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ACOUSTICAL IMPEDANCE OF A CIRCULAR MEMBRANE VIBRATING UNDER THE INFLUENCE OF A FORCE WITH A UNIFORM SURFACE DISTRIBUTION*

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The problem of acoustic impedance was analyzed for a circular membrane being acted on by a time-harmonic surface force with constant density. The membrane is innuersed in a lose-loss gaseous medium and the edges of the membrane are assumed to be rigid and fixed. Emploing the integral HUYGENS-RAYLEIGH formulas the exact formulae were obtained for the acoustic pressure and power. These formulae are especially convinient for digital computer calculations in the situation where the propagation velocity of the wave on the membrane surface is much smaller than the velocity of the acoustic wave propagation through the surronnoling medium. The acoustic impedance is presented as a function of an interference parameter.

1. Introduction

Although the problems connected with the acoustical field of a vibrating circular membrane are clasical problems of acoustics, up to now they have not been analyzed theoretically in detail with the application of mathematical methods. Among others a comprehensive analysis of the acoustical impedance of a circular membrane vibrating harmonically under the influence of a force with a konwn surface distribution, is lacking.

The knowledge of the acoustical impedance of a circular piston with a uniform distribution of the vibration velocity [6] and a piston with a nonuniform velocity distribution [5], is not sufficient to infer about the impedance of a circular membrane.

* This investigation was carried out within the problem MR. I. 24.

Skudrzyk [9] presented the problem of acoustical power radiated by a membrane for a determined vibration mode, but only for small interference parameters.

In paper [7] the author conducted an analysis of the acoustical pressuer in the far field of a circular membrane vibrating harmonically under the influence of a force with a uniform surface distribution.

The investigation of the radiation impedance of a vibrating membrane, done in this paper, is the next stage of the study on the acoustical properties of a vibrating membrane. The expression for the vibration velocity [2], [7], obtained by solving the non-uniform vibration equation for not damped and harmonic in time effects, was used. It was assumed, that a membrane is stretched on the circumference and placed in a perfectly rigid and flat acoustical baffle, and the gascous medium, in which it radiates, is non-dissipative. The exact expression for the radiation impedane was reached here on the grounds of the integral HUYGENS-RAYLEIGH equation. Obtained equations were given a thorough analysis in the domain of small interference parameters. Calculation resuls have been also presented graphically.

Notation

a – membrane radius,
b – radius of the central membrane surface, effected by a non-zero normal component of the exciting force,
c – propagation velocity of a wave in a fluid medium,
c_M – propagation velocity of a wave in the membrane,
f – surface density of the force exciting vibrations (1),
f_0 – time independent constant density of the force forcing vibrations,
J_m – m-order BESSEL function,
k – $\omega\sqrt{\frac{\eta}{T}}$,
k_0 – $2\pi/\lambda$,
M – characteristic function of the source (A7),
N – acoustical power radiated by the membrane (A3),
N_m – m-order NEUMANN function,
p – acoustical pressure (A4),
r_0 – radial variable of the membrane surface point in a polar coordinate system,
S_n – n-order STRUVE function,
T – force stretching the membrane, related to a unit length,
t – time,
U – function described by formula (A22),
v – normal component of the vibration velocity of the points on the surface of the membrane,
$\langle|v|^2\rangle$ – value of the quadratic mean of the vibration velocity (A2),
Z – mechanical impedance (A1),

α – function described by formula (17),
ζ – normalized relative impedance (A8),
η – surface density of the membrane,
Θ – normalized relative resistance (A8),
λ – wave length in a fluid medium,
ξ – transverse displacement of the membrane surface points,
ϱ_0 – rest denisty of the fluid medium,
σ_0 – membrane surface,
χ – normalized relative reactance (A8),
ω – angular frequency of the force forcing vibrations.

