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MULTI-DIMENSIONAL TRANSFER FUNCTIONS FOR A NON-DISSIPATIVE BURGERS' EQUATION

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The propagation of acoustic disturbances in a continuum medium was analyzed under the assumption that the non-dissipative Burgers' equation is a reasonable mathematical model of the phenomenon under study. Regarding the propagation as a transformation of the time dependence of the acoustic velocity in a system with an input signal and employing the Banta's solution, the non-linear Burgers-Banta system was obtained. This system was described in the form of Volterra's series; the kernels of the series being determined with the help of the method of harmonic excitations. The r-dimensional Volterra's kernels given in the paper and their Fourier transforms (transfer functions) enable the parameters and probabilistic characteristics of the output signal to be determined under the condition that the input signal is known.

1. Non-dissipative Burgers' equation

Navier-Stokes equations [1, 2] define the dynamics of a viscous gas medium with the consideration of heat conduction. These can be reduced to one equation for the potential of the acoustic velocity, with general assumptions concerning the disturbances of the medium [3]:

$$c_0^2\nabla^2\Phi-\frac{\partial^2\Phi}{\partial t^2}+\left(2+\frac{\eta'}{\eta}+\frac{\gamma-1}{Pr}\right)\nu\nabla^2\frac{\partial\Phi}{\partial t}=2\nabla\frac{\partial\Phi}{\partial t}\nabla\Phi+(\gamma-1)\frac{\partial\Phi}{\partial t}\nabla^2\Phi \tag{1}$$

where besides typical notations, there also are:
c_0 — adiabatic sound velocity, γ — exponent of the adiabate ($=c_p/c_w$), η — first coefficient of viscosity (coefficient of dynamic viscosity), η' — second

coefficient of viscosity, ν — coefficient of kinematic viscosity ($= \eta/\varrho_0$), Pr — Prandtl number.

Applying the approximation of the theory valid for waves with a small but finite amplitude and limiting the case to a one-dimension problem, the above equation can be written in the following form:

$$\left(c_0 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \frac{\partial \Phi}{\partial t} - \frac{1}{2}\,\delta \frac{\partial^3 \Phi}{\partial x^2 \partial t} + \frac{1}{2}(\gamma+1) \frac{\partial \Phi}{\partial x} \frac{\partial^2 \Phi}{\partial x \partial t} = 0 \tag{2}$$

which, by integrating in terms of t and differentiating in terms of x, can be converted to the equation for acoustic velocity:

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2} u\right) \frac{\partial u}{\partial x} = \frac{1}{2}\,\delta \frac{\partial^2 u}{\partial x^2} \tag{3}$$

δ in equations (2) and (3) marks the coefficient of sound dissipation:

$$\delta = \nu \left(\frac{4}{3} + \frac{\zeta}{\eta} + \frac{\gamma - 1}{Pr}\right) \tag{4}$$

where $\zeta = \eta' + \frac{2}{3}\eta$ is the total coefficient of viscosity. The coefficient of sound dissipation represents losses in the medium due to viscosity and heat conduction.

This paper is concerned with such a case of propagation of disturbances, in which the right side of equation (3) can be neglected. Thus, equation (3) is replaced by the non-dissipative Burgers' equation:

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2} u\right) \frac{\partial u}{\partial x} = 0. \tag{5}$$

2. Banta's solution

The unconventional solution of equation (5) given by Banta [4], has the following form:

$$u(x, t) = \varphi + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!} \frac{d^{n-1}}{dt^{n-1}} [F^n(\varphi)\dot{\varphi}] \tag{6}$$

where

$$F(\varphi) = \frac{1}{c_0 + \beta\varphi}, \quad \dot{\varphi} = \frac{d}{dt}\varphi, \quad \varphi = \varphi(t) = u(0, t). \tag{6a}$$

In further considerations a slightly different expression for F will be used; applying the assumption about finite but small (with respect to c_0) acoustic velocities it can be accepted, that

$$F(\varphi) = \frac{1}{c_0+\beta\varphi} \cong \frac{1}{c_0} - \frac{\beta}{c_0^2}\varphi; \tag{6b}$$

in expressions (6a) and (6b): $\beta = (\gamma+1)/2$.

The approximation in (6b) is sufficient; e.g. if the level of acoustic pressure equals 174 dB (re $2\cdot10^{-5}$ Pa) what corresponds to the velocity of the acoustical particle of 0.1 c_0, the approximation error in (6b) does not exceed 1.5% [5].

3. Application of the harmonic input method in the construction of a transfer function of a system defined by Banta's series

The phenomenon of non-linear propagation, defined by Banta's series (6), can be presented in the form of a system with an input signal $X(t) = \varphi(t) = u(0,t)$ and output signal $Y(t) = u(x,t)$ (Fig. 1) [6].

