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THE SOUND POWER OF A CIRCULAR PLATE FOR HIGH-FREQUENCY WAVE RADIATION

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This paper gives an analysis of the sound power of a circular plate which vibrates at a frequency much higher than the resonance one. This analysis was carried out for Bessel axially-symmetric distributions of vibration velocity on the surface of a source placed in a rigid, planar baffle. An exact expression of the sound power of the vibrating circular plate was given in Hankel representation. It was assumed in a specific case that the source radiated waves at frequencies much higher than the resonance ones, permitting simplifications to be introduced in the subintegral function. As the final result of the analysis, an approximate expression was derived using the Cauchy theorem on residua. The expressions derived here are very useful and convenient for numerical calculations.

Notation

a — plate radius
c_o — sound wave propagation velocity in a medium of density ϱ_0
$H_n^{(1)}(x)$ — Hankel function of the nth order of the first kind
$H_n^{(2)}(x)$ — Hankel function of the nth order of the second kind
$I_n(x)$ — modified Bessel function of the nth order of the first kind
$J_n(x)$ — Bessel function of the nth order
k — wave number
$K_n(x)$ — cylindrical MacDonald function of the nth order
N — real component of source acoustic power (A2), (A3)
N_0 — real power of source for $k \to \infty$ (A10)
p — sound pressure (A1)
r — radial variable
v — vibration velocity amplitude of source surface points
v_n — vibration velocity amplitude of source points of the plate (1)
v'_{0n} — vibration velocity amplitude of the central point of the plate
W — characteristic function of circular source (A8)

W_n – characteristic function of circular plate for (0, n) vibration mode (4)
$Z_n(x)$ – cylindrical function of the nth order
λ – sound wave length
ϱ_0 – density of gaseous medium
σ – source surface area
ω – angular frequency

1. Introduction

The problem of the impendance and sound power of circular sources with an irregular vibration velocity distribution has been the object of analysis in a large number of papers in the field of acoustic wave generation by surface sources. In terms of subject papers [3-8] are above all most related to the problems considered in the present paper (a full bibliography of this problem was given in paper [3]). Most of the investigation results obtained could be used in partical applications only when using computers. Of the results obtained, only the expressions of impedance and sound power appeared to be convenient in a small number of cases, above all and most frequently for very small interference parameters.

These have been to date a lack of elaborations giving the form of the expressions of the sound power of a circular plate in a speciafic case which would be convenient for numerical calculations, namely for high-frequency wave radiation. The investigations reported on in the present paper have given such relationships.

The present considerations of the radiation of a circular plate refer to the results obtained in paper [6], where the object of investigation also included the problem of the sound power of a circular membrane for frequencies much higher than the resonance ones.

In terms of the possibility of practical applications, analysis was carried out on the axially-symmetric vibration of a circular plate clamped on the circumference to an ideal rigid and planar baffle. Linear processes harmonic in time were considered.

Taking as the basis the Huygens-Rayleigh integral formula, exact expressions were introduced for sound power in the form of a single integral. It was assumed in a specific case that the plate radiated waves at frequencies much higher than the resonance ones. This permitted simplifications in the subintegral function and subsequently integration using the Cauchy residua theorem.

Very useful and convenient expressions were derived for numerical calculations.

2. Exact calculation of the sound power

In considering the linear phenomena sinusoidally dependent on time, the axially – symmetric proper vibration of a circular plate clamped on the circumference can be described in the following way [8]:

$$v_n(r) = v_{0n}\left\{J_0(r\beta_n) - \frac{J_0(a\beta_n)}{J_0(ia\beta_n)} J_0(ir\beta_n)\right\}, \tag{1}$$

where a is the radius of the plate, v_{0n} is the vibration velocity amplitude of points of the plate, r is a radial variable, J_0 is a Bessel function of *zero*th order and $a\beta_n$ is the nth root of the equation

$$J_0(a\beta_n)I_1(a\beta_n) = -J_1(a\beta_n)I_0(a\beta_n), \tag{2}$$

where I_s is a modified Bessel function of the sth order. The constant v_{0n} can be expressed by the vibration velocity of the central point of the plate v'_{0n}, from the following relation

$$v'_{0n} = v_{0n}\left[1 - \frac{J_0(a\beta_n)}{J_0(ia\beta_n)}\right]. \tag{2a}$$