2. Analysis assumptions

A circular membrane, stretched with an equal force on circumference of a raduis a, is placed on a plane forming a rigid acoustical baffle, in an unlimited, ideal fluid medium with a rest density ϱ_0. The membrane is excited to vibrate transversally by an axially-symmetrical force (e. g. with the aid of two flat circular electrodes with a raduis, b parallel to the membrane surface), having the surface density equal to:

$$f(r,\,t) = \begin{cases} f_0 e^{i\omega t} & \text{for} \quad 0 < r < b, \\ 0 & \text{for} \quad b < r < a, \end{cases} \tag{1}$$

where f_0 is a constant, r – radial variable in a polar coordinate system, t – time, ω – angular frequency of the force forcing vibrations, b – radius of the circular membrane surface, on which the non-zero normal component of the force forcing vibrations acts.

The equation of the circular membrane vibrations [3] is as follows:

$$T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi(r,\,t)}{\partial r}\right) - \frac{\partial^2 \xi(r,\,t)}{\partial t^2} = -f(r,\,t), \tag{2}$$

where ξ is the distribution of the transverse vibrations of the membrane surface, $T = c_M^2\eta$ – force stretching the membrane, related to a unit length, η – membrane surface density, c_M – wave propagation velocity on the membrane, f – surface density of the axially-symmetrical force forcing vibrations.

The solution of equation (2) for a membrane excited to vibrate by force (1) has the form [2], [7]

$$\xi_1(r,\,t) = \frac{f_0}{\eta\omega^2}\left\{\frac{\pi kb}{2}\left[\frac{N_0(ka)}{J_0(ka)}J_1(kb) - N_1(kb)\right]J_0(kr) - 1\right\}e^{i\omega t} \tag{3}$$

for $0 < r < b$,

$$\xi_2(r,\,t) = \frac{f}{\eta\omega^2}\frac{\pi kb}{2}J_1(kb)\left[\frac{N_0(ka)}{J_0(ka)}J_0(kr) - N_0(kr)\right]e^{i\omega t} \tag{4}$$

for $b < r < a$, where $k = \omega\sqrt{\eta/T}$, J_m – the Bessel function, N_m – Neumann function, both (m-order). The solution presented here satisfies the boundary condition $\xi_2(a, t) = 0$ and the conformity conditions $\xi_1(b, t) = \xi_2(b, t)$ and

$$\frac{\partial \xi_1(r, t)}{\partial r} = \frac{\partial \xi_2(r, t)}{\partial r} \quad \text{for } r = b.$$

The normal component of the vibration velocity is obtained after taking into account, that

$$v(r, t) = \frac{\partial \xi(r, t)}{\partial t}, \tag{5}$$

while

$$\xi(r, t) = \xi_0(r) e^{i\omega t}.$$

3. Exact calculation of the radiation impedance

The characteristic function (A7) for a circular membrane which vibrates according to dependences (3) and (4), is equal to:

$$M(\vartheta) = \int_0^b v_1(r_0) J_0(k_0 r_0 \sin\vartheta) r_0 dr_0 + \int_b^a v_2(r_0) J_0(k_0 r_0 \sin\vartheta) r_0 dr_0. \tag{6}$$

We use the formula for a indefinite integral [11]

$$\int wJ_0(hw) Z_0(lw)\, dw = \frac{1}{h^2 - l^2} [hwJ_1(hw) Z_0(lw) - lwJ_0(hw) Z_1(lw)], \tag{7}$$

where Z_n is a cylindrical m-order function. We obtain

$$M(\vartheta) = \frac{if_0 b^2}{\eta\omega} \frac{1}{1 - \left(\frac{k_0}{k}\right)^2 \sin^2\vartheta} \left[\frac{J_1(kb)}{kb} \frac{J_0(k_0 a \sin\vartheta)}{J_0(ka)} - \frac{J_1(k_0 b \sin\vartheta)}{k_0 b \sin\vartheta} \right] \tag{8}$$

taking into account the Wronskian [11]