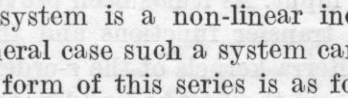

Fig. 1. Illustration of the "input – output" relations for propagation described by the Banta series

The Burgers-Banta system is a non-linear inertial system without the hysteresis effect. In a general case such a system can be described by Volterra series [7, 8]; the general form of this series is as follows:

$$Y(t) = \sum_{r=1}^{\infty} \frac{1}{r!} \int_{R^r} h_r(\tau_1, \ldots, \tau_r) \prod_{i=1}^{r} x(t-\tau_i)\, d\tau^r \tag{7}$$

where h_r are the Volterra kernels of the r-order. Their analytic form depends on the properties of the system; the integration domain R^r is the r-multiple Cartesian product of $R = \{\tau : \tau \in (-\infty, \infty)\}$ and $d\tau^r = d\tau_1 \ldots d\tau_r$.

This paper is aimed at the determination of kernels h_r and their r-dimensional Fourier transforms, i.e. r-multiple transfer functions of a system presented in Fig. 1, which is described by series (6). $h_r(t_1, \ldots, t_r)$ denotes the Volterra kernel and $\bar{h}_r(t_1, \ldots, t_r)$ denotes the kernel of the Burgers-Banta system. The harmonic input method [6, 8, 9] was used to determine the set $\{h_r\}$. In order to apply this method effectively the series (6) should be converted to a form more convenient for further calculations. From

$$\frac{d^{n-1}}{dt^{n-1}}\,[F^n(\varphi)F'(\varphi)\dot{\varphi}] = \frac{1}{n+1}\,\frac{d^n}{dt^n}\,[F^{n+1}(\varphi)]$$

series (6) can be converted to

$$u(x,t) = \varphi(t) + \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(n+1)!}\,\frac{d^n}{dt^n}\,[F^{n+1}[\varphi(t)]],$$

and then using equation (6b) and taking into consideration that

$$F^{n+1}(\varphi) = \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{c_0^{n+1-k}}\left(-\frac{\beta}{c_0^2}\right)^k\varphi^k$$

we have

$$u(x,t) = \varphi(t) + \frac{c_0^2}{\beta x}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{(n+1)!}\,\frac{d^n}{dt^n}\sum_{k=0}^{n+1}\frac{1}{c_0^{n+1-k}}\left(-\frac{\beta}{c_0^2}\right)^k\varphi^k(t). \tag{8}$$

This new form of the Banta series is particularily useful in the generation of Volterra kernels with the harmonic input method. This method consists in the determination of coefficients of exponential factors of the $\exp[j(\omega_1+\ldots+\omega_r)]$ type in the input signal, under the assumption that the signal $\exp(j\omega_1 t)+\ldots$ $\ldots+\exp(j\omega_r t)$ acts at the input. As it has been proved in paper [8] these coefficients are r-dimensional transfer functions and their r-dimensional inverse Fourier transforms are Volterra kernels of the r-order. Thus, in order to determine the transfer functions of the first order (denoted by $\bar{H}_1(\omega, x)$) it was accepted that $\varphi(t) = \exp(j\omega t)$. Finding the coefficient of the $\exp(j\omega t)$ factor in series (8), $H_1(\omega, x)$ is obtained. Making the substitution in expression (8), we obtain the series:

$$u(x,t) = e^{j\omega t} + \frac{c_0^2}{\beta x}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{(n+1)!}\,\frac{d^n}{dt^n}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{c_0^{n+1-k}}\left(-\frac{\beta}{c_0^2}\right)^k e^{jk\omega t}$$

which, after differentiating, has the form:

$$u(x,t) = e^{j\omega t} + \frac{c_0^2}{\beta x}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{c_0^{n+1-k}}\left(-\frac{\beta}{c_0^2}\right)^k(jk\omega)^n e^{jk\omega t}.$$

The sought coefficient can be derived from the above expression by accepting $k = 1$; then the expression for H_1 will be:

$$\bar{H}_1(\omega, x) = 1 + \frac{c_0^2}{\beta x} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{(n+1)!} (n+1) \frac{1}{c_0^n} \left(-\frac{\beta}{c_0^2}\right) (j\omega)^n =$$

$$= \exp\left(-j\frac{x}{c_0}\right)\omega,$$

hence

$$h_1(t, x) = F^{-1}\{\bar{H}_1(\omega, x)\} = F^{-1}\left\{\exp\left(-j\frac{x}{c_0}\omega\right)\right\} = \delta\left(t - \frac{x}{c_0}\right)$$

where $\delta(\cdot)$ denotes the Dirac delta. In order to obtain the transfer function of the second order we have to accept

$$\varphi(t) = e^{j\omega_1 t} + e^{j\omega_2 t}$$

and then we have to find the coefficient of the harmonic factor with a $\omega_1 + \omega_2$ pulsation in series (8) with the $\varphi(t)$ function accepted as above. The series of calculations (as above) leads to the following expression for the transfer function of the second order:

$$\bar{H}_2(\omega_1, \omega_2, x) = j(\omega_1 + \omega_2) \frac{\beta x}{c_0^2} \exp\left[-\frac{x}{c_0} j(\omega_1 + \omega_2)\right].$$

