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MUTUAL IMPEDANCE OF AXIALLY-SYMMETRIC MODES OF A CIRCULAR PLATE

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This paper presents an exact calculation of the mutual radiation impedance of axially-symmetric modes of a fixed at the edge circular plate. Linear and harmonic processes in respect to time have been considered and it has been accepted that the plate radiates acoustic waves into a lossless gas medium. Included here expressions for the mutual impedance in the form of single integrals have been adopted on the basis of several simplifying assumptions to numerical calculations for low and high frequencies of radiated waves. Achieved results are used in the analysis of the impedance and sound power radiated by a circular plate excited to vibrate by a known (from the assumption) superficial distribution of the exciting force.

Notations

a — plate radius
B — flexural rigidity
b_n — constant quantity for n-mode (10)
c — propagation velocity of a wave in a gas medium
f_n — frequency of free vibrations for mode $(0, n)$ (4)
h — plate thickness
$H_m^{(1)}(x)$ — first type, m-order Hankel function
$H_m^{(2)}(x)$ — second type, m-order Hankel function
$I_m(x)$ — first type, m-order modified Bessel function
$J_m(x)$ — m-order Bessel function
k_0 — wave number
$K_m(x)$ — m-order cylindrical MacDonald function
$N_m(x)$ — m-order Neumann function
N_{ns} — mutual power of modes, $(0, n)$ and $(0, s)$, of the circular plate (6)
p_{ns} — acoustic pressure produced by the vibrating plate through mode $(0, n)$ and exerted on the same plate through mode $(0, s)$
r — radial variable of point on the surface of the plate, in polar coordinates
$S_m(x)$ — m-order Struve function

t — time
v_n — vibration velocity of points on the surface of the plate for mode $(0, n)$ (2)
Z_{ns} — mechanical impedance of modes, $(0, n)$ and $(0, s)$ of the circular plate (3)
γ_n — n-root of the characteristic equation (3)
δ_{nm} — Kronecker delta
ζ_{ns} — normalized mutual impedance (12)
θ_{ns} — normalized mutual resistance (13)
λ — length of an acoustic wave in a gas medium
ξ — transverse dislocation of points on the surface of the plate
ϱ — density of the material of the plate
ϱ_0 — rest density of the gas medium
σ — area of the plate
χ_{ns} — normalized mutual reactance (14)
ω_n — angular frequency of free vibrations, corresponding to mode $(0, n)$

1. Introduction

Only few published papers in the field of the generation of acoustic waves by superficial sources are concerned with the problem of acoustic mutual interactions of plates or circular membranes. The carried out analysis was done for a system of two plates or circular membranes for a case of axially-symmetric free vibrations.

Besides theoretical work on acoustic mutual interactions between two sources, research is also performed on acoustic interactions of two different vibration modes of only one source. Results of the analysis of a circular membrane are presented in papers [6] and [7].

Hitherto the problem for a circular plate has not been solved.

This paper undertakes the problem of acoustic interactions by calculating the mutual impedance of two different axially-symmetric, $(0, n)$ and $(0, s)$, vibration modes of a circular plate fixed at the edge, which radiates acoustic waves into a lossless gas medium. Linear and harmonic in time processes have been examined.

Obtained expressions for mutual impedance can be a basis for further investigations of the radiation impedance of a circular plate with a determined superficial distribution of the force exciting vibrations.

2. Superficial distribution of the vibration velocity

The motion equation for free axially-symmetric vibrations of a circular plate, made from a homogeneous material of density ϱ, and of small in respect to the diameter $2a$ thickness h, is as follows [2]:

$$B\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]^2 \xi(r, t)+\varrho h\,\frac{\partial^2\xi(r, t)}{\partial t^2} = 0 \tag{1}$$

where ξ is the transverse dislocation of points on the surface of the plate, B — flexural rigidity of the plate.

