ARCHIVES OF ACOUSTICS
7, 1, 61–70 (1982)

ULTRASONIC WAVE PROPAGATION IN A LAYERED MEDIUM UNDER DIFFERENT BOUNDARY CONDITIONS

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This paper presents an analysis of the possibility of using the phase velocity measurements of surface or plate waves for the evaluation of the adhesive bond strength, i.e. of the adhesion degree in layered joints. Dispersive curves are determined for phase velocities in layer on base and layer on layer systems with two kinds of boundary conditions, i.e. welded and smooth ones, by numerical solutions of the characteristic equations. The procedure of deriving these equations for any number of layers is given.

1. Introduction

Ultrasonic methods using the phenomenon of ultrasonic wave propagation in elastic layered media area can be used not only for detecting the unbounded but also for evaluating the bond strength [1].

Seismologists and geophysicists have long been interested in elastic waves propagating in layered media [2-4]. Waves propagating parallel to boundary surfaces in layered media can, for the sake of simplicity and by analogy to nondestructive testing terminology, be called below surface waves in the case of a layer, or layers on a base, and plate waves in the case of one or more solid layers. A base means a solid medium of thickness exceeding several times the wavelength of a surface mode, while a solid medium of thickness comparable with the wavelength of a surface mode is regarded as a layer. For example, a metal sheet glued to thick rubber involves surface waves, while a simple lap adhesive joint involves plate waves. Both can, however, have a common mathematical approach (cf. next section).

The previous attempts at using surface waves [5] or plate waves [6] for evaluating the bond strength of adhesive joints consisted in the measurements of decay in these waves after they have passed through a controlled section. The aim of the present paper is to analyze the possibility of evaluating the adhesion degree in layered joints made by different techniques, on the basis of velocity measurements of waves propagating along the connection. For this purpose dispersive curves of phase velocity can be determined numerically for different boundary conditions. The knowledge of these characteristics permits not only the estimation of the sensitivity of the acoustic parameter, i.e. the phase velocity, to changes in the strength of a connection, but also makes it easier to conduct a purposive experimental research.

2. Characteristic equation

Mathematically, this problem can be formulated in the following way for flat parallel layers (homogeneous, isotropic and ideally elastic media). The solutions of the twodimensional wave equations [7] are sought

$$\nabla^2\varphi = \frac{1}{c_L^2}\varphi_{,tt}, \quad \nabla^2\psi = \frac{1}{c_T^2}\psi_{,tt}, \quad \nabla^2 = \partial_{,xx}+\partial_{,zz}; \tag{1}$$

with the assumption that their solutions, i.e. their scalar potentials φ and ψ are sought in the form

$$\varphi(x, z, t) = \varphi^*(z)\exp[ik(x+ct)], \quad \psi(x, z, t) = \psi^*(z)\exp[ik(x+ct)]. \tag{2}$$

This signifies that the wave is harmonic in time and moves in the negative direction of the x axis. In these formulae c_L and c_T are the velocities of longitudinal and transverse waves, respectively, while k is the wave number $k = \omega/c$. Insertion of formulae (2) into (1) gives the simple differential equations

$$(\partial_{,zz}-k^2s^2)\varphi^* = 0, \quad (\partial_{,zz}-k^2q^2)\psi^* = 0, \tag{3}$$

where

$$s = [1-(c/c_L)^2]^{1/2}, \quad q = [1-(c/c_T)^2]^{1/2}. \tag{4}$$

Therefore, the general solution of equations (1) for the mth layer (Fig. 1) can be given in the form

$$\varphi_m = [A_{4m-3}\cosh(ksz)+A_{4m-2}\sinh(ksz)]\exp[ik(x+ct)], \tag{5}$$

$$\psi_m = [A_{4m-1}\cosh(kqz)+A_{4m}\sinh(kqz)]\exp[ik(x+ct)] \tag{6}$$

while in the half-space (medium $n+1$) the solution of equations (1) can be given as

