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# DETERMINATION OF CIRCULAR MEMBRANE PARAMETERS FROM ITS RESONANCE FREQUENCIES

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Circumferential forces, speed of elastic wave and dissipation of energy factor are calculated from resonance frequencies of circular membranes made of tantalum and nickel-chromium steel. The results obtained for a mathematical model without dissipation are compared with the results obtained for a model with dissipation of energy. The assignment of the coefficient of partial differential equation is presented for this second case. An algorithm applied to computer simulation of membrane vibrations is based on the "leap frog" difference method.

Znając częstotliwości rezonansowe membran kołowych można obliczyć ich obwodowe siły napięcia, prędkości propagacji fal sprężystych i współczynnik dyssypacji energii. Obliczenia przeprowadzono posługując się danymi uzyskanymi dla membrany tantalowej i membran ze stali niklowo-chromowej. Porównane są wyniki obliczeń dla modelu matematycznego bez dyssypacji z rezultatami dla modelu uwzględniającego rozpraszanie energii. Dla tego drugiego przypadku przedstawiony jest również sposób identyfikacji współczynników równania różniczkowego cząstkowego na podstawie danych eksperymentalnych. Algorytm zastosowany do komputerowego modelowania drgań membran opiera się na metodzie różnicowej „leap frog".

## 1. Introduction

Euler gave a mathematical model of circular membrane vibrations [1, 4]. This is the differential equation of the hyperbolic type for axial strain as a function of three variables: time $t$ and polar co-ordinates, that is the distance from the centre of membrane and angle $\phi$

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial r^2}-\frac{1}{r}\frac{\partial z}{\partial r}-\frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}=0, \tag{1}$$

where the constant coefficient $c$ is the speed of elastic wave.

It is most frequently assumed that the membrane is fixed stiff on the circumference. This means that the boundary conditions for equation (1) are equal to zero. Thus the solution of the homogeneous differential equation (1) presents nondamped vibrations as a result of nonzero initial conditions. If they are well assumed it is possible to simulate steady and symmetrical axial strains. Using the classical method of separation of variables [1] [2] we obtain the relation

$$c = \frac{\omega_i R}{x_i} \tag{2}$$

which enables to calculate the speed of elastic wave $c$. $R$ is the radius of membrane, $x_i$ are roots of Bessel's function of first kind and zero order, $\omega_i$ are resonance frequencies. Next, from the speed of elastic wave it is possible to calculate the circumferencial force. This well known formula enables to calculate the value of the force which is difficult to measure. Usually we can measure a few resonance frequencies. For each of them we obtain from (2) an estimator for the speed of an elastic wave. Differences between these estimators usually differ at the second decimal place. This justifies a verification of the usability of a more complex mathematical model with dissipation of energy.

## 2. Mathematical model of membrane

Assuming the axial symmetry of forces which deform membrane and introducing into equation (1) the term for dissipation of energy and term for forced vibration, we obtain nonhomogeneous partial differential equation of hyperbolic type

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial r^2} + a\frac{\partial z}{\partial t} - \frac{1}{r}\frac{\partial z}{\partial r} = u, \tag{3}$$

where $z$ — transverse strain of membrane [m], $c$ — speed of elastic wave travelling along the radius of membrane [m/s], $a$ — positive coefficient of dissipation of the energy [s/m$^2$], $u$ — force vibration function of time $t$ and space variable $r$ [m$^{-1}$], $t$ — time [s].

To obtain a unique solution of the equation (3) we assume initial conditions

$$z(r,0) = 0, \quad \left.\frac{\partial z}{\partial t}\right|_{t=0} = 0, \tag{4}$$

and boundary conditions

$$\left.\frac{\partial z}{\partial r}\right|_{r=0} = 0, \quad z(R,t) = 0, \tag{5}$$

where $R$ [$m$] is the radius of membrane. The first boundary condition follows from the axial symmetry of strain and the second one means that the membrane is fixed stiff on circumference.

Using the classical method we assume that the solution of initial boundary value problem (3), (4), (5) can be described as an infinite series

$$z(r,t) = \sum_{i=1}^{\infty} T_i(t) R_{i(r)}. \tag{6}$$

A function forcing the vibration of membrane with frequency $\omega$ can also be written as an infinite series with separable variables

$$u(r,t) = \sum_{i=1}^{\infty} p_i R_i(r) \sin\omega t, \tag{7}$$

where $p_i$ are constant coefficients.