Solving this equation for effects which are sinusoidal in time, in the case of a plate fixed at the edge leads to a formula for the vibration velocity [2]

$$v_n(r) = v_{0n}\left\{J_0\left(\frac{r}{a}\gamma_n\right) - \frac{J_0(\gamma_n)}{I_0(\gamma_n)} I_0\left(\frac{r}{a}\gamma_n\right)\right\}. \tag{2}$$

In paper [2], v_{0n} denotes the maximal value of the vibration velocity of the central point of the plate for mode $(0, n)$. Occurring here special function $I_0(x)$ is a zero order modified Bessel function of the first type, which can be expressed by a Bessel function $J_0(ix)$ of an imaginary argument, i.e. $I_m(x) = i^{-m}J_m(ix)$ for $m = 0, 1, 2, \ldots$

From the frequency equation (2)

$$I_0(\gamma_n)J_1(\gamma_n) + I_1(\gamma_n)J_0(\gamma_n) = 0 \tag{3}$$

we obtain an infinite number of values $k = k_n(ka = \gamma_n)$, which determine frequencies of free vibrations

$$f_n = \frac{1}{2\pi a^2}\gamma_n^2\sqrt{\frac{B}{\varrho n}} \tag{4}$$

while for $n = 1, 2, 3$ we have (e.g. [3]): $\gamma_1 = 3.195\ldots$; $\gamma_2 = 6.306\ldots$; $\gamma_3 = 9.439\ldots$ If n is sufficiently large, then according to relationship [3] $\gamma_n \simeq n\pi$, instead of $ka = \frac{2\pi}{\lambda}a = \gamma_n$ we have $n\lambda = 2a$.

3. Integral expression for mutual impedance

The mechanical mutual impedance between axially-symmetric free vibration modes, $(0, n)$ and $(0, s)$, of a circular plate placed in a rigid and flat acoustic baffle is calculated on the basis of the definition (compare [7])

$$Z_{ns} = \frac{1}{2\sqrt{\langle|v_n|^2\rangle\langle|v_s|^2\rangle}}\int_\sigma p_{ns}v_s\,d\sigma \tag{5}$$

where p_{ns} is the acoustic pressure produced by the vibrating plate through mode $(0, n)$ and exerted on the same plate through vibration mode $(0, s)$,

$$N_{ns} = \frac{1}{2}\int_\sigma p_{ns}v_s\,d\sigma \tag{6}$$

is the mutual power of modes, (0, n) and (0, s), of the circular plate, while

$$\langle |v_n|^2 \rangle = \frac{1}{2\sigma} \int_{\sigma} v_n^2(r)\, d\sigma \tag{6a}$$

is the mean of the square of velocity of the vibration mode (0, n).

On the basis of paper [4] the mutual impedance (5) can be expressed by the following formula

$$Z_{ns} = \frac{\pi \varrho_0 c k_0^2}{\sqrt{\langle |v_n|^2 \rangle \langle |v_s|^2 \rangle}} \int_0^{\pi/2+i\infty} M_n(\vartheta) M_s(\vartheta) \sin\vartheta\, d\vartheta \tag{7}$$

where

$$M_n(\vartheta) = v_{0n} \int_0^a \left\{ J_0\left(\frac{r}{a}\gamma_n\right) - \frac{J_0(\vartheta_n)}{I_0(\gamma_n)} I_0\left(\frac{r}{a}\gamma_n\right) \right\} J_0(k_0 r \sin\vartheta)\, r dr \tag{8}$$

c_0 — propagation velocity of a wave in a gas medium of a rest density of ϱ_0, $k_0 = 2\pi/\lambda$ — wave number, λ — acoustic wave length in a gas medium. Applying the integral formula (A3) and the frequency equation (3), we achieve

$$M_n(\vartheta) = 2\, v_{0n} \frac{a^2}{\gamma_n} \frac{J_0(\gamma_n)}{1 - \left(\frac{k_0 a}{\gamma_n}\right)^4 \sin^4\vartheta} \left\{ b_n J_0(k_0 a \sin\vartheta) - \frac{k_0 a}{\gamma_n} \sin\vartheta J_1(k_0 a \sin\vartheta) \right\}, \tag{9}$$