$$\varphi_{n+1} = A_{4n+1}\exp(-ksz)\exp[ik(x+ct)], \tag{7}$$

$$\psi_{n+1} = A_{4n+2}\exp(-kqz)\exp[ik(x+ct)]. \tag{8}$$

The displacements u and w and also the stresses σ_{zz} and σ_{xz} are related to the functions φ and ψ by the relations

$$u = \varphi_{,x} - \psi_{,z}, \quad w = \varphi_{,z} + \psi_{,x}, \tag{9}$$

$$\sigma_{zz} = G[(c_L/c_T)^2 - 2]\varphi_{,xx} + G(c_L/c_T)^2\varphi_{,zz} + 2G\psi_{,xz}, \tag{10}$$

$$\sigma_{xz} = 2G\varphi_{,xz} - G\psi_{,zz} + G\psi_{,xx}. \tag{11}$$

Insertion of (5) and (6) and also of (7) and (8) into (9)-(11) gives

$$u_m = \{ik[A_{4m-3}\cosh(ks_m z) + A_{4m-2}\sinh(ks_m z)] - -kq_m[A_{4m-1}\sinh(kq_m z) + A_{4m}\cosh(kq_m z)]\}\exp[i(\omega t + kx)], \tag{12}$$

$$w_m = \{ks_m[A_{4m-3}\sinh(ks_m z) + A_{4m-2}\cosh(ks_m z)] + +ik[A_{4m-1}\cosh(kq_m z) + A_{4m}\sinh(kq_m z)]\}\exp[i(\omega t + kx)], \tag{13}$$

$$(\sigma_{zz})_m = \{G_m k^2 r_m[A_{4m-3}\cosh(ks_m z) + A_{4m-2}\sinh(ks_m z)] + +i2G_m k^2 q_m[A_{4m-1}\sinh(kq_m z) + A_{4m}\cosh(kq_m z)]\}\exp[i(\omega t + kx)], \tag{14}$$

$$(\sigma_{xz})_m = \{i2G_m k^2 s_m[A_{4m-3}\sinh(ks_m z) + A_{4m-2}\cosh(ks_m z)] - -G_m k^2 r_m[A_{4m-1}\cosh(kq_m z) + A_{4m}\sinh(kq_m z)]\}\exp[i(\omega t + kx)], \tag{15}$$

$$u_{n+1} = [A_{4n+1} ik\exp(-ks_{n+1}z) + A_{4n+2} kq_{n+1}\exp(-kq_{n+1}z)]\exp[i(\omega t + kx)], \tag{16}$$

$$w_{n+1} = [-A_{4n+1} ks_{n+1}\exp(-ks_{n+1}z) + A_{4n+2} ik\exp(-kq_{n+1}z)]\exp[i(\omega t + kx)], \tag{17}$$

$$(\sigma_{zz})_{n+1} = [A_{4n+1}G_{n+1}k^2 r_{n+1}\exp(-ks_{n+1}z) - A_{4n+2}2G_{n+1}ik^2 q_{n+1}\exp(-kq_{n+1}z)\times \times\exp[i(\omega t + kx)], \tag{18}$$

$$(\sigma_{xz})_{n+1} = [-A_{4n+1}2G_{n+1}ik^2 s_{n+1}\exp(-ks_{n+1}z) - A_{4n+2}G_{n+1}k^2 r_{n+1}\exp(-kq_{n+1}z)]\times \times\exp[i(\omega t + kx)]. \tag{19}$$

In these formulae the following quantity was introduced

$$r = 2 - (c/c_T)^2. \tag{20}$$

Subsequently, using the relevant boundary conditions for the coordinate $z_0 = -(h_1 + \ldots + h_m + \ldots + h_n), \ldots, z_m = -(h_{m+1} + \ldots + h_n), \ldots, z_n = 0$ given in Table 1, a system of homogeneous linear equations can be obtained,

$$\sum_{j=1}^{4n+2(4n)} a_{ij}A_j = 0 \qquad [i = 1, 2, \ldots, (4n), \ldots, (4n+2)], \tag{21}$$

where for n layers on base there are $(4n+2)$ equations with $(4n+2)$ unknowns and for n layers $(4n)$ equations with $(4n)$ unknowns.