The expression of the vibration velocity (1) can be inserted into relationship (A8) and the following integral property [9] used:

$$\int_0^u wJ_0(hw)J_0(lw)\,dw = \frac{u}{h^2-l^2}\{hJ_1(hu)J_0(lu) - lJ_0(hu)J_1(lu)\}, \tag{3}$$

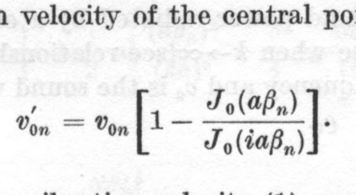


Fig. 1. Integration in the plane of the complex variable $\vartheta = \vartheta' + i\vartheta''$ for expression (A4), $\varepsilon \to 0$

as a result of which the characteristic function $W_n(\vartheta)$ of the circular plate for a $(0, n)$ vibration mode is

$$W_n(\vartheta) = v_{0n}\frac{2\beta_n^2}{\beta_n^4 - k^4\sin^4\vartheta}\{a\beta_n J_1(a\beta_n)J_0(ka\sin\vartheta) - \\ - ka\sin\vartheta J_0(a\beta_n)J_1(ka\sin\vartheta)\}. \tag{4}$$

The real power radiated by the circular plate by the $(0, n)$ vibration mode can be calculated from relationship $(A9)$. This involves the substitution

$x = ka\sin\vartheta$, giving

$$N_n = 4(a\beta_n)^6 N_0 \int_0^{ka} \left\{ \frac{a_n J_0(x) - \dfrac{x}{a\beta_n} J_1(x)}{(a\beta_n)^4 - x^4} \right\}^2 \frac{x dx}{\sqrt{1-(x/ka)^2}}, \tag{5}$$

where

$$a_n = \frac{J_1(a\beta_n)}{J_0(a\beta_n)}; \tag{6}$$

whereas

$$N_0 = \varrho_0 c_0 \pi a^2 v_{0n}^2 J_0^2(a\beta_n) \tag{7}$$

is an expression of the sound power radiated by a circular plate by the $(0, n)$ vibration mode in the case when $k \to \infty$(see relationship $(A11)$), with $k = \omega/c_0$, where ω is the angular frequency and c_0 is the sound wave propagation velocity in a medium of density ϱ_0.

3. Approximate calculation of the sound power

In a specific case, when the wave radiation frequency is much higher than the resonance frequency ($k \gg \beta_n$), the approximate formula

$$\left[1 - \left(\frac{x}{ka}\right)^2\right]^{-1/2} \simeq 1 + \frac{1}{2}\left(\frac{x}{ka}\right)^2 + \frac{3}{8}\left(\frac{x}{ka}\right)^4 \tag{8}$$

can be used and integration in expression (5) extended from finite $(0 \leqslant x \leqslant ka)$ to infinite limits $(0 \leqslant x < \infty)$.

The expression derived for the sound power

$$N_n = 4(a\beta_n)^6 N_0 \int_0^{\infty} \left\{ \frac{a_n J_0(x) - \dfrac{x}{a\beta_n} J_1(x)}{(a\beta_n)^4 - x^4} \right\}^2 \left[1 + \frac{1}{2}\left(\frac{x}{ka}\right)^2 + \frac{3}{8}\left(\frac{x}{ka}\right)^4\right] x dx \tag{9}$$

can be given in the form of an integral sum calculated from the integral formula $(A14)$.