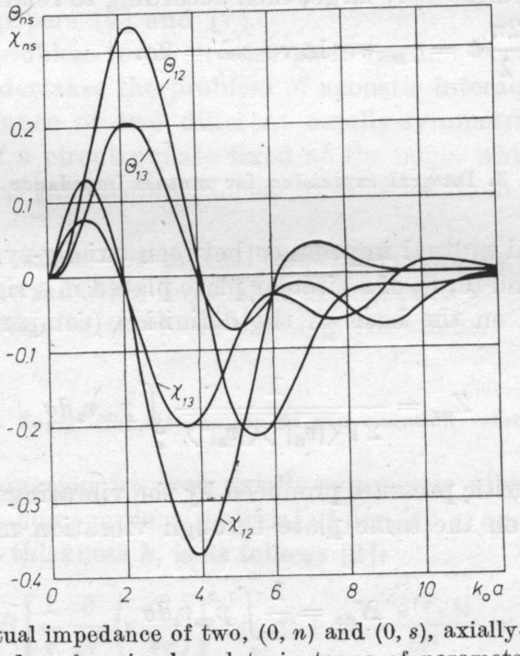

Fig. 1. Normalized mutual impedance of two, (0, n) and (0, s), axially-symmetric vibrations modes of a circular plate in terms of parameter $k_0 a$

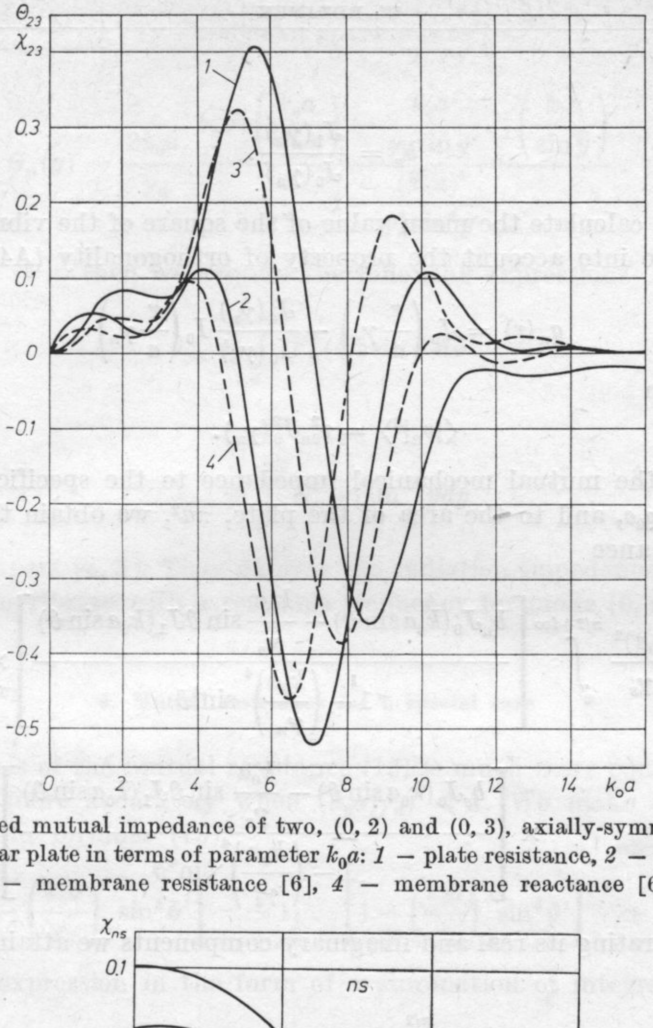

Fig. 2. Normalized mutual impedance of two, (0, 2) and (0, 3), axially-symmetric vibration modes of a circular plate in terms of parameter k_0a: *1* — plate resistance, *2* — plate reactance, *3* — membrane resistance [6], *4* — membrane reactance [6]