$$J_1(x) N_0(x) - J_0(x) N_1(x) = \frac{2}{x}. \tag{9}$$

The value of the quadratic mean of the vibration velocity, $\langle |v|_0^2 \rangle$, occuring in the formulas for the relative impedance, is determined for a case, when the distribution of the force forcing vibrations is not equal zero (uniform) on the whole membrane surface, so $0 \leqslant r \leqslant a$. This also means, that the relative impedance is normalized in such a way, that its real component tends to one when the wave number $k_0 = 2\pi/\lambda$ tends to an infinitelly great value and the whole membrane surface is excited to vibrate. In formulas (3) and (4) we accept $a = b$ and apply the definition (5), and then $v_2(r_0, t)|_{b=a} = 0$ and

$$v_1(r_0, t)|_{b=a} = v_0(r_0, t) = \frac{if_0}{\eta\omega}\left[\frac{J_0(kr_0)}{J_0(ka)} - 1\right] e^{i\omega t}. \tag{10}$$

On the basis of definition (A2) we achieve

$$\langle |v_0|^2 \rangle = \left(\frac{f_0}{a\eta\omega}\right)^2 \int_0^a \left[\frac{J_0(kr_0)}{J_0(ka)} - 1\right]^2 r_0\, dr_0. \tag{11}$$

Taking into account formulas [11] for indefinite integrals

$$\int J_0(hw)\, w\, dw = \frac{w}{h} J_1(hw), \tag{12}$$

$$\int J_0^2(hw)\, w\, dw = \frac{1}{2} w^2 [J_0^2(hw) + J_1^2(hw)], \tag{13}$$

we obtain

$$\langle |v_0|^2 \rangle = \left(\frac{f_0}{\eta\omega}\right)^2 \left[1 + \frac{1}{2}\frac{J_1^2(ka)}{J_0^2(ka)} - \frac{2}{ka}\frac{J_1(ka)}{J_0(ka)}\right]. \tag{14}$$

Exact expressions for the relative impedance will be reached by placing the calculation results of the characteristic function $M(\vartheta')$, $M(\vartheta'')$ and the quadratic mean of the vibration velocity $\langle |v_0|^2 \rangle$ in formulas (A9) and (A12). So

$$\Theta = (k_0 b)^2 \left(\frac{b}{a}\right)^2 \alpha^{-1} \int_0^{\pi/2} \left\{ \frac{1}{1 - \left(\frac{k_0}{k}\right)^2 \sin^2\vartheta'} \left[\frac{J_1(kb)}{kb}\frac{J_0(k_0 a \sin\vartheta')}{J_0(ka)} - \frac{J_1(k_0 b \sin\vartheta')}{k_0 b \sin\vartheta'}\right]\right\}^2 \sin\vartheta'\, d\vartheta', \tag{15}$$

$$\chi = (k_0 b)^2 \left(\frac{b}{a}\right)^2 \alpha^{-1} \int_0^{\pi/2} \left\{ \frac{1}{1 - \left(\frac{k_0}{k}\right)^2 \frac{1}{\sin^2 y}} \left[\frac{J_1(kb)}{kb}\frac{J_0\left(\frac{k_0 a}{\sin y}\right)}{J_0(ka)} - \frac{\sin y}{k_0 b} J_1\left(\frac{k_0 b}{\sin y}\right)\right] \frac{1}{\sin y}\right\}^2 dy, \tag{16}$$

where

$$\alpha = 1 + \frac{1}{2}\frac{J_1^2(ka)}{J_0^2(ka)} - \frac{2}{ka}\frac{J_1(ka)}{J_0(ka)}. \tag{17}$$