System (21) has a non-trivial solution when

$$D_n = \det[a_{ij}] = 0. \tag{22}$$

The determinant D_n is of order $(4n+2)$ or $(4n)$. The matrix $[a_{ij}]$ can be divided into rectangular matrices in the following way

$$[a_{ij}] = \begin{pmatrix} [a_{ij}]_0 & [0] & & \\ [0] & [a_{ij}]_m & [0] & \\ & [0] & [a_{ij}]_n & \end{pmatrix}, \tag{23}$$

where the particular matrices result from the assumption of relevant boundary conditions (Table 1).

Table 1. Boundary conditions for the particular matrices

Matrices		Boundary conditions	
symbol	dimen-sions	welded	smooth
free surface $[a_{ij}]_0$	2×4	$(\sigma_{xz})_1 = (\sigma_{zz})_1 = 0$	$(\sigma_{xz})_1 = (\sigma_{zz})_1 = 0$
mth interface $[a_{ij}]_m$	4×8	$(\sigma_{xz})_m = (\sigma_{xz})_{m+1}$ $(\sigma_{zz})_m = (\sigma_{zz})_{m+1}$ $u_m = u_{m+1}$ $w_m = w_{m+1}$	$(\sigma_{xz})_m = 0$ $(\sigma_{zz})_m = (\sigma_{zz})_{m+1}$ $(\sigma_{xz})_{m+1} = 0$ $w_m = w_{m+1}$
nth interface $[a_{ij}]_n$	4×6	$(\sigma_{xz})_n = (\sigma_{xz})_{n+1}$ $(\sigma_{zz})_n = (\sigma_{zz})_{n+1}$ $u_n = u_{n+1}$ $w_n = w_{n+1}$	$(\sigma_{xz})_n = 0$ $(\sigma_{zz})_n = (\sigma_{zz})_{n+1}$ $(\sigma_{xz})_{n+1} = 0$ $w_n = w_{n+1}$
free surface $[a_{ij}]_n$	2×4	$(\sigma_{xz})_n = (\sigma_{zz})_n = 0$	$(\sigma_{xz})_n = (\sigma_{zz})_n = 0$

For the purposes of the present work, after NICKERSON's suggestion [8], boundary conditions can be divided into welded and smooth. The welded conditions assume a continuity of displacements and stresses, both normal and tangent, corresponding to an ideal connection of two solids (welded contact). The smooth conditions allow a decrease in stresses tangent to boundary conditions, i.e. correspond to smooth contact. Such a case can be conceived, after ACHENBACH [9], as two solids separated by an inviscid liquid of infinitely small thickness. It can be assumed that real bonds of different adhesion degree fall between these two extreme cases of boundary conditions.

Therefore, using formulae (12)-(20) and Table 1, a determinant can be determined for any layered joint with different boundary conditions. Equating this determinant to zero leads to a characteristic equation from which the phase velocity can be determined for predetermined values of the density ϱ, the modulus of longitudinal elasticity E, the modulus of transverse elasticity G, thicknesses of the particular layers and frequency.

For example, for a single layer ($n = 1$), from Table 1 boundary conditions are chosen only for free surfaces, giving the determinant $\det[a_{ij}]$ of the fourth order, which when equated to zero becomes a characteristic equation for the problem solved by LAMB [10].

From the point of view of the evaluation of adhesion in layered joints, the following cases are of most interest here: layer on base, layer on layer, two layers on base and three layers. The first two cases correspond to bimetals, while the other two to adhesive bonded joints or soldered joints.