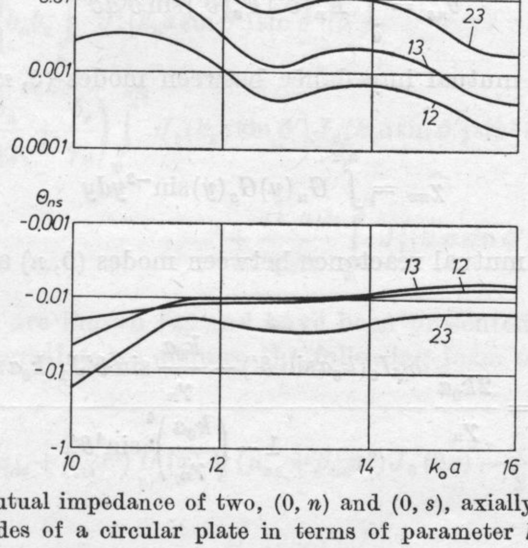

Fig. 3. Normalized mutual impedance of two, $(0, n)$ and $(0, s)$, axially-symmetric vibration modes of a circular plate in terms of parameter k_0a

where

$$b_n = \frac{J_1(\gamma_n)}{J_0(\gamma_n)}. \tag{10}$$

In order to calculate the mean value of the square of the vibration velocity $\langle |v_n|^2 \rangle$ we take into account the property of orthogonality (A4) for function

$$q_n(r) = J_0\left(\frac{r}{a}\gamma_n\right) - \frac{J_0(\gamma_n)}{I_0(\gamma_n)} I_0\left(\frac{r}{a}\gamma_n\right)$$

and we obtain

$$\langle |v_n|^2 \rangle = v_{0n}^2 J_0^2(\gamma_n). \tag{11}$$

Relating the mutual mechanical impedance to the specific resistance of the medium, $\varrho_0 c$, and to the area of the plate, πa^2, we obtain the normalized mutual impedance

$$\zeta_{ns} = \frac{(2k_0a)^2}{\gamma_n\gamma_s} \int\limits_0^{\pi/2+i\infty} \left[\frac{b_n J_0(k_0 a \sin\vartheta) - \dfrac{k_0 a}{\gamma_n}\sin\vartheta J_1(k_0 a \sin\vartheta)}{1 - \left(\dfrac{k_0 a}{\gamma_n}\right)^4 \sin^4\vartheta}\right] \times$$

$$\times \left[\frac{b_s J_0(k_0 a \sin\vartheta) - \dfrac{k_0 a}{\gamma_s}\sin\vartheta J_1(k_0 a \sin\vartheta)}{1 - \left(\dfrac{k_0 a}{\gamma_s}\right)^4 \sin^4\vartheta}\right] \sin\vartheta d\vartheta \tag{12}$$

and after separating its real and imaginary components we attain the following expressions

$$\theta_{ns} = \int\limits_0^{\pi/2} F_n(\vartheta') F_s(\vartheta') \sin\vartheta' d\vartheta' \tag{13}$$

for the normalized mutual impedance between modes $(0, n)$ and $(0, s)$ of the circular plate, and

$$\chi_{ns} = \int\limits_0^{\pi/2} G_n(y) G_s(y) \sin^{-2} y dy \tag{14}$$

for the normalized mutual reactance between modes $(0, n)$ and $(0, s)$ of the circular plate, where

$$F_n(\vartheta') = \frac{2k_0a}{\gamma_n} \frac{b_n J_0(k_0 a \sin\vartheta') - \dfrac{k_0 a}{\gamma_n}\sin\vartheta' J_1(k_0 a \sin\vartheta')}{1 - \left(\dfrac{k_0 a}{\gamma_n}\right)^4 \sin^4\vartheta'}, \tag{15}$$

$$G_n(y) = \frac{2k_0a}{\gamma_n} \frac{b_n J_0\left(\frac{k_0a}{\sin y}\right) - \frac{k_0a}{\gamma_n \sin y} J_1\left(\frac{k_0a}{\sin y}\right)}{1 - \frac{(k_0a)^4}{\gamma_n^4 \sin^4 y}}. \tag{16}$$

If we accept $s = n$, then we acquire the following expressions

$$\theta_{nn} = \int_0^{\pi/2} F_n^2(\vartheta') \sin\vartheta' d\vartheta' \tag{17}$$

and

$$\chi_{nn} = \int_0^{\pi/2} G_n^2(y) \sin^{-2} y dy \tag{18}$$

known from papers [4, 5]. They express the radiation impedance of a circular plate excited to vibrate with a resonans frequency for mode $(0, n)$.