4. Radiation impedance at resonance frequencies

In a particular case, when the frequency of radiated waves is equal to the frequency of free vibrations of the membrane, the resonance effect occurs. At resonance frequencies $ka = x_n$, where quantities $x_1, x_2, x_3 = 2.4048 \ldots, 5.5201 \ldots, 8.6557 \ldots$ are the not of equation $J_0(x_n) = 0$. If $ka \to x_n$, $kb = (b/a)ka \to (b/a)x_n$,

$$\lim_{ka\to x_n} \alpha = \lim_{ka\to x_n} \frac{1}{2}\frac{J_1^2(x_n)}{J_0^2(ka)}, \tag{18}$$

then the radiation impedance (15), (16) are of the form

$$\lim_{ka\to x_n} \Theta = \Theta_n' = 2\left(\frac{k_0 a}{x_n}\right)^2 \left[\frac{b}{a}\frac{J_1(kb)}{J_1(x_n)}\right]^2 \int_0^{\pi/2} \frac{J_0^2(k_0 a \sin\vartheta')\sin\vartheta'}{\left[1-\left(\frac{k_0 a}{x_n}\right)^2 \sin^2\vartheta'\right]^2} d\vartheta', \tag{19}$$

$$\lim_{ka\to x_n} \chi = \chi_n' = 2\left(\frac{k_0 a}{x_n}\right)^2 \left[\frac{b}{a}\frac{J_1(kb)}{J_1(x_n)}\right]^2 \int_0^{\pi/2} \frac{J_0^2\left(\frac{k_0 a}{\sin y}\right)}{\left[1-\left(\frac{k_0 a}{x_n}\right)^2 \frac{1}{\sin^2 y}\right]^2} \frac{dy}{\sin^2 y}. \tag{20}$$

The obtained expressions present the radiation resistance and radiation reactance of a circular membrane of a radius a, excited by a force with uniform distribution on its central circular surface of a radius b.

If we then put $b = a$, we reach expressions:

$$\Theta_n = 2\left(\frac{k_0 a}{x_n}\right)^2 \int_0^{\pi/2} \frac{J_0^2(k_0 a \sin\vartheta')\sin\vartheta'}{\left[1-\left(\frac{k_0 a}{x_n}\right)^2 \sin^2\vartheta'\right]^2} d\vartheta' \tag{21}$$

and

$$\chi_n = 2\left(\frac{k_0 a}{x_n}\right)^2 \int_0^{\pi/2} \frac{J_0^2\left(\frac{k_0 a}{\sin y}\right)}{\left[1-\left(\frac{k_0 a}{x_n}\right)^2 \frac{1}{\sin^2 y}\right]^2} \frac{dy}{\sin^2 y} \tag{22}$$

which are known from paper [8] and are the formulae for the radiation impedance of a circular membrane, excited to vibrate axially-symmetrically, so for $a(0, n)$ vibration mode.

5. Radiation resistance in a particular case

The radiation resistance is easier to analyze, when $k_0/k < 1$ and $b = a$. In such a case approximate formulae can be used.

For $(k_0/k)^2 \ll 1$, a reduction can be applied

$$\left[1-\left(\frac{k_0}{k}\right)^2 \sin\vartheta'\right]^{-2} \simeq 1+2\left(\frac{k_0}{k}\right)^2 \sin^2\vartheta' \tag{23}$$

and including $b = a$

$$\Theta' = \frac{(k_0 a)^2}{\alpha} \int_0^{\pi/2} \left[1+2\left(\frac{k_0}{k}\right)^2 \sin^2\vartheta'\right]\left[\frac{J_1(ka)}{ka} \frac{J_0(k_0 a \sin\vartheta')}{J_0(ka)} - \frac{J_1(k_0 a \sin\vartheta')}{k_0 a \sin\vartheta'}\right]^2 \sin\vartheta' d\vartheta'. \tag{24}$$