For the first derivatives of the special functions the following relations can be used [9]:

$$\begin{gathered} J_0'(x) = -J_1(x), \quad xJ_1'(x) = xJ_0(x) - J_1(x), \\ H_0^{(1)\prime}(x) = -H_1^{(1)}(x), \quad xH_1^{(1)\prime}(x) = xH_0^{(1)}(x) - H_1^{(1)}(x), \\ I_0'(x) = I_1(x), \quad xI_1'(x) = xI_0(x) - I_1(x), \\ K_0'(x) = -K_1(x), \quad xK_1'(x) = -xK_0(x) - K_1(x), \end{gathered} \tag{10}$$

where $H_0^{(1)}(x)$ and $H_1^{(1)}(x)$ are Hankel functions of the first kind, whereas $K_0(x)$ and $K_1(x)$ are cylindrical MacDonald functions, both pairs being respectively of the *zero*th and first orders.

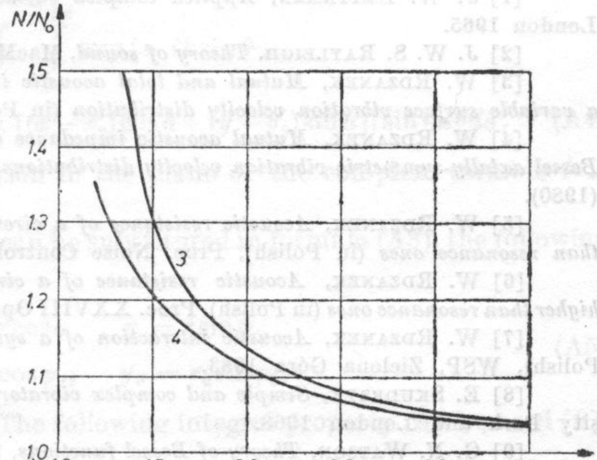
When in addition the characteristic equation (2), determination (6) and the wronskians [9]

$$H_0^{(1)}(x)J_1(x)-J_0(x)H_1^{(1)}(x)=\frac{2i}{\pi x},$$
$$K_0(x)I_1(x)+I_0(x)K_1(x)=\frac{1}{x}, \tag{11}$$

are taken into consideration, finally thus

$$N_n = N_0\left\{1+\frac{1}{2}\,a_n^2\,\frac{(a\beta_n)^2}{(ka)^2}-\frac{3}{4}\,a_n\,\frac{(a\beta_n)^3}{(ka)^4}\right\}, \tag{12}$$

if $k \gg \beta_n$.



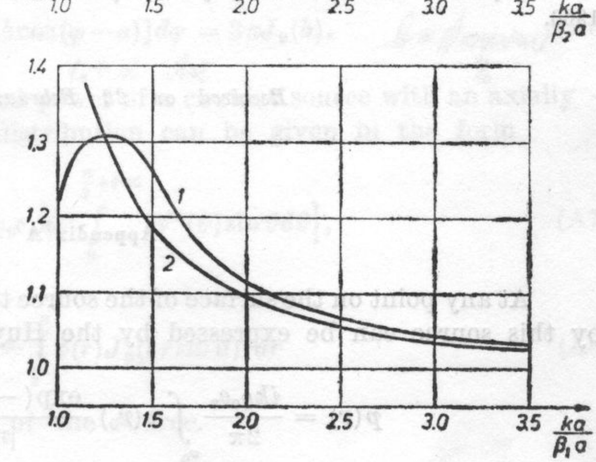


Fig. 2. The relative sound power N/N_0 of the circular plate for (0, 1) and (0, 2) axially-symmetic vibration modes, depending on $ka/\beta_n a$

Curves 1 and 3 have been plotted from the exact formula (5); curves 2 and 4, from the approximate formula (12). It is assumed that $\beta_1\,a = 3.195$ and $\beta_2\,a = 6.306$

4. Conclusions

The approximate expression (12) derived is very convenient for numerical calculations of the sound power radiated by a circular plate with axially-symmetric vibration modes and can be used with less demanding assumptions than $ka \gg a\beta_n$. E.g. with $ka > 3a\beta_n$ the sound power for the first few vibration modes involves relative error not exceeding 1 per cent (Fig. 2).