4. Mutual resistance for a special case

The analysis of the mutual resistance (13) is much more convenient when $k_0a/\gamma_n < 1$, or more accurately when $(k_0a/\gamma_n)^4 \ll 1$. We make the following simplifications in formula (13)

$$\left[1 - \left(\frac{k_0a}{\gamma_n}\right)^4 \sin^4\vartheta'\right]^{-1} \simeq 1, \quad \left[1 - \left(\frac{k_0a}{\gamma_s}\right)^4 \sin^4\vartheta'\right]^{-1} \simeq 1 \tag{19}$$

and reach an expression in the form of a summation of integrals

$$\Theta_{ns} \simeq \frac{(2k_0a)^2}{\gamma_n\gamma_s} \Bigg\{ b_n b_s \int_0^{\pi/2} J_0^2(k_0a \sin\vartheta') \sin\vartheta' d\vartheta' - $$

$$- k_0a \left(\frac{b_n}{\gamma_s} + \frac{b_s}{\gamma_n}\right) \int_0^{\pi/2} J_1(k_0a\sin\vartheta') J_0(k_0a\sin\vartheta') \sin^2\vartheta' d\vartheta' +$$

$$+ \frac{(k_0a)^2}{\gamma_n\gamma_s} \int_0^{\pi/2} J_1^2(k_0a\sin\vartheta') \sin^3\vartheta' d\vartheta' \Bigg\}. \tag{20}$$

These integrals are known [4] and have been presented in formulae (A5), (A6) and (A7). Integrating we acquire the following form of the mutual resistance (20)

$$\theta_{ns} \simeq (2x)^2 \left[(a_{ns} + \beta_{ns} x^2)\, U(x) + (\mu_{ns} + \beta_{ns} x^2) J_0(2x) - \frac{3}{2} \beta_{ns} x J_1(2x) \right] \tag{21}$$

where: $x = k_0 a$, $U(x) = \frac{\pi}{2}[J_1(2x)S_0(2x) - J_0(2x)S_1(2x)]$, $S_m(x)$ is the m-order Struve function,

$$\alpha_{ns} = \frac{1}{\gamma_n\gamma_s}\left[b_n b_s + \frac{1}{2\gamma_n\gamma_s}\left(\frac{3}{4} - b_n\gamma_n - b_s\gamma_s\right)\right], \tag{22}$$

$$\beta_{ns} = \frac{1}{2\gamma_n^2\gamma_s^2}, \tag{23}$$

$$\mu_{ns} = \frac{b_n b_s}{\gamma_n\gamma_s}. \tag{24}$$

If moreover $x = k_0 a \ll 1$, then we can apply approximate formulae, (A9) and (A11), and thus

$$\theta_{ns} \simeq (2x)^2 \frac{b_n b_s}{\gamma_n\gamma_s}\left[1 - \frac{1}{3}\left(1 + \frac{1}{\gamma_s b_s} + \frac{1}{\gamma_n b_n}\right)x^2\right] \tag{25}$$

whereas for $n = s$

$$\theta_{nn} = (2x)^2\left(\frac{b_n}{\gamma_n}\right)^2\left[1 - \frac{1}{3}\left(1 + \frac{2}{\gamma_n b_n}\right)x^2\right]. \tag{26}$$

We will also analyse the mutual reistance when $k_0 a > \gamma_n, \gamma_s$, or more accurately when $(k_0 a)^4 \gg \gamma_n^4, \gamma_s^4$.

