$$\mu_{gr1} = \sqrt{\frac{1}{\varrho_{(0)}} \varrho''_{(0)}}, \tag{10}$$

$$\mu_{gr2} = \sqrt{\frac{1}{\varrho_{(a_l)}} \varrho''_{(a_l)}}, \tag{11}$$

where the dashes denote differentiation with respect to the dimensionless abscissa a.

In order to consider more closely the phenomena occurring in the frequency interval $[\mu_{gr2}, \mu_{gr1}]$ one can consider a frequency μ_p within this interval. It can be seen from Fig. 1 that the inequality $\mu_p^2 < V_{(a)}$ occurs here over the inlet section of the horn of length a_p. The wave function F is aperiodic in this region of the horn, although from a_p to the outlet this function has an oscillatory character, since $\mu_p^2 > V_{(a)}$. However, with decreasing frequency, for $\mu < \mu_p$, the outlet section of the horn where F is aperiodic becomes increasingly longer. Finally, at the lower threshold frequency ($\mu = \mu_{gr2}$) this section expands to the whole horn, from 0 to a_l. In summary it can be stated that with increasing frequency below the upper threshold of the horn, the transmission properties of the waveguide gradually worsen.

This phenomenon can be considered more precisely by using the analogy between the reduced form of the Webster equation (1) and the onedimensional Schrödinger equation known from quantum mechanics [2]. This analogy occurs particularly distinctly when we write the Webster equation (1) using equation (9)

$$\frac{d^2F}{da^2} + (\mu^2 - V_{(a)})F = 0. \tag{12}$$

Equation (12) indicates that the function $V_{(a)}$ giving information about the geometry of the horn is an analogue of the potential energy function in quantum theory. Thus, the problem of the transmission properties of the horn in the interval $[\mu_{gr2}, \mu_{gr1}]$ is analogous to the problem of the penetration of particles through the potential barrier in quantum mechanics. Since, as in quantum mechanics, one can speak of an energy barrier (potential barrier) for elementary particles, one can now speak of a "geometric" barrier formed by the horn for a wave of a given wavelength. The dashed region in Fig. 1 can, therefore, be considered to be a measure of the size of the barrier for a wave of a given frequency $\mu = \mu_p$.

Using the WKB approximation known from quantum mechanics [2, 3] one can, taking into consideration the analogy mentioned above, use the formula that is employed in the WKB method for the coefficient, D_t, of transmission through the potential barrier. In the present case it will be the coefficient of power transmission by the horn. For example, when $\mu = \mu_p$ and μ_p^2 lies below the peak of the barrier the formula for D_t has the form

$$D_t = \exp\left[-2\int_0^{a_p} \sqrt{V_{(a)} - \mu_p^2}\, da\right]. \tag{13}$$

It can be seen from Fig. 1 that the size of the barrier increases with decreasing frequency, since the barrier becomes increasingly higher and wider. This is accompanied by an increase in the integral in formula (13) and accordingly a decrease in the transmission, coefficient D_t. Below the lower threshold frequency, the barrier no longer changes its width which is a_l, but its height continues to increase with decreasing μ. This causes a further increase in the value of the integral in formula (13) and decrease in the transmission coefficient D_t. It should be noted that this coefficient is, in this case, different from zero, which means that even below the lower threshold frequency power transmission by the horn is possible. This transmission vanishes completely for $\mu < \mu_{gr2}$ only

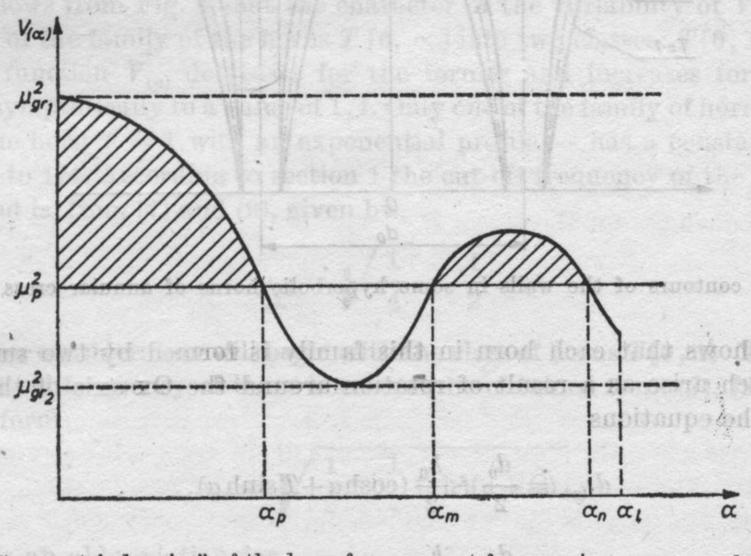

Fig. 2. The "geometric barrier" of the horn for waves at frequencies $\mu < \mu_{gr1}$ when $V_{(a)}$ is not a monotonic function