Computing the partial derivatives of function (6) and substituting them into (3) we obtain conditions

$$\frac{1}{c^2}\frac{T_i''}{T_i} + a\frac{T_i'}{T_i} - \frac{p_i}{T_i}\sin\omega t = \frac{R_i''}{R_i} + \frac{1}{r}\frac{R_i'}{R_i}, \tag{8}$$

where the upper indexes denote the differentiation with respect to time for the left side of (8) and differentiation with respect to the space variable for the right side of (8). Denoting the value of both sides by $-k_i^2$ we obtain two sets of ordinary differential equations

$$R_i'' + \frac{1}{r} R_i' + k_i R_i = 0, \tag{9}$$

$$\frac{1}{c^2} T_i'' + a T_i' + k_i^2 T_i = p_i \sin\omega t. \tag{10}$$

The boundary condition for equation (9) is obtained from (5)

$$R_i'(0) = 0, \quad R_i(R) = 0. \tag{11}$$

In this way we improve the constrains for basis functions $R_i$. It follows that function $u$ must fulfil conditions

$$\left.\frac{\partial z}{\partial r}\right|_{r=0} = 0, \quad z(R) = 0,$$

for convergence of series (7).

From (4) we obtain the initial conditions for equation (10)

$$T_i(0) = 0, \quad T_i'(0) = 0. \tag{12}$$

The solution of differential equation (9) is Bessel's function of first kind and zero order which can be defined either by

$$R_i(r) = \sum_{j=0}^{\infty} (-1)^j \frac{k_i^{2j}}{j!j!} \left(\frac{r}{2}\right)^{2j} \tag{13}$$

or in an equivalent form

$$R_i(r) = \sum_{j=0}^{\infty} a_j (k_i r)^{2j},$$

$$a_0 = 1, \quad a_{j+1} = \frac{a_j}{4j^2}. \tag{14}$$

From (13) we find that the first condition of (11) is always satisfied. The second condition requires

$$k_i = \frac{x_i}{R}, \tag{15}$$

where $x_i$ are roots of Bessel's function.

The solution of equation (10) with the initial conditions (12) has the form

$$T_i = A_i e^{-\alpha t} \sin(\omega_i t + \phi_i) + B_i \sin(\omega t + \psi_i). \tag{16}$$

The frequency of damping vibrations is given by

$$\omega_i = c\sqrt{k_i^2 - \frac{a^2 c^2}{4}}. \tag{17}$$

Introducing auxiliary variable

$$s_i = \sqrt{\left(k_i^2 - \frac{\omega^2}{c^2}\right)^2 + a^2 \omega^2}, \tag{18}$$

we obtain initial amplitude for damping vibrations

$$A_i = \frac{\omega p_i}{\omega_i s_i}. \tag{19}$$

The phase displacement of these vibrations is obtained from formula

$$\phi_i = \arcsin \frac{a\omega_i}{s_i} \tag{20}$$

and the damping coefficient from

$$\alpha = \frac{ac^2}{2}. \tag{21}$$

The amplitude of steady vibration with forced frequency is given by

$$B_i = \frac{p_i}{s_i} \tag{22}$$

and its phase displacement

$$\psi_i = -\arcsin \frac{a\omega}{s_i} \tag{23}$$

The greatest amplitudes of the steady vibrations occur for resonance frequencies

$$t_i^r = \frac{c}{2\pi}\sqrt{k_i^2 - \frac{a^2 c^2}{2}} \tag{24}$$

and are given by

$$B_i^r = \frac{p_i}{a\omega_i}, \tag{25}$$

where $\omega_i$ is defined by (17). The coefficient of dissipation must be small enough for the $i$-th resonance vibrations to occur, that is

$$a < \frac{\sqrt{2}\,k_i}{c}. \tag{26}$$

### 3. Determination of membrane parameters

The formula which connects resonance frequencies with the parameters of membrane is obtained from (15) and (24). Assuming that we have managed to measure $N$ first resonance frequencies, which we denote now by $\omega_i$, we obtain the set of equations

$$\omega_i^2 = c^2\left[\left(\frac{x_i}{R}\right)^2 - \frac{a^2 c^2}{2}\right]. \tag{27}$$

This means that the resonance frequencies are the functions of three parameters: speed of elastic wave, coefficient of dissipation and radius of membrane. The Jacobian determinant of function (27) has the form

$$\mathrm{J} = \begin{vmatrix} 2c\left(\frac{x_1}{R}\right)^2 - 2a^2c^3 & -c^4 a & -\frac{2c^2 x_1^2}{R^3} \\ 2c\left(\frac{x_2}{R}\right)^2 - 2a^2c^3 & -c^4 a & -\frac{2c^2 x_2^2}{R^3} \\ \cdots & \cdots & \cdots \\ 2c\left(\frac{x_N}{R}\right)^2 - 2a^2c^3 & -c^4 a & -\frac{2c^2 x_N^2}{R^3} \end{vmatrix}. \tag{28}$$

Only two columns of determinant (28) are linearly independent, therefore only two parameters can be calculated from resonance frequencies. As it is easy to measure the radius of membrane, so we will calculate, from (27), the speed of elastic wave and coefficient of dissipation.