The obtained expression for the radiation resistance is expressed by a sum of intefrals calculated from formulas (A17), (A19), (A20), (A21) and (A22). After integrating

$$2\alpha\Theta' = 1-\frac{J_1(2x)}{x} - \frac{2}{y}\frac{J_1(y)}{J_0(y)}[1-J_0(2x)] + (2\varepsilon)^2\left\{-\frac{J_1(2x)}{x} - \frac{J_1(y)}{yJ_0(y)}U(x) + \left[1+\frac{1}{2}\frac{J_1^2(y)}{J_0^2(y)}\right][J_0(2x)+U(x)]\right\}, \tag{25}$$

where

$$x = k_0 a,\ y = ka, \qquad \varepsilon = x/y = k_0/k, \qquad U(x) = \frac{\pi}{2}[J_1(2x)S_0(2x) - J_0(2x)S_1(2x)].$$

In formula (25) only the small terms of second order with respect to ε have been taken into account.

6. Example and conclusions

The diagrams of the radiation resistance and reactance of a circular membrane excited to vibrate by a force with a uniform surface distribution are shown in Fig. 1. $k_0/k = c_M/c = 1$ was taken, so the propagation velocity of a wave in a fluid medium is equal to the wave propagation velocity on the membrane.

Fig. 2 presents diagrams of radiation resistance fo r$k_0/k = 1, 2$ and 4, while Fig. 3 for $k_0/k = 1/2$.

In the analyzed example it was supposed that a non zero uniform distribution of the force forcing vibrations occurs on the whole surface of the membrane.

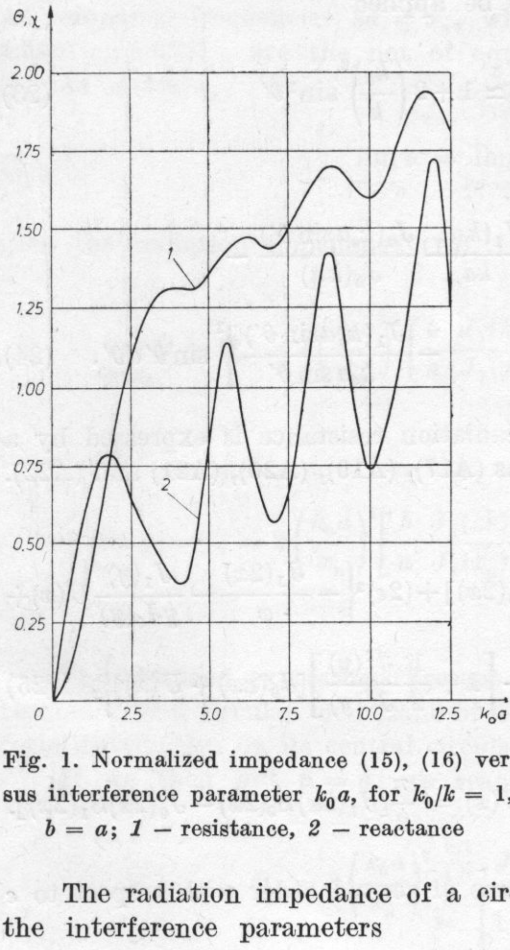

Fig. 1. Normalized impedance (15), (16) versus interference parameter k_0a, for $k_0/k = 1$, $b = a$; *1* – resistance, *2* – reactance