when the horn is a waveguide of infinite length, since in this case a_l tends to infinity and the value of the integral in formula (13) also becomes infinitely large, resulting in the coefficient D_l taking a value of zero.

In the case where $V_{(a)}$ is an increasing function, analogous considerations can be made. The more general version, where $V_{(a)}$ is a continuous function, but is not monotonic, does not contribute any new elements to the problem under consideration and only requires more development in terms of calculation. Thus, for example, for the function $V_{(a)}$ shown in Fig. 2 it has to be considered in evaluating formula (13) that the "geometric" barrier of the horn for a wave of frequency μ_p occurs not only in the interval $[0, a_p]$, but also in the interval $[a_m, a_n]$.

3. Blurred cut-off frequency of hyperbolic horns of annular cross section

In order to illustrate the general considerations in section 2 one can discuss the phenomenon of the blurred cut-off frequency for the family of hyperbolic horns of annular cross section.

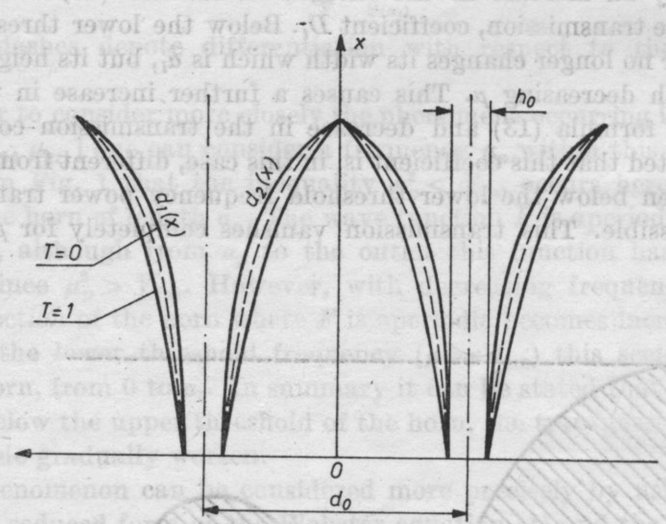

Fig. 3. The contours of the walls in some hyperbolic horns of annular cross section

Fig. 3 shows that each horn in this family is formed by two surfaces of rotation which arise as a result of rotation around the Ox axis of the curves defined by the equations

$$d_{1(x)} = \frac{d_0}{2} + \frac{h_0}{2}(\cosh a + T \sinh a), \tag{14}$$

$$d_{2(x)} = \frac{d_0}{2} - \frac{h_0}{2}(\cosh a + T \sinh a), \tag{15}$$

where d_0 is the central diameter of the annular channel of the horn, and h_0 is the width of the channel of the horn at the inlet. The constant T defines the shape of the profile of the walls of the horn. For the family of waveguides considered here this constant lies in the interval $[0, \infty)$. For $T = 0$ the horn has a catenoidal profile, and an exponential one for $T = 1$. The profile of the horn takes other shapes for other values of T, one of which is shown as an example by the dashed line in Fig. 3.

It follows from geometrical considerations that the area of the cross section S of the family of waveguides under discussion is defined by the formula

$$S = S_0(\cosh a + T \sinh a), \tag{16}$$

where

$$S_0 = \pi d_0 h_0 \tag{17}$$

is the area of the inlet of the horn.

The horns of the form (16) have not been discussed to date in the literature, although they have been used in practice, e.g. in axial dynamic flow generators [4].

The expression for the function $V_{(a)}$ can be obtained from formulae (6), (16) and (9)

$$V_{(a)} = \frac{1}{2} - \frac{1}{4}\left(\frac{\sinh a + T\cosh a}{\cosh a + T \sinh a}\right)^2. \tag{18}$$

The behaviour of the function $V_{(a)}$ for different values of the parameter T is shown in Fig. 4.