In a boundary case, for $ka \to \infty$, it can be shown from formula (12) that the relative sound power N/N_0 tends to unity.

When $ka < 3a\beta_n$, or when high accuracy is required of results, calculations can be carried out by computers from the integral formula (5).

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Appendix A

At any point on the surface of the source the sound pressure $p(\boldsymbol{r})$ generated by this source can be expressed by the Huygens-Rayleigh formula [2]

$$p(\boldsymbol{r}) = \frac{ik\varrho_0 c_0}{2\pi} \int_{\sigma_0} v(\boldsymbol{r}_0) \frac{\exp(-ik|\boldsymbol{r}-\boldsymbol{r}_0|)}{|\boldsymbol{r}-\boldsymbol{r}_0|} d\sigma_0, \tag{A1}$$

where $|\boldsymbol{r}-\boldsymbol{r}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$ is the distance between any two points on the surface of the source, c_0 is the sound wave propagation velocity in a medium of density ϱ_0, $k = 2\pi/\lambda$ is a wave number and λ is wavelength.

The real component of the sound power emitted by the source is

$$N = \frac{1}{2}\,\mathrm{Re}\left\{\int\limits_{\sigma} p(\boldsymbol{r})v(\boldsymbol{r})d\sigma,\right. \tag{A2}$$

or, considering relation (A1),

$$N = \mathrm{Re}\left\{\frac{ik\varrho_0 c_0}{4\pi}\int\limits_{\sigma}\int\limits_{\sigma_0} v(\boldsymbol{r})v(\boldsymbol{r}_0)\,\frac{\exp(-ik|\boldsymbol{r}-\boldsymbol{r}_0|)}{|\boldsymbol{r}-\boldsymbol{r}_0|}\,d\sigma_0 d\sigma\right\}. \tag{A3}$$

This formula represents the real power emitted by the source into the surrounding space, i.e. the energy flux radiated by the source over one full period.

The surface integrals in formula (A3) can be calculated using the following expansion [3], [7],

$$\frac{\exp(-ik|\boldsymbol{r}-\boldsymbol{r}_0|)}{|\boldsymbol{r}-\boldsymbol{r}_0|} = -\frac{ik}{2\pi}\int\limits_{0}^{\frac{\pi}{2}+i\cdot\infty}\int\limits_{0}^{2\pi}\exp\{-ik\sin\vartheta\times$$
$$\times[(x-x_0)\cos a+(y-y_0)\sin a]\}\sin\vartheta d\vartheta da. \tag{A4}$$

The course of the integration in the plane of the complex variable ϑ is given in Fig. 1 (see p. 333).

The integral function (A4) can be substituted in formula (A3), the following polar coordinates introduced:

$$\begin{aligned} x &= r\cos\varphi, \quad & y &= r\sin\varphi,\\ x_0 &= r_0\cos\varphi_0, \quad & y_0 &= r_0\sin\varphi_0, \end{aligned} \tag{A5}$$

changing the integration order. The following integral property can be used [9]:

$$\int\limits_{0}^{2\pi}\exp[\pm ib\cos(\varphi-a)]d\varphi = 2\pi J_0(b). \tag{A6}$$

The expression of the sound power of a circular source with an axially — symmetric vibration velocity distribution can be given in the form

$$N = \mathrm{Re}\left\{\varrho_0 c_0 \pi k^2\int\limits_{0}^{\frac{\pi}{2}+i\cdot\infty} W^2(\vartheta)\sin\vartheta d\vartheta\right\}, \tag{A7}$$

where

$$W(\vartheta) = \int\limits_{0}^{a} v(r)J_0(kr\sin\vartheta)rdr \tag{A8}$$

is the characteristic function of the source.