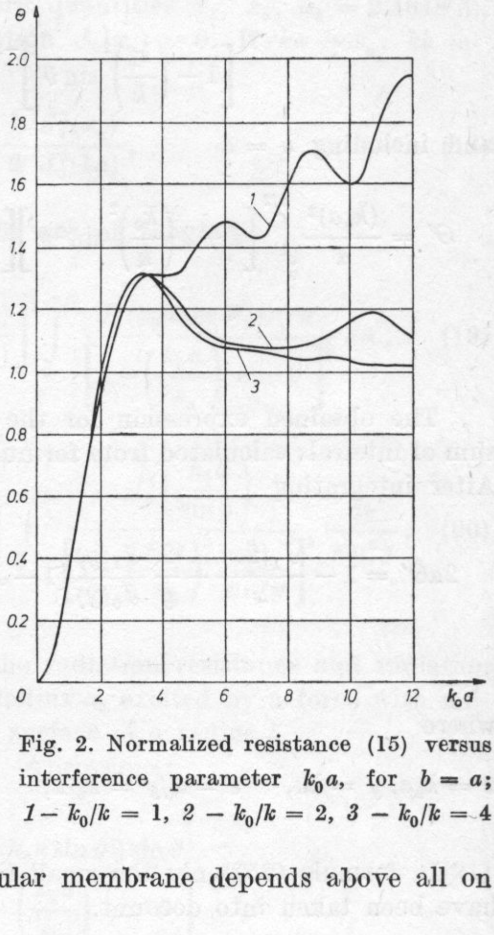

Fig. 2. Normalized resistance (15) versus interference parameter k_0a, for $b = a$; *1* – $k_0/k = 1$, *2* – $k_0/k = 2$, *3* – $k_0/k = 4$

The radiation impedance of a circular membrane depends above all on the interference parameters

$$ka = \frac{\omega}{c_M} a, \quad k_0 a = \frac{\omega}{c} a$$

and at fixed dimensions of the membrane (fixed diameter a) on the frequency of the radiated waves, the propagation velocity in the fluid medium and the propagation velocity of a wave on the membrane.

Unfavourable radiation conditions take place for $c_M/c_0 < 1$. Maximum values of the real component of the radiation impedance are lower from the corresponding resistance values for $k_0/k = 1$. Also such values of the k_0a parameter occur, for which radiation with the employment of the real impedance imponent is not present (Fig. 3).

It results from the above analysis that the radiation impedance is of a finite value for every quantity of the interference parameter k_0a. The acoustical radiation power and vibration velocity for the analyzed membrane were infinitely high for resonance frequencies.

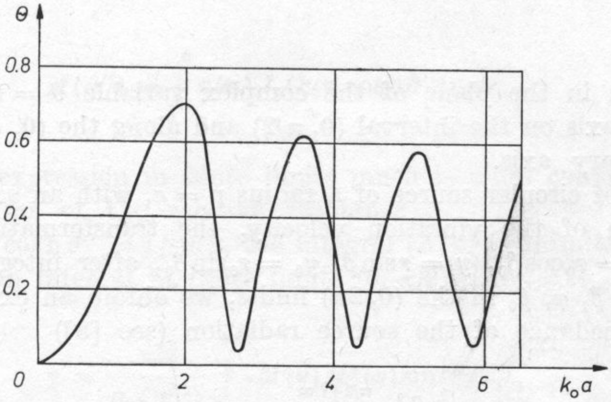

Fig. 3. Normalized resistance (15) versus interference parameter $k_0 a$, for $k_0/k = 1/2$, $b = a$

Appendix A

The calculation of the mechanical impedance of a vibrating membrane is calculated in accordance to definition [10]

$$Z = \frac{N}{\langle |v|^2 \rangle}, \tag{A1}$$

where

$$\langle |v|^2 \rangle = \frac{1}{2\sigma} \int_\sigma v^2(r)\, d\sigma \tag{A2}$$

is the quadratic mean value of the vibration velocity, while

$$N = \frac{1}{2} \int_\sigma p(\vec{r})\, v^*(\vec{r})\, d\sigma \tag{A3}$$

s the expression for the acoustical power radiated by the membrane v^* marks the value conjugated to the complex quantity of the vibration velocity v.