From two arbitrary equations (27) we obtain

$$c_{ij} = R\sqrt{\frac{w_i^2 - w_j^2}{x_i^2 - w_j^2}}, \quad 1 \leqslant i, \; j \leqslant N, \; i \neq j, \tag{29}$$

where $c_{ij}$ is one of the estimates of the speed of elastic wave. For $N$ resonance frequencies, the average value of estimates is equal to

$$c = \frac{2(N-2)!}{N!} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_{ij}. \tag{30}$$

Next we can calculate the force stretching the membrane [N/m]

$$F = q d c^2, \tag{31}$$

where $q$ – mass density of material of membrane [kg/m³], $d$ – membrane thickness [m].

From (27) we can calculate estimates for the coefficient of dissipation

$$a_i = \frac{\sqrt{2}}{c}\sqrt{\frac{x_i^2}{R^2} - \frac{\omega_i^2}{c^2}} \tag{32}$$

and obtain finally

$$a = \frac{1}{N}\sum_{i=1}^{N} a_i.$$

If it is possible to measure the amplitudes $B_i^r$ of vibrations of the centre of membrane, we can calculate additionally the amplitudes of forced vibrations [1/m]

$$p_i = B_i^r ac\sqrt{\frac{x_i^2}{R^2} - \frac{a^2c^2}{4}}. \tag{33}$$

## 4. Measurement of resonance frequencies

The vibrations of circular membrane, stretched with the same force on the whole circumference, can be stimulated by a sonic waves. In this way, the resonance frequencies of membrane were assigned. A variable frequency generator supplied a loudspeaker and was connected with frequency meter. The sonic waves from the loudspeaker reached microphone through the membrane fixed between the stretching rings. The signal from the microphone was amplified and next measured by the digital voltage meter. Experiments were started from the possible lowest frequency and afterward it was increased gradually. The first resonance frequency was found when the sound intensity indicated by the microphone had the well-marked greatest value. Next, by increasing the frequency of generator once again, the other resonance vibrations were found.

Measurements were carried out for membranes made of tantalum and nickel-chromium steel. The first five resonance frequencies were found for every membrane. The results are presented in Table 1.

**Table 1.** Measurement results

| Membrane No | 1 | 2 | 3 | |
|---|---|---|---|---|
| Material | tantalum | steel | steel | |
| Radius [mm] | 27.8 | 27.8 | 40 | |
| Thickness [mm] | 0.025 | 0.127 | 0.127 | |
| Roots of Bessel's function | Resonance frequencies [Hz] | | | Amplitude [mm] |
| 2.40483 | 434 | 1043 | 695 | 1.25 |
| 5.52008 | 1050 | 2594 | 1940 | 0.54 |
| 8.65373 | 1658 | 4115 | 3130 | 0.34 |
| 11.79153 | 2264 | 5630 | 4285 | 0.25 |
| 14.93092 | 2870 | 7141 | 5428 | 0.19 |

Sometimes the vibrations of the center of membrane have amplitude great enough to be measured. For this purpose the measuring position was equipped additionally with a micrometer screw. Its end was placed above the center of the membrane. The junction of micrometer screw with the metal membrane was signalled by an ohmmeter as an electric short circuit. After putting the membrane into resonance vibrations, the micrometer screw was dropped until the junction of the micrometer screw with the membrane during its greatest deflection. Next, the generator was switched off and the micrometer screw was dropped once again until it reached the membrane. The difference between the positions of micrometer screw in both these cases was equal to the amplitude of membrane vibrations. These measurements were made only for the third membrane which had the greatest amplitudes of vibration.

Substituting the values of the resonance frequencies into (2) we obtain for Euler equation (1) the estimates of coefficient $c$. The results of these calculations for data from Table 1 are presented in Table 2. For every membrane there were measured five resonance frequencies, therefore we obtained five estimates of parameter $c$. The last line of Table 2 presents the mean values of wave speeds.

From formula (29) we obtain for each membrane ten estimates of the speed of elastic wave. The results are presented in Table 3 and their mean values are written in the last line. The circumferencial forces were calculated in accordance with (31).