We perform a change of variables in formulae (13) and (15)

$$\theta_{ns} = 4(\gamma_n\gamma_s)^3 \int_0^{k_0 a} \frac{\left[b_n J_0(t) - \frac{t}{\gamma_n}J_1(t)\right]\left[b_s J_0(t) - \frac{t}{\gamma_s}J_1(t)\right]}{(\gamma_n^4 - t^4)(\gamma_s^4 - t^4)} \frac{t\,dt}{\sqrt{1 - \left(\frac{t}{k_0 a}\right)^2}}. \tag{27}$$

We use the approximate formula

$$\left[1 - \left(\frac{t}{k_0 a}\right)^2\right]^{-1/2} \simeq 1 + \frac{1}{2}\left(\frac{t}{k_0 a}\right)^2 + \frac{3}{8}\left(\frac{t}{k_0 a}\right)^4 + \ldots \tag{28}$$

and this results in the expression

$$\theta_{ns} \simeq \int_0^x A_n(t)A_s(t)\left[1 + \frac{1}{2}\left(\frac{t}{x}\right)^2 + \frac{3}{8}\left(\frac{t}{x}\right)^4\right] t\,dt \tag{29}$$

where

$$A_n(t) = 2\gamma_n^3 \frac{b_n J_0(t) - \frac{t}{\gamma_n} J_1(t)}{\gamma_n^4 - t^4}. \tag{30}$$

Three terms of the series (28) have been taken into account in order to ensure the convergence of integral (29) for very large values of $x = k_0 a$. If $t = \gamma_n$, then function $A_n(t)$ is an indeterminate symbol, which has the following limit

$$\lim_{t\to\gamma_n} A_n(t) = 2\gamma_n^3 \lim_{t\to\gamma_n} \frac{b_n J_0(t) - \frac{t}{\gamma_n} J_1(t)}{\gamma_n^4 - t^4} = \frac{1}{2}\,\frac{J_1^2(\gamma_n) + J_0^2(\gamma_n)}{J_0(\gamma_n)}. \tag{31}$$

Integral (29) within limits $(0, x)$ is presented in the form of a difference of integrals, i.e.

$$\int_0^x = \int_0^\infty - \int_x^\infty \tag{32}$$

The integral within limites $(0, \infty)$ is calculated from formula (A14), while the value of the integral within limits (x, ∞) can be neglected, because it is a small quantity in comparison to the value of the integral within limits $(0, \infty)$. Moreover if we take into account the characteristic equation (3) and Wronskians, (A1) and (A2), then finally we obtain

$$\theta_{ns} \simeq b_{ns} x^{-2}, \tag{33}$$

where $x = k_0 a > \gamma_n, \gamma_s$,

$$h_{ns} = 2\frac{(\gamma_n\gamma_s)^2}{\gamma_n^4 - \gamma_s^4}\left[\gamma_n \frac{J_1(\gamma_n)}{J_0(\gamma_n)} - \gamma_s \frac{J_1(\gamma_s)}{J_0(\gamma_s)}\right] \tag{34}$$

for $n \neq s$. In order to achieve higher accuracy of calculations, the integral within limits (x, ∞) has to subtracted in expression (33). The approximate value

Table 1. Coefficients h_{ns}, a_{ns}, β_{ns}, and μ_{ns}

n, s	1, 2	1, 3	2, 3
h_{ns}	-1.7275	-1.459	-3.504
a_{ns}	$4.904\cdot10^{-2}$	$3.278\cdot10^{-2}$	$1.678\cdot10^{-2}$
β_{ns}	$1.232\cdot10^{-3}$	$5.499\cdot10^{-4}$	$1.412\cdot10^{-4}$
μ_{ns}	$3.773\cdot10^{-2}$	$2.599\cdot10^{-2}$	$1.459\cdot10^{-2}$

of the integral for $(k_0a)^4 = x^4 \gg \gamma_n^4, \gamma_s^4$ is

$$\int_x^\infty \simeq \frac{33}{10} \frac{(\gamma_n \gamma_s)^2}{\pi} x^{-5}. \tag{35}$$

Values of several coefficients are gathered in Tab. 1 to facilitate numerical calculations.

Though there is a value of coefficient (34) within limits for $n = s$, but expression (33) for $n = s$ can not be used in calculations of the self-resistance. In this case an approximate formula, given in paper [8] should be applied.