The acoustical pressure [6]

$$p(\vec{r}) = \frac{ik_0 \varrho_0 c}{2\pi} \int_{\sigma_0} V(\vec{r}_0) \frac{e^{-ik_0|\vec{r} - \vec{r}_0|}}{|\vec{r} - \vec{r}_0|}\, d\sigma_0 \tag{A4}$$

– radiated by the membrane – is rearranged with the application of the Fourier expansion [8]

$$\frac{e^{-ik_0|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} = \frac{-ik_0}{2\pi} \int\limits_0^{\pi/2+i\cdot\infty} \exp\{-ik_0 \sin\vartheta[(x-x_0)\cos\varphi+(y-y_0)\sin\varphi]\}\sin\vartheta\, d\vartheta\, d\varphi. \tag{A5}$$

Integration in the plane of the complex variable $\vartheta = \vartheta' + i\vartheta''$ is done along the real axis on the interval $(0, \pi/2)$ and along the $(0, \infty)$ line, parallel to the imaginary axis.

In a case of circular source of a radius $r = a$, with an axially-symmetrical distribution of the vibration velocity, the transformation coordinates: $x = r\cos\beta$, $x_0 = r_0\cos\beta_0$, $y = r\sin\beta$, $y_0 = r_0\sin\beta_0$, after integrating over angular variables β, φ, β_0 in the $(0, 2\pi)$ limits, we obtain an expression for the mechanical impedance of the source radiation (see [8])

$$Z = \varrho_0 c \frac{\pi k_0^2}{\langle |v|^2 \rangle} \int\limits_0^{\pi/2+i\cdot\infty} M(\vartheta) M^*(\vartheta) \sin\vartheta\, d\vartheta, \tag{A6}$$

where

$$M(\vartheta) = \int\limits_0^a v(r_0) J_0(k_0 r_0 \sin\vartheta) r_0\, dr_0 \tag{A7}$$

is the characteristic function of the source.

The relative impedance is

$$Z/\varrho_0 c\sigma_0 = \zeta = \Theta + i\chi, \tag{A8}$$

where Θ, χ are the relative resistance and reactance, respectively, c – wave propagation velocity in a fluid medium, and $\sigma_0 = \pi a^2$.

The real component of the relative impedance, i. e., the relative resistance, is acquired from expression (A6), when the integration in the complex variable $\vartheta = \vartheta' + i\vartheta''$ is performed on the interval on the real axis ϑ' in the $(0, \pi/2)$ limits, i. e.,

$$\Theta = \frac{\pi k_0^2}{\sigma\langle |v|^2 \rangle} \int\limits_0^{\pi/2} M(\vartheta') M^*(\vartheta') \sin\vartheta'\, d\vartheta'. \tag{A9}$$

In order to isolate the imaginary component of the relative impedance in formula (A6), calculations have to be limited to the calculation of the integral over a half-line parallel to the imaginary axis in the complex variable plane, $\vartheta = \vartheta' + i\vartheta''$. Accepting $\vartheta' = \pi/2$, we obtain $\vartheta = \pi/2 + i\vartheta''$, $0 \leqslant \vartheta'' < \infty$, and then

$$\chi = \frac{\pi k_0^2}{\sigma\langle|v^2\rangle}\int_0^\infty M(\vartheta'')M^*(\vartheta'')\cos h\vartheta''\,d\vartheta'' \tag{A10}$$

while

$$M(\vartheta'') = \int_0^a v(r_0)J_0(k_0r_0\cos\mathrm{h}\vartheta'')r_0dr_0. \tag{A11}$$

An integral expression in finite limits much is more convenient for numerical calculations of the relative reactance.

Substituting $\cos\mathrm{h}\,\vartheta'' = 1/\sin y$, the integral (A10) in infinite limits $(0, \infty)$ is converted to an integral in finite limits $(0, \pi/2)$, i. e.,

$$\chi = \frac{\pi k_0^2}{\sigma_0\langle|v|^2\rangle}\int_0^{\pi/2} M(y)M^*(y)\sin^{-2}y\,dy, \tag{A12}$$

where

$$M(y) = \int_0^a v(r_0)J_0\left(\frac{k_0r_0}{\sin y}\right)r_0dr_0. \tag{A13}$$