**Table. 2.** Speeds of elastic waves calculated from Euler's model [m/s]

| No of resonance frequency | Membrane No | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 1 | 31.52 | 75.76 | 72.63 |
| 2 | 33.23 | 82.08 | 88.33 |
| 3 | 33.47 | 83.06 | 90.90 |
| 4 | 33.54 | 83.40 | 91.33 |
| 5 | 33.58 | 83.54 | 91.37 |
| average | 33.07 | 81.57 | 86.91 |

**Table 3.** Speeds of elastic waves [m/s] and circumferencial forces [N/m] calculated from the model with dissipation

| Estimate No | Membrane | | | | | |
|---|---|---|---|---|---|---|
| | 1 | | 2 | | 3 | |
| | speed | force | speed | force | speed | force |
| 1 | 33.61 | 469 | 83.49 | 7120 | 91.62 | 8580 |
| 2 | 33.62 | 469 | 83.64 | 7150 | 92.27 | 8700 |
| 3 | 33.62 | 469 | 83.72 | 7160 | 92.06 | 8660 |
| 4 | 33.63 | 469 | 83.74 | 7160 | 91.81 | 8610 |
| 5 | 33.63 | 469 | 83.72 | 7160 | 92.63 | 8770 |
| 6 | 33.62 | 469 | 83.77 | 7170 | 92.16 | 8680 |
| 7 | 33.63 | 469 | 83.77 | 7170 | 91.84 | 8620 |
| 8 | 33.62 | 469 | 83.79 | 7170 | 91.83 | 8620 |
| 9 | 33.63 | 469 | 83.78 | 7170 | 91.60 | 8570 |
| 10 | 33.64 | 470 | 83.77 | 7170 | 91.43 | 8540 |
| average | 33.63 | 469 | 83.72 | 7160 | 91.93 | 8640 |

**Table 4.** Values of dissipation coefficients [s/m$^2$]

| No of resonance frequency | Membrane No | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 1 | 1.27 | 0.622 | 0.567 |
| 2 | 1.29 | 0.660 | 0.588 |
| 3 | 1.27 | 0.659 | 0.495 |
| 4 | 1.29 | 0.626 | 0.514 |
| 5 | 1.24 | 0.594 | 0.631 |
| average | 1.27 | 0.632 | 0.559 |

The mass density for tantalum was assumed $q = 16.6$ [g/cm$^3$], while for nickel-chromium steel $q = 8.045$ [g/cm$^3$].

Table 4 presents the values of the dissipation coefficient calculated from (32) for five resonance frequencies and their mean values. We obtain the damping coefficient by putting mean values of $c$ and $a$ into formula (21). For data presented in Tables 3 and 4 we obtain values

$$\alpha_1 = 718\,[1/\text{s}], \quad \alpha_2 = 2215\,[1/\text{s}], \quad \alpha_3 = 2362\,[1/\text{s}].$$

For the third membrane were measured the amplitudes of vibrations additionally (Table 1), therefore it was possible to compute amplitudes of forced vibrations according to (33). The results are presented in Table 5.

**Table 5.** Amplitudes of forced vibrations

| No of resonance frequency | Amplitudes [1/m] |
|---|---|
| 1 | 0.349 |
| 2 | 0.376 |
| 3 | 0.375 |
| 4 | 0.377 |
| 5 | 0.364 |

### 5. Computer modelling of membrane vibrations

The "leap frog" difference method is frequently used to solve numerically the partial differential equation of hyperbolic type. In the adequate distribution of knots of the space grid the "wave character" of the occurred phenomena is taken into account. The second valuable advantage of this method is the simplicity of its algorithm. For these reasons the leap frog method was used to solve numerically equation (3) which is the mathematical model of vibrating membrane.

Dividing the membrane along its radius into $N$ segments, we define the arrangement of knots of discrete space

$$\Delta = \left\{(r_l, t_k): r_l = R-(l-1)\,h, t_k = (k-2)\frac{h}{c};\ h = \frac{2R}{2N-1};\ l = 1, 2, \ldots, N+1; \quad k = 1, 2, \ldots, \frac{Tc}{h}\right\}, \tag{34}$$

where $T$ denotes the final time. The knots are distributed in such a way that the coefficient of derivative in equation (3) with respect to membrane radius is limited,