The present paper gives a schematic procedure for determination of a characteristic equation in the case of a flat parallel layer on base and of two layers, for two kinds of boundary conditions. Using the KEILIS-BOROK [4] notation these equations can be written as

$$D_1^{(1,2)} = 0 \quad \text{and} \quad D_2^{(1,2)} = 0,$$

where the lower index denotes the number of layers, the index in brackets the subsequent layers or, possibly, the base, while D is the determinant (see equation (22)).

3. Layer on base $[D_1^{(1,2)} = 0]$

(a) *Welded contact.* The starting point for derivation of the characteristic equation in the case of layer on base for welded boundary conditions is, according to Table 1, the six boundary conditions

$$(\sigma_{xz})_1 = (\sigma_{zz})_1 = 0 \tag{24}$$

for $z = z_0 = -h$;

$$(\sigma_{xz})_1 = (\sigma_{xz})_2, \qquad (\sigma_{zz})_1 = (\sigma_{zz})_2, \tag{25}$$

$$u_1 = u_2, \qquad w_1 = w_2 \tag{26}$$

for $z = z_1 = 0$.

Writing the stresses and displacements occurring in equations (24)-(26) by means of formulae (12)-(20) gives a system of six homogeneous equations relative to the six unknowns $A_1, A_2, \ldots, A_6$. By forming the determinant from indices of the unknowns and equating it to zero, after slight transformations [11], the characteristic equation can be obtained

$$\begin{vmatrix} -2s_1\sinh S_1 & 2\cosh S_1 & -r_1\cosh Q_1 & r_1\dfrac{\sinh Q_1}{q_1} & 0 & 0 \\ r_1\cosh S_1 & -r_1\dfrac{\sinh S_1}{s_1} & 2q_1\sinh Q_1 & -2\cosh Q_1 & 0 & 0 \\ 0 & 2 & -r_1 & 0 & 2gs_2 & gr_2 \\ r_1 & 0 & 0 & -2 & -gr_2 & -2gq_2 \\ 1 & 0 & 0 & -1 & -1 & -q_2 \\ 0 & 1 & -1 & 0 & s_2 & 1 \end{vmatrix} = 0, \tag{27}$$

where for the sake of brief notation new symbols were introduced; namely

$$S_1 = -ks_1z_0, \quad Q_1 = -kq_1z_0, \quad g = G_2/G_1. \tag{28}$$

(b) *Smooth contact.* In this case the starting point for derivation of the characteristic equation can be, according to Table 1, the following six boundary conditions

$$(\sigma_{xz})_1 = (\sigma_{zz})_1 = 0 \tag{29}$$

for $z = z_0 = -h$;

$$(\sigma_{xz})_1 = (\sigma_{xz})_2 = 0, \tag{30}$$

$$(\sigma_{zz})_1 = (\sigma_{zz})_2, \quad w_1 = w_2 \tag{31}$$

for $z = z_1 = 0$.

By a procedure analogous to point *a*), after transformations, the characteristic equation can be obtained

$$\begin{vmatrix} -2s_1\sinh S_1 & 2\cosh S_1 & -r_1\cosh Q_1 & r_1\dfrac{\sinh Q_1}{q_1} & 0 & 0 \\ r_1\cosh Q_1 & -r_1\dfrac{\sinh S_1}{s_1} & 2q_1\sinh Q_1 & -2\cosh Q_1 & 0 & 0 \\ 0 & 2 & -r_1 & 0 & 0 & 0 \\ r_1 & 0 & 0 & -2 & -gr_2 & -2gq_2 \\ 0 & 0 & 0 & 0 & 2s_2 & r_2 \\ 0 & 1 & -1 & 0 & s_2 & 1 \end{vmatrix} = 0, \tag{32}$$

4. Layer on laxer $[D_2^{(1,2)} = 0]$

The same procedure as in point 3a gives characteristic equations whose left sides, in view of eight boundary conditions (two for each of free surfaces and four for the interface), are determinants of dimensions 8×8. A diagram of the left sides of these equations, with plotted zero elements and those characteristic of a given type of boundary conditions, is shown in Fig. 2. There is the following set of the particular elements:

– The elements a_{ij} common to welded and smooth boundary conditions:

$a_{11} = -2s_1\sinh S_1,\quad a_{12} = 2\cosh S_1,\quad a_{13} = -r_1\cosh Q_1,\quad a_{14} = r_1\sinh Q_1/q_1,$
$a_{21} = r_1\cosh S_1,\quad a_{22} = -r_1\sinh S_1/s_1,\quad a_{23} = 2q_1\sinh Q_1,\quad a_{24} = -2\cosh Q_1,$
$a_{31} = -2s_1\sinh \bar{S}_1,\quad a_{32} = 2\cosh \bar{S}_1,\quad a_{33} = -r_1\cosh \bar{Q}_1,\quad a_{34} = r_1\sinh \bar{Q}_1/q_1,$
$a_{41} = r_1\cosh \bar{S}_1,\quad a_{42} = -r_1\sinh \bar{S}_1/s_1,\quad a_{43} = 2q_1\sinh \bar{Q}_1,\quad a_{44} = -2\cosh \bar{Q}_1,$
$a_{45} = -r_2g\cosh S_2,\quad a_{46} = r_2g\sinh S_2/s_2,\quad a_{47} = -2gq_2\sinh Q_2,\quad a_{48} = 2g\cosh Q_2,$
$a_{61} = -s_1\sinh \bar{S}_1,\quad a_{62} = \cosh \bar{S}_1,\quad a_{63} = -\cosh \bar{Q}_1,\quad a_{64} = \sinh \bar{Q}_1/q_1,$
$a_{65} = s_2\sinh S_2,\quad a_{66} = -\cosh S_2,\quad a_{67} = \cosh Q_2,\quad a_{68} = -\sinh Q_2/q_2,$
$a_{75} = a_{78} = a_{86} = a_{87} = 0,\quad a_{76} = 2,\quad a_{77} = -r_2,\quad a_{85} = r_2,\quad a_{88} = -2.$

– The elements a_{ij} characteristic of welded boundary conditions:
$a_{35} = 2s_2 g\sinh S_2$, $a_{36} = -2g\cosh S_2$, $a_{37} = r_2 g\cosh Q_2$, $a_{38} = -r_2 g\sinh Q_2/$
$/q_2$, $a_{51} = \cosh \bar{S}_1$, $a_{52} = -\sinh \bar{S}_1/s_1$, $a_{53} = q_1 \sinh \bar{Q}_1$, $a_{54} = -\cosh\bar{Q}_1$,
$a_{55} = -\cosh S_2$, $a_{56} = \sinh S_2/s_2$, $a_{57} = -q_2\sinh Q_2$, $a_{58} = \cosh Q_2$.

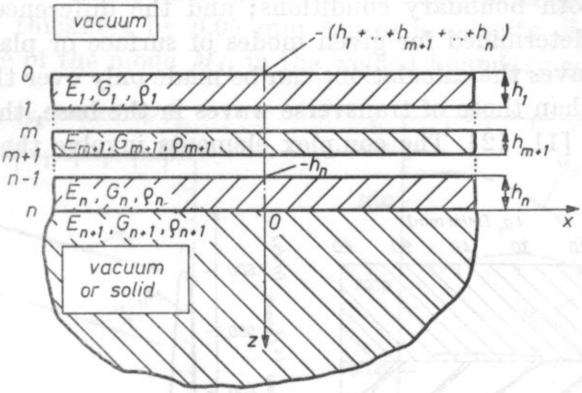

Fig. 1. A schematic diagram of a layered medium

– The elements a_{ij} characteristic of smooth boundary conditions:
$a_{55} = 2s_2\sinh S_2$, $a_{56} = -2\cosh S_2$, $a_{57} = r_2\cosh Q_2$, $a_{58} = -r_2\sinh Q_2/q_2$.