It follows from Fig. 4 that the character of the variability of $V_{(a)}$ suggests a division of the family of the horns $T\,[0, \infty]$ into two classes: $T[0, 1)$ and $T(1, \infty)$. The function $V_{(a)}$ decreases for the former and increases for the latter tending asymptotically to a value of 1/4. Only one of the family of horns discussed here – the horn $T = 1$ with an exponential profile – has a constant value of $V_{(a)}$ equal to 1/4. According to section 1 the cut-off frequency of the horn is not blurred and is, from (7) and (9), given by

$$\mu_{gr} = \sqrt{\frac{1}{4}} = \frac{1}{2}. \tag{19}$$

Further consideration will begin with the class of horns $T[0, 1)$. The formula for the cut-off frequency of these horns can be obtained from (7), (9) and (18). It has the form

$$\mu_{gr} = \sqrt{\frac{1}{2} - \frac{1}{4}\,\mathrm{tgh}^2(a + \Omega)}\,, \tag{20}$$

where Ω is an abbreviation for

$$\Omega = \mathrm{artgh}\, T. \tag{21}$$

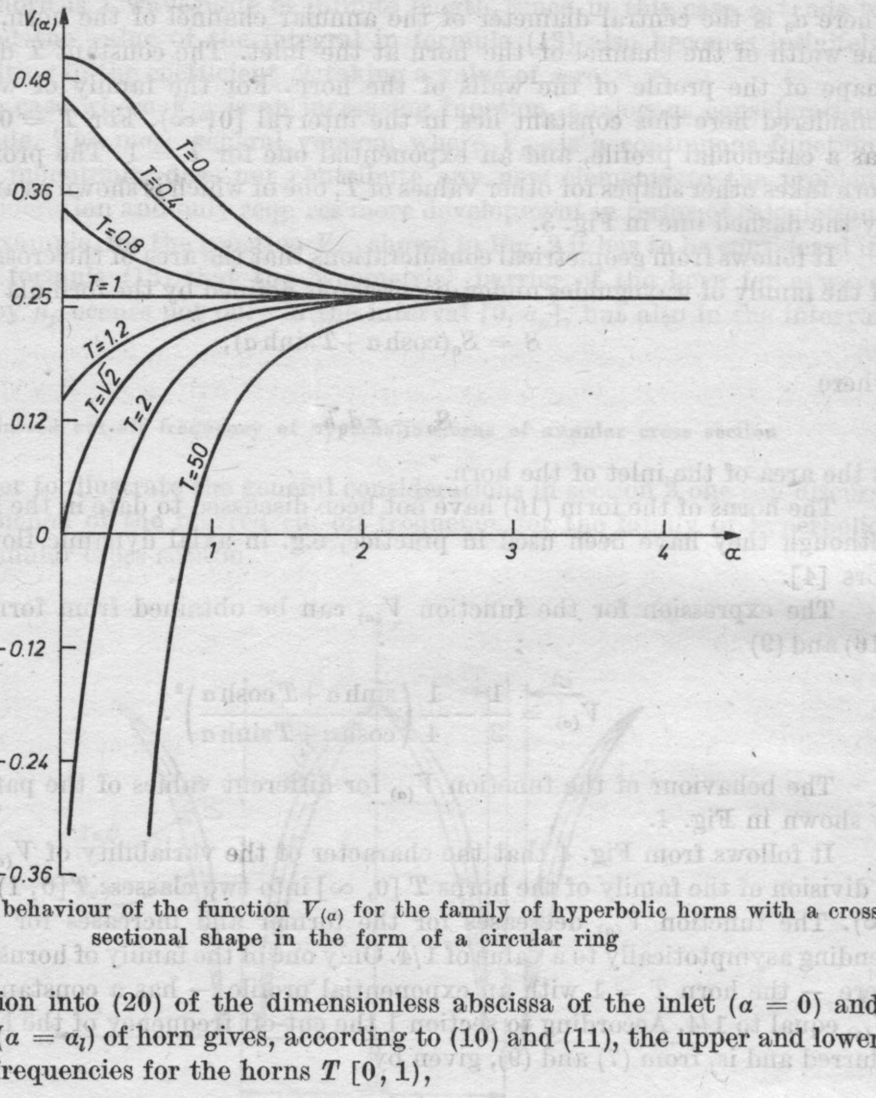

Fig. 4. The behaviour of the function $V_{(\alpha)}$ for the family of hyperbolic horns with a cross sectional shape in the form of a circular ring