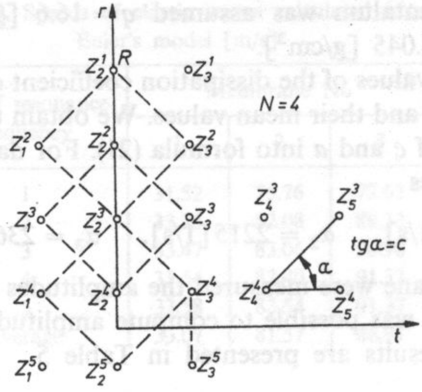


FIG. 1. Distribution of knots in space grid

that is $r_l \neq 0$. Thanks to that, the coefficients of difference equation are also limited. The derivatives from equation (3) are approximated by difference quotients in the following terms

$$\frac{\partial z}{\partial t} \sim (z_{k+1}^l - z_{k-1}^l)\frac{c}{2h}, \quad \frac{\partial^2 z}{\partial t^2} \sim (z_{k+1}^l - 2z_{k-1}^l + z_{k-1}^l)\frac{c^2}{h^2},$$
$$\frac{\partial z}{\partial r} \sim (z_k^{l-1} - z_k^{l+1})/2h, \quad \frac{\partial^2 z}{\partial r^2} \sim (z_k^{l-1} - 2z_k^l + z_k^{l+1})/h^2. \tag{35}$$

For the interior knots defined by (34) the approximate solution of the differential equation (3) is calculated from formula

$$z_{k+1}^l = z_{k-1}^l\, w_1 + z_k^{l+1}\, w_2 + z_k^{l-1}\, w_3 + w_4\, u_{(R-(l-1)h,kh/c)}, \tag{36}$$

where the coefficients have values

$$w_4 = \left(\frac{ac}{2h} + \frac{1}{h^2}\right)^{-1}, \quad w_1 = \left(\frac{ac}{2h} - \frac{1}{h^2}\right) w_4,$$
$$w_2 = \left(\frac{1}{h^2} - \frac{1}{2h[R-(l-1)h]}\right) w_4, \quad w_3 = \left(\frac{1}{h^2} + \frac{1}{2h[R-(l-1)h]}\right) w_4.$$

For the boundary knots we obtain from (5)

$$z_k^1 = 0, \quad z_k^{N+1} = z_k^N, \quad k = 3, 4, \ldots \tag{37}$$

The leap frog method is a three-step difference scheme. Therefore, at the beginning we put values into the first two steps

$$z_1^l = z_2^l = 0, \quad l = 1, 2, \ldots, N+1. \tag{38}$$

In this way we take into account the initial conditions (4).

The mesh width results from definition (34)

$$\Delta r = \frac{2R}{2N-1}, \quad \Delta t = \frac{2R}{c(2N-1)}. \tag{39}$$

The number of all knots is

$$I = (2N^2+N-1)\frac{Tc}{2R}. \tag{40}$$

It means that the time of computations increases considerably for large $N$. To obtain correct results of computer modelling of membrane vibrations with the frequency $f$ [Hz] it is sufficient to take

$$N \approx 40\frac{Rf}{c}. \tag{41}$$

Vibrations with the fifth resonance frequency for the membrane number 3 were simulated taking $N = 100$, in other words

$$\Delta r = 402.01\,[\mu\text{m}], \quad \Delta t = 4.373\,[\mu\text{s}].$$

For this example the time-constant of unsteady state, equal to inverse of $\alpha$, is equal to 0.42 [ms]. Therefore we can assume that unsteady state vanishes after time 1.7 [ms].

## 6. Conclusions

In Table 2 there are presented the results of calculations for the classical mathematical model without dissipation. The estimates for the speed of propagation of elastic wave were obtained for each resonance vibration. Differences between them and their mean values are considerable, especially for low frequencies. These differences amount from a few percents for the first and second membrane until 16% for the third membrane. On the other hand, the estimates of speed of elastic wave calculated for the model with dissipation (Table 3) have small deviations (less than 1%). The mean values of speed of elastic waves for the model with dissipation are greater than the mean values for the model without dissipation. The differences are considerable, they are equal to a few percents.

The coefficient of dissipation introduced as a new parameter into the mathematical model, enable to fit better the mathematical model to the experimental data. This possibility exists in general, even if there is no physical justification for such a treatment. However, we must remember that the formula (32) can be used only if $k_i \geqslant \omega_i/c$.

For the first membrane there are small differences between the estimates of coefficient of dissipation (Table 4). The greatest difference between the mean value and the estimate, occurs for the fifth resonance frequency of third membrane. Its value is equal to 13%.

The generator of electric sinusoidal oscillations had constant amplitude for all frequencies. Therefore the amplitudes of forced vibrations presented in Table 5 are not far each from the other.

The additional advantage of taking into account the dissipation of energy consists in obtaining formulae for numerical solution of the differential equation with better property of numerical stability.

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