The kernel of the second order will be expressed by:

$$\bar{h}_2(t_1, t_2, x) = F_2^{-1}\left\{j(\omega_1 + \omega_2)\frac{x}{c_0^2} \exp\left[-\frac{x}{c_0} j(\omega_1 + \omega_2)\right]\right\} =$$

$$= \frac{\beta x}{c_0^2}\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) \delta\left(t_1 - \frac{x}{c_0}\right) \delta\left(t_2 - \frac{x}{c_0}\right).$$

In a general case, the following expression for the transfer function of the r-order is obtained:

$$H_r(\omega_1, \ldots, \omega_r, x) = \left(\frac{\beta x}{c_0^2}\right)^{r-1} [j(\omega_1 + \ldots + \omega_r)]^{r-1} \exp\left[-j\frac{x}{c_0}(\omega_1 + \ldots + \omega_r)\right] \tag{9}$$

hence the general form of the Volterra kernel is:

$$\bar{h}_r(t_1, \ldots, t_r, x) = F_r^{-1}[\bar{H}_r(\omega_1, \ldots \omega_r, x)] =$$

$$= \left(\frac{\beta x}{c_0^2}\right)^{r-1} \left(\frac{\partial}{\partial t_1} + \ldots + \frac{\partial}{\partial t_r}\right)^{r-1} \delta\left(t_1 - \frac{x}{c_0}\right) \ldots \delta\left(t_r - \frac{x}{c_0}\right). \tag{10}$$

The obtained expression for the general form of Volterra kernels leads to a more compact than in (6) form of the Banta series [4]. Namely, substituting (10) in (7), expressions for succeeding terms of the Volterra series are obtained. Thus, let V_r denote the r-term of series (7), i.e.

$$V_r = \frac{1}{r!}\int\limits_{R^r} h_r(t_1, \ldots, t_r)\prod_{i=1}^{r} X(t-\tau_i)\,d\tau^r \tag{11}$$

and then in case of the Burgers-Banta series it is:

$$h_r(t_1, \ldots, t_r) = \bar{h}_r(t_1, \ldots, t_r, x),\ X(t) = \varphi(t),\ Y(t) = u(x,t).$$

The final result is:

$$V_1 = \int\limits_{-\infty}^{\infty} \delta\left(\tau - \frac{x}{c_0}\right)\varphi(t-\tau)\,d\tau = \varphi\left(t-\frac{x}{c_0}\right),$$

$$V_2 = \frac{1}{2!}\frac{Bx}{c_2^0}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left[\delta'\left(\tau_1-\frac{x}{c_0}\right)\delta\left(\tau_2-\frac{x}{c_0}\right)+\delta\left(\tau_1-\frac{x}{c_0}\right)\delta'\left(\tau_2-\frac{x}{c_0}\right)\right]\times$$
$$\times\varphi(t-\tau_1)\varphi(t-\tau_2)\,d\tau_1 d\tau_2 = \frac{\beta x}{c_0^2}\varphi\left(t-\frac{x}{c_0}\right)\dot{\varphi}\left(t-\frac{x}{c_0}\right) = \frac{1}{2!}\frac{\beta x}{c_0^2}\frac{d}{dt}\varphi^2\left(t-\frac{x}{c_0}\right),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$V_r = \frac{1}{r!}\left(\frac{\beta x}{c_0^2}\right)^{r-1}\frac{d^{r-1}}{dt^{r-1}}\varphi^r\left(t-\frac{x}{c_0}\right).$$

Hence, a different, more compact form of the Banta series is achieved:

$$u(x,t) = \sum_{r=1}^{\infty}\left(\frac{\beta x}{c_0^2}\right)^{r-1}\frac{1}{r!}\frac{d^{r-1}}{dt^{r-1}}\varphi^r\left(t-\frac{x}{c_0}\right). \tag{12}$$

4. Conclusions

The approach applied in this paper to the description of the "input-output" relations of a Burgers-Banta system consists in treating the propagation of an intensive acoustic signal from the point of view of the analysis of non-linear changes of the signal initiating disturbances in the medium, i.e. signal $X(t) = u(0,t) = \varphi(t)$, where the signal $Y(t) = u(x,t)$ reflects these non-linear changes. Such a formulation of the problem suggests that the non-linear propagation phenomenon should be treated as a non-linear system; and the method of Volterra serieses was used, because of its versatality.

New elements of the description of lossless non-linear propagation have been achieved. A compact analytical Volterra description of the Burgers-Banta system was derived; the form of kernels (10) shows that the said system is quasi-memoryless.

Analytical forms of r-dimensional Volterra kernels and transfer functions, presented in this paper, make it possible to determine easily all parameters of the output signal, when the input signal is known. Also the construction of all probabilistic characteristics (e.g. multi-dimensional probability distributions, power spectrum density) of the output signal is possible, when the input signal is a stationary Gaussian process [7]. This can find application in investigations of non-linear propagation of intensive acoustic noises [2, 10].

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