Insertion into (20) of the dimensionless abscissa of the inlet ($\alpha = 0$) and the outlet ($\alpha = \alpha_l$) of horn gives, according to (10) and (11), the upper and lower threshold frequencies for the horns $T\,[0, 1)$,

$$\mu_{\mathrm{gr1}} = \sqrt{\frac{1}{2} - \frac{T^2}{4}}, \tag{22}$$

$$\mu_{\mathrm{gr2}} = \sqrt{\frac{1}{2} - \frac{1}{4}\,\mathrm{tgh}^2(\alpha_l + \Omega)}. \tag{23}$$

In the case when the dimensionless length of the horn α_l is so large that the hyperbolic tangent in formula (23) can, to a good approximation, be taken as equal to unity, formula (23) can be written in the simpler form

$$\mu_{\mathrm{gr2}} \cong \sqrt{\frac{1}{4}} = \frac{1}{2}. \tag{24}$$

The same value is obtained here as in the particular case of the horn $T = 1$ (cf. formula (19)). It should be noted, however, that an equality of the dimensionless frequencies does not signify the equality of their absolute values, since the horns with different profiles most frequently have differing coefficients of the divergence of the walls, x_0, which leads (cf. formula (5)) to differences in the values of the constant f_0. This constant which was introduced when μ was defined (formula (4)) must be considered in the conversion of the dimensionless frequencies into absolute frequencies and vice versa.

Formulae (22) and (23) indicate that the upper and lower dimensionless threshold frequencies are less than unity for the family of the horns $T\,[0, 1)$. Thus it can be seen from (4) that both the absolute upper threshold frequency $f_{gr1} = \mu_{gr1} f_0$, and the absolute lower threshold frequency $f_{gr2} = \mu_{gr2} f_0$ are, for these horns, always lower than f_0. This fact differentiates these horns distinctly from the Salmon family of horns, that are similar to them in terms of geometry, [6], but whose cut-off frequency is not blurred and is equal to the constant f_0. Moreover, it can be seen from formulae (22) and (23) that the width of the blurred interval of the cut-off frequency $[\mu_{gr2}, \mu_{gr1}]$, for a horn of a given length depends on the parameter T which characterizes the shape of the profile of the horn. The horn $T = 0$ has the most blurred cut-off frequency.

In order to investigate the phenomena for values of T higher than unity one needs consider the family of horns $T\,(1, \infty)$. In this case, after consideration of formulae (7), (9), (18) one obtains the formula for the cut-off frequency

$$\mu_{gr} = \sqrt{\frac{1}{2} - \frac{1}{4}\operatorname{ctgh}^2(\alpha + \hat{\Omega})}, \tag{25}$$

where $\hat{\Omega}$ is an abbreviation for

$$\hat{\Omega} = \operatorname{artgh}\left(\frac{1}{T}\right). \tag{26}$$

The threshold frequencies can be found from relation (25) using (10) and (11)

$$\mu_{gr1} = \sqrt{\frac{1}{2} - \frac{1}{4}\operatorname{ctgh}^2\hat{\Omega}}, \tag{27}$$

$$\mu_{gr2} = \sqrt{\frac{1}{2} - \frac{1}{4}\operatorname{ctgh}^2(\alpha_l + \hat{\Omega})}. \tag{28}$$

It can be noted that in the case analyzed here, contrary to the situation for horns of the class $T\,[0, 1)$, μ_{gr1} is lower than μ_{gr2}. Thus the magnitude of μ_{gr1}, which for the horns $T\,[0, 1)$ played the role of the upper threshold frequency, now becomes the lower threshold frequency, while μ_{gr2} changes conversely. Both these quantities decrease with increasing T. It follows from formulae (26) and (27) that for $T = \sqrt{2}$, the lower threshold frequency μ_{gr1} is equal to zero, while the upper threshold frequency μ_{gr2} is equal to zero for $T = T_l$ which satisfies the equation

$$\operatorname{ctgh}^2\left[a_l + \operatorname{artgh}\left(\frac{1}{T_l}\right)\right] = 2. \tag{29}$$

In order to determine T_l from formula (29) it is necessary to find first the relation of the quantities a_l and T from relation (16). After manipulation one obtains

$$a_l = \frac{l}{x_0} = \operatorname{arsinh}\left(\frac{S_w}{S_0\sqrt{T^2-1}}\right) - \operatorname{artgh}\left(\frac{1}{T}\right), \tag{30}$$

where S_w is the area of the outlet of the horn.