5. Conclusions

Mutual acoustic interactions between vibration modes, $(0, n)$ and $(0, s)$, of a single circular plate take place for determined intervals of parameter k_0a. Extreme values of the mutual impedance occur for k_0a near γ_n and γ_s. For higher modes maximal interactions occur when the linear dimensions $2a$ of the plate are comparable with the integral multiple of the length of radiated waves, $n\lambda$.

It is characteristic that acoustic interactions suddenly decay for wave lengths λ slightly differing from $2a/n$. When the wave length is decreased still, then the mutual resistance also decreases assuming negative values and within the limit for $\lambda \to 0$ it equals zero. The mutual reactance also decreases with the frequency increase of radiated waves. It assumes positive values and within the limit for $k_0 = 2\pi/\lambda \to \infty$ approaches zero.

Acoustic interactions through a fixed mode $(0, n)$ and an arbitrary different mode $(0, s)$ are the smaller, the higher the value of $n-s$. If the value of $m-s$ is fixed, then acoustic interactions decrease when higher and higher modes, $(0, n)$ and $(0, s)$, are considered.

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Appendix A

The following Wronskians are known for the Bessel function $J_m(x)$, Neumann function $N_m(x)$ and the MacDonald function $K_m(x)$ [9]:

$$J_1(x)N_0(x)-J_0(x)N_1(x) = \frac{2}{\pi x}, \tag{A1}$$

$$I_1(x)K_0(x)+I_0(x)K_1(x) = \frac{1}{x}. \tag{A2}$$

The indefinite integral [9]

$$\int wJ_0(hw)J_0(lw)dw = \frac{w}{h^2-l^2}\{hJ_1(hw)J_0(lw)-lJ_0(hw)J_1(lw)\} \tag{A3}$$

can be applied also for complex quantities h, l.

Using the indefinite integral (A3) it can be proved that eigenfunctions

$$q_n(r) = J_0\left(\frac{r}{a}\gamma_n\right)-\frac{J_0(\gamma_n)}{I_0(\gamma_n)}I_0\left(\frac{r}{a}\gamma_n\right)$$

are orthogonal for $0 \leqslant r \leqslant a$ in the sense of the Kronecker delta, i.e.

$$\int_0^a q_n(r)q_m(r)rdr = a^2J_0^2(\gamma_n)\delta_{nm} \tag{A4}$$

if γ_n is the n-root of the characteristic equation (3).

The following definite integrals [5] are found in expression (20):

$$A_{00} \equiv \int_0^{\pi/2} J_0^2(x\sin t)\sin t dt = J_0(2x)+ \\ +\frac{\pi}{2}[J_1(2x)S_0(2x)-J_0(2x)S_1(2x)], \tag{A5}$$

$$A_{01} \equiv \int_0^{\pi/2} J_1(x\sin t) J_0(x\sin t)\sin^2 t\, dt = \frac{1}{2x}\,[A_{00}(x) - J_0(2x)] \tag{A6}$$

and

$$A_{11} \equiv \int_0^{\pi/2} J_1^2(x\sin t)\sin^3 t\, dt =$$

$$= \frac{1}{2} A_{00}(x) + \frac{3}{2}\frac{1}{(2x)^2}\,[A_{00}(x) - 2xJ_1(2x) - J_0(2x)]. \tag{A7}$$

It is convenient to introduce function

$$U(x) \equiv \frac{\pi}{2}\,[J_1(2x)S_0(2x) - J_0(2x)S_1(2x)] \tag{A8}$$

which for $x \ll 1$ can be approximated by the expression

$$U(x) \simeq \frac{2}{3}x^2\left(1 - \frac{3}{10}x^2\right) \tag{A9}$$

if we use approximate formulas for Struve and Bessel functions [3]:

$$S_0(x) \simeq \frac{2}{\pi}x\left(1 - \frac{x^2}{9}\right), \quad S_1(x) \simeq \frac{2}{3\pi}x^2\left(1 - \frac{x^2}{15}\right), \tag{A10}$$

$$J_0(x) \simeq 1 - \frac{x^2}{4}, \quad J_1(x) \simeq \frac{x}{2}\left(1 - \frac{x^2}{8}\right). \tag{A11}$$

Appendix B

The contour function (compare [9], [8])

$$\frac{1}{2\pi i}\int_c z^{\varrho-1} Z_\mu(bz)\,\frac{H_\nu^{(1)}(az)\,dz}{(z^4 - r^4)(z^4 - s^4)} \tag{A12}$$

where $a > b > 0$; r, s — complex numbers; Z_μ — μ-order cylindrical function; $|\mu| + |\nu| < \varrho < 10$, can be expressed in the form of a sum of residues in poles of the integrand. When $a = b$, then $\varrho < 9$.