Appendix B

In order to determine the value of the integral

$$\mathrm{A}_{10} = \int_0^{\pi/2} J_0(x\sin t)J_1(x\sin t)\,dt \tag{A14}$$

the product of two Bessel function is expanded into a series [4], [11]

$$J_p(u)J_q(u) = \sum_{n=0}^\infty(-1)^n\frac{(2n+p+q)!(\frac{1}{2}u)^{2n+p+q}}{n!(n+p)!(n+q)!(n+p+q)!} \tag{A15}$$

and, putting $\mathrm{p} = 0$, $q = 1$, $u = x\sin t$ we have

$$J_0(x\sin t)J_1(x\sin t) = \sum_{n=0}^\infty(-1)^n\frac{(2n+1)!(\frac{1}{2}x\sin t)^{2n+1}}{(n!)^2[(n+1)!]^2}. \tag{A16}$$

The alternating series (A16) is substituted in the integral in expression (A14). Integrating term by term we obtain

$$A_{10} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!(\frac{1}{2}x)^{2n+1}}{[n!(n+1)!]^2} \int_0^{\pi/2} \sin^{2n+1} t\, dt$$

$$= \frac{1}{2x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{[(n+1)!]^2} = \frac{1}{2x}[1 - J_0(2x)], \quad \text{(A17)}$$

where the intagrate property was applied

$$\int_0^{\pi/2} \sin^{2n+1} t\, dt = \frac{1}{2} \frac{\Gamma(n+1)\sqrt{\pi}}{\Gamma(n+3/2)} = \frac{2^{2n}(n!)^2}{(2n+1)!}. \quad \text{(A18)}$$

In a similar way the other integrals, essential for the radiation resistance analysis, can be calculated. Most of them is considered in papers [1] and [4]:

$$A_{00} = \int_0^{\pi/2} J_0^2(x \sin t) \sin t\, dt = \frac{1}{2x} \int_0^{2x} J_0(t)\, dt$$

$$= J_0(2x) + \frac{\pi}{2}[J_1(2x) S_0(2x) - J_0(2x) S_1(2x)], \quad \text{(A19)}$$

where S_n is the n-order STRUVE function.

$$A_{11} = \int_0^{\pi/2} \frac{J_1^2(x \sin t)}{\sin t}\, dt = \frac{1}{2}\left[1 - \frac{J_1(2x)}{x}\right], \quad \text{(A20)}$$

$$B_{10} = \int_0^{\pi/2} J_1(x \sin t) J_0(x \sin t) \sin^2 t\, dt = \frac{\pi}{4x}[J_1(2x) S_0(2x) -$$

$$- J_0(2x) S_1(2x)], \quad \text{(A21)}$$

$$B_{11} = \int_0^{\pi/2} J_1^2(x \sin t) \sin t\, dt = J_0(2x) - \frac{J_1(2x)}{x} +$$

$$+ \frac{\pi}{2}[J_1(2x) S_0(2x) - J_0(2x) S_1(2x)]. \quad \text{(A22)}$$

It is convenient to introduce function

$$U(x) = \frac{\pi}{2}[J_1(2x) S_0(2x) - J_0(2x) S_1(2x)] \quad \text{(A23)}$$

which can be approximated for $x \ll 1$ by the expression

$$U(x) \simeq \frac{2}{3} x^2 \left(1 + \frac{3}{10} x^2\right) \quad \text{(A24)}$$

if we use the approximate formulae for the STRUVE and BESSEL functions [4]:

$$S_0(x) \simeq \frac{2}{\pi} x \left(1 - \frac{x^2}{9}\right), \quad S_1(x) \simeq \frac{2}{3\pi} x^2 \left(1 - \frac{x^2}{15}\right), \tag{A25}$$

$$J_0(x) \simeq 1 - \frac{x^2}{4}, \quad J_1(x) \simeq \frac{x}{2}\left(1 - \frac{x^2}{8}\right). \tag{A26}$$

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