New symbols which occur in the elements given above denote

$$\bar{S}_1 = -ks_1z_1, \quad \bar{Q}_1 = -kq_1z_1, \quad S_2 = -ks_2z_1, \quad Q_2 = -kq_2z_1,$$

where $z_1 = -h_2$ (see Fig. 1).

The quantities S_1 and Q_1 are defined by formulae (28), where $z_0 = -(h_1+h_2)$.

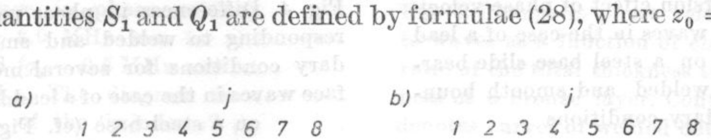

Fig. 2. A schematic representation of the left sides of characteristic equations in the case of layer on layer $[D_2^{(1,2)}]$ for boundary conditions a) welded, b) smooth

the crisscrossed area denotes zero elements, the area dashed horizontally denotes elements characteristic of welded conditions, the area dashed vertically corresponds to those of smooth conditions; the other elements are common

5. Results of numerical calculations

A numerical programme in Algol 1204 permits the phase velocities to be calculated for a given product of frequency and thickness of the superficial (first) layer, which product is the parameter of the calculations, in the case of welded and smooth boundary conditions; and the differences between these velocities to be determined for given modes of surface or plate waves. In the case of surface waves the calculations can be made only over the range of phase velocities lower than those of transverse waves in the base, thus corresponding to real elements [11, 12]. The complex elements involve the so-called "leaky

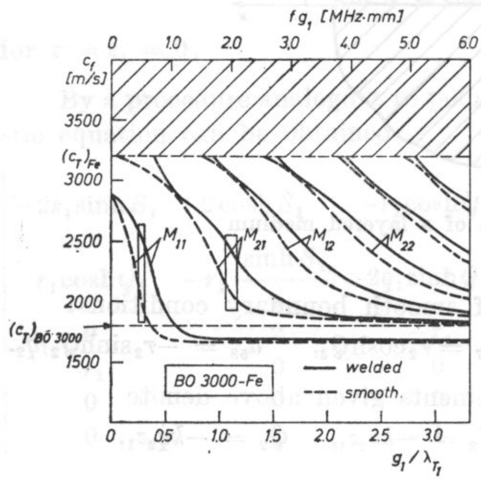

Fig. 3. Dispersion effect of phase velocity of surface waves in the case of a lead bronze layer on a steel base slide bearings in the welded and smooth boundary conditions

M_{ij} – modes, g_1 – layer thickness, λ_{T1} – the wavelength of the transverse mode in layer, f – frequency. The area dashed corresponds to leaky waves; – welded, – – – smoth

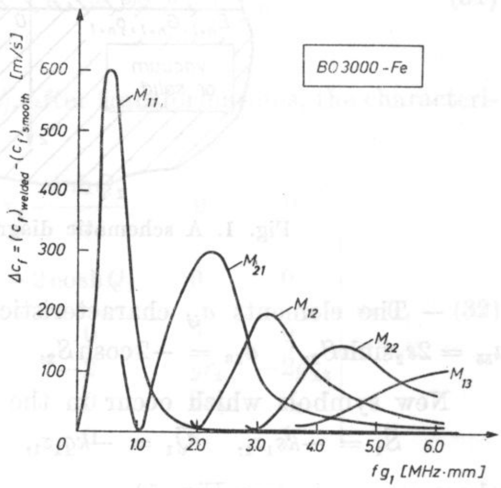

Fig. 4. Differences in phase velocities corresponding to welded and smooth boundary conditions for several modes of surface waves in the case of a lead bronze layer on a steel base (cf. Fig. 3)