With the application of the Jordan lemat and the Cauchy residuum theorem [1], the integration contour can be closed in the top half-plane of the complex variable z. Four poles of the integrand, for $z = r$, $z = ir$, $z = s$ and $z = is$, are enclosed during integration. We obtain

$$\frac{1}{2\pi i}\int\limits_0^\infty \{Z_\mu(bx)H_\nu^{(1)}(ax)-\exp(\varrho\pi i)Z_\mu[bx\exp(\pi i)]H_\nu^{(1)}[ax\exp(\pi i)]\}\times$$

$$\times\frac{x^{\varrho-1}dx}{(x^4-r^4)(x^4-s^4)}=\frac{1}{4(r^4-s^4)}\{r^{\varrho-4}Z_\mu(br)H_\nu^{(1)}(ar)-s^{\varrho-4}Z_\mu(bs)\times$$

$$\times H^{(1)}(as)+i^\varrho[r^{\varrho-4}Z_\mu(ibr)H_\nu^{(1)}(iar)-s^{\varrho-4}Z_\mu(ibs)H_\nu^{(1)}(ias)]\}. \quad \text{(A13)}$$

For a special case, when $Z_\mu = J_\mu$, $a = b = 1$, taking into account relations

$$J_\mu(ix)=\exp\left(i\mu\frac{\pi}{2}\right)I_\mu(x),\ H_\nu^{(1)}(ix)=\frac{2}{\pi}\exp\left[-i(\nu+1)\frac{\pi}{2}\right]K_\nu(x),$$

$$J_\mu[x\exp(\pi i)]=\exp(\mu\pi i)J_\mu(x),\qquad H_\nu^{(1)}[x\exp(\pi i)]=-\exp(-i\nu\pi)H_\nu^{(2)}(x),$$

$$H_\nu^{(1)}(x)=J_\nu(x)+iN_\nu(x),\qquad H_\nu^{(2)}(x)=J_\nu(x)-iN_\nu(x)$$

in place of (A13) we have

$$\int\limits_0^\infty J_\mu(x)J_\nu(x)\frac{x^{\varrho-1}dx}{(x^4-r^4)(x^4-s^4)}=\frac{\pi}{4(r^4-s^4)}\Big\{-r^{\varrho-4}J_\mu(r)H_\nu^{(1)}(r)-$$

$$-s^{\varrho-4}J_\mu(s)H_\nu^{(1)}(s)+\frac{2}{\pi}\cos(\varrho+\mu-\nu)\frac{\pi}{2}[r^{\varrho-4}I_\mu(r)K_\nu(r)-s^{\varrho-4}I_\mu(s)K_\nu(s)]\Big\} \quad \text{(A14)}$$

for $\varrho+\mu-\nu=2n$, $n=1,2,3,\ldots$ and

$$\int\limits_0^\infty J_\mu(x)N_\nu(x)\frac{x^{\varrho-1}dx}{(x^4-r^4)(x^4-s^4)}=\frac{\pi}{4(r^4-s^4)}\Big\{r^{\varrho-4}J_\mu(r)H_\nu^{(1)}(r)-$$

$$-s^{\varrho-4}J_\mu(s)H_\nu^{(1)}(s)+\frac{2}{\pi}\sin(\varrho+\mu-\nu)\pi/2[r^{\varrho-4}I_\mu(r)K_\nu(r)-s^{\varrho-4}I_\mu(s)K_\nu(s)]\Big\} \quad \text{(A15)}$$

for $\varrho+\mu-\nu=2n+1$, $n=0,1,2,\ldots$