Insertion of (30) into (29) leads to

$$T_l = \sqrt{\left(\frac{S_w}{S_0}\right)^2 + 1}. \tag{31}$$

A negative number is obtained under the square root sign in formula (25) for $T > T_l$, which signifies an imaginary value of μ_{gr}. Since an imaginary value of the cut-off frequency would be physically meaningless, it must therefore be concluded that for $T > T_l$ the horn transmits all the frequencies of the wave.

Finally it is possible to give a numerical example which shows the dependence of the absolute upper and lower threshold frequencies on the parameter T, for a horn of specified dimensions (cf. Fig. 5).

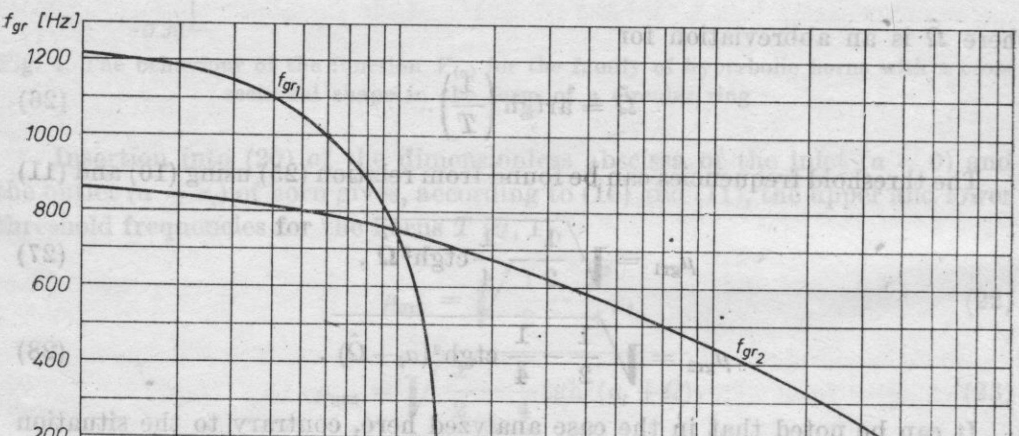

Fig. 5. The dependence of the upper and lower threshold frequency on the value of T for specific diameters of the inlet and the outlet, and a specific length of the horn

It was assumed for the calculations for Fig. 5 that, irrespective of the change in the shape of the profile T, the sizes of the inlet and the outlet of the horn and its length are constant and given by:
- the width of the channel of the horn at the inlet $h_0 = 1.5 \cdot 10^{-3}$ m;
- the central diameter of the annular channel of the horn $d_0 = 10^{-1}$ m;
- the radius of the outlet of the horn $h_l = 10^{-1}$ m;
- the length of the horn $l = 1.5 \cdot 10^{-1}$ m.

Fig. 5 thus shows a special case of the results of the general considerations of this section for the family of hyperbolic horns with annular cross section. It can be seen that with increasing T in the interval $(0, 1)$, the blurred interval of the cut-off frequency becomes narrower and at the same time the upper and lower threshold frequencies decrease. The upper threshold frequency decreases more rapidly and accordingly, for $T = 1, f_{gr1} = f_{gr2}$. For $T > 1$ the threshold frequencies change places: f_{gr1} is now the lower and f_{gr2} the upper threshold frequency. Both f_{gr1} and f_{gr2} tend, in this case, to zero with increasing T. The lower threshold frequency f_{gr1} reaches a value of zero for $T = \sqrt{2}$, while the upper threshold frequency f_{gr2} does so for $T = T_l$, where T_l is defined by formula (31). For $T > T_l$ the horn should transmit all the frequencies of the waves propagating in it.

4. Conclusions

The phenomenon of the blurred cut-off frequency occurs in horns for which the function $V_{(a)}$ in the propagation equation (12) does not take a constant value. These horns are used in practice. For specific dimensions of the horn (the diameter of the inlet and of the outlet, and the length) the blurred cut-off frequency depends on the profile of the walls of the waveguide.

The equation of wave propagation in the horn has an analogous form to a onedimensional Schrödinger equation in quantum mechanics. Because of this formal analogy, the blurred cut-off frequency of the horn and its transmission properties in the blurred interval can be determined only by way of discussion and not by solving the propagation equation.

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