waves" which are accompanied by a leakage of elastic energy to the base The design of the programme assures that by way of changing the procedure for calculation of values of the elements of the determinant, i.e. the left side of the characteristic equation, it can be used for any number of layers. Fig. 3 shows, as an example, the solution for a layer of B03000 bearing alloy on a steel base [13]. The curve of the differences in the phase velocities calculated ni this case with two types of boundary conditions for the particular modes of surface waves is shown in Fig. 4. From the point of view of the usefulness of velocity measurements for the estimation of adhesion degree it can be seen that there are optimum frequencies for a given layer thickness, i.e. those at which the differences in velocity are greatest. At the same time a further analysis shows that at those optimum frequencies the modes M_{11} and M_{21} are fairly sensitive to changes in layer thickness. For example, at frequency of 1 MHz a change in layer thickness by 0.05 mm causes a velocity change of 50 m/s (Fig. 5) in the case of the mode M_{11} in the welded boundary conditions.

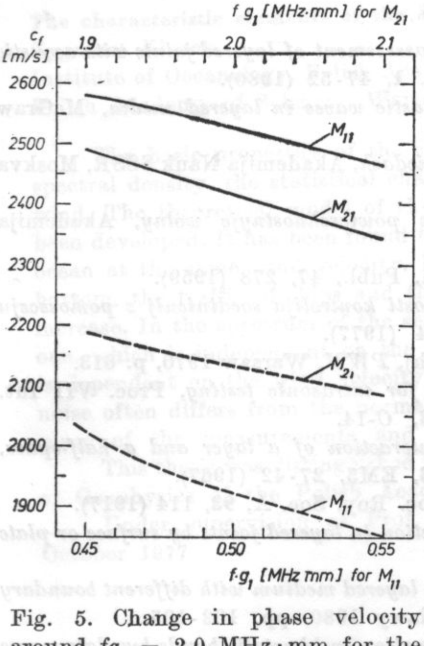

Fig. 5. Change in phase velocity around $fg_1 = 2.0$ MHz·mm for the mode M_{21} and $fg_1 = 0.5$ MHz·mm for the mode M_{11}. The diagram is a magnification of the "windows" (in Fig. 3)

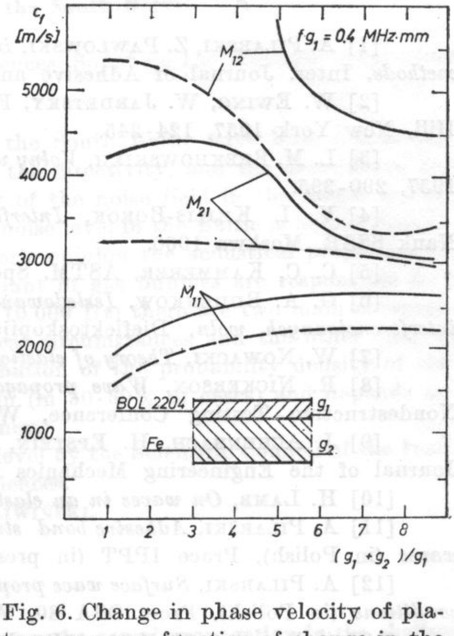

Fig. 6. Change in phase velocity of plate waves as a function of change in the ratio of the total thickness to the thickness of a bronze layer. Continuous line denotes curves of welded boundary conditions, discontinuous line corresponds to smooth boundary conditions
M_{ij} – successive modes of plate waves

When the base turns into a layer, which occurs when the thickness of the base is comparable to the wavelength of the longitudinal mode in the base, a velocity change is caused in addition by a change in the ratio of thicknesses of those two layers. Fig. 6 shows curves of such changes calculated for a BOL-2204 lead bronze on steel in the case when the product fg_1 is equal to 0.4 MHz·mm for the first three modes of plate waves.

6. Conclusions

Taking into consideration the practical possibilities of using the phase velocity measurements of surface or plate waves for the evaluation of adhesion degree (e.g. in bimetals), it can be stated that the knowledge of numerically determined dispersive curves for different boundary conditions facilitates a purposive selection of investigation parameters and permits the estimation of the effect on the quantity measured of such other factors as change in layer thickness or change in acoustic parameters characteristic of media connected.

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Received on January 27, 1981 ; revised version on July 29, 1981.