The real component of sound power, i.e. the real power, can be determined from expression ($A7$) when the integration in the plane of the complex variable is carried out over a section on the real axis ϑ in the limits $(0, \pi/2)$, i.e.

$$N = \varrho_0 c_0 \pi k^2 \int\limits_0^{\pi/2} W^2(\vartheta) \sin\vartheta d\vartheta. \tag{A9}$$

In numerical calculations it is convenient to use the concept of relative sound power N/N_0, where N_0 can be assumed to be the real power of the source for $k \to \infty$. When $k \to \infty$, $p(\boldsymbol{r}) = \varrho_0 c_0 v(\boldsymbol{r})$, and then, according to formula ($A2$),

$$N_0 = \lim_{k\to\infty} N = \frac{1}{2}\, \varrho_0 c_0 \int\limits_\sigma v^2(\boldsymbol{r})\, d\sigma. \tag{A10}$$

When the sound source is circular and the vibration velocity distribution axially – symmetric,

$$N_0 = \pi \varrho_0 c_0 \int\limits_0^a v^2(r) r dr, \tag{A11}$$

where a is the radius of the circular plate.

Appendix B

The contour integral (see [9])

$$\frac{1}{2\pi i} \int\limits_C z^{\varrho-1} Z_\mu(bz) \frac{\mathrm{H}_\nu^{(1)}(az)\, dz}{(z^4 - r^4)^2}, \tag{A12}$$

where $a > b > 0$, r is a complex number, Z_μ is a cylindrical function of the order μ, $|\mu| + |\nu| < \varrho < 10$, can be expressed in the form of the sum of the residua at the poles of the subintegral function. When $a = b$, then $\varrho < 9$.

Using the Jordan lemma and Cauchy's residua theorem [1] the integration contour C can be closed in the upper half-plane of the complex variable z. This integration covers the two poles of the subintegral function for $z = r$ and $z = ir$. This gives

$$\frac{1}{2\pi i} \int\limits_0^\infty \{Z_\mu(bx) H_\nu^{(1)}(ax) - \exp(\varrho\pi i) Z_\mu[bx \exp(\pi i)] \times$$

$$\times H_\nu^{(1)}[ax \exp(\pi i)]\} \frac{x^{\varrho-1} dx}{(x^4 - r^4)^2} = \frac{1}{16 r^3} \frac{d}{dr} \{r^{\varrho-4} [Z_\mu(br) H_\nu^{(1)}(ar) +$$

$$+ i^{\varrho-4} Z_\mu(ibr) H_\nu^{(1)}(iar)]\}. \tag{A13}$$

In a specific case for $Z_\mu = J_\mu$, $a = b = 1$, considering the relationships

$$J_\mu(ir) = \exp\left(i\mu\frac{\pi}{2}\right) I_\mu(r),\ H_\nu^{(1)}(ir) = \frac{2}{\pi}\exp\left[-i(\nu+1)\frac{\pi}{2}\right] K_\nu(r),$$

$$J_\mu(-x) = \exp(\mu\pi i) J_\mu(x), \qquad H_\nu^{(1)}(-x) = -\exp(-i\nu\pi) H_\nu^2(x),$$

(A13) becomes

$$\int\limits_0^\infty J_\mu(x) J_\nu(x)\frac{x^{\varrho-1}dx}{(x^4-r^4)^2} = \frac{1}{8r^3}\frac{d}{dr}\left\{r^{\varrho-4}\left[\frac{\pi i}{2} J_\mu(r) H_\nu^{(1)}(r) + \right.\right.$$

$$\left.\left. + \exp\left[i(\varrho+\mu-\nu)\frac{\pi}{2}\right] I_\mu(r) K_\nu(r)\right]\right\}, \qquad \text{(A14)}$$

when $|\mu| + |\nu| < \varrho < 9$.