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IMPEDANCE OF THE SEMI-INFINITE UNBAFFLED CYLINDRICAL WAVE-GUIDE OUTLET

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The paper presents the exact formula for the impedance of the outlet of a semi-infinite cylindrical wave-guide derived by considering the propagation of an arbitrary Bessel mode towards the outlet and accounting for the generation of all permissible mode due to the diffraction at the open end. For this purpose, the formula of acoustic potential as well as the expressions for the reflection and transformation coefficient were used.

The results of numerical calculations of the real and imaginary part and the moduli of impedance for the diffraction parameter *ka* in the range 0–20 were presented on graphs.

Wprowadzono ścisłą relację dla impedancji półnieskończonego falowodu cylindrycznego na podstawie teorii uwzględniającej dowolny mod propagacyjny w kierunku otwartego końca falowodu i wszystkich możliwych modów generowanych na tej nieciągłości falowodu. Wyniki zilustrowano w funkcji parametru *ka* w zakresie 0–20.

Introduction

In the practical applications of acoustics, the phenomena occurring at the open ends of wave-guides seem to be important because we come across such elements in different acoustical equipment, e.g. measuring pipes, acoustic horns, tubes. The investigations of the problem were introduced by Lord Rayleigh [1] who calculated the impedance of the outlet provided additionally with an infinitely rigid acoustic baffle and assuming that only the plane wave propagates towards the end. As a result, Rayleigh obtained the well known "correction for the open end". The further step in solving the problem was made when H. Levine and J. Schwinger [2] derived the acoustical potential of incident plane wave inside the unbaffled semi-infinite cylindrical pipe. However, they neglected the "higher modes effect", i.e. they assumed that only the plane wave is reflected. It is obvious that such assumption is valid only when the wave length is not smaller than the diameter of the pipe, what strongly restricts the applications of the results. In 1948 [3] Wajnshtejn developed an analytical theory of the acoustic field of semi-infinite cylindrical wave-guide applying the factorization method of solving the Wiener–Hopf integral equation. In his paper he worked out the exact formula for the acoustical potential inside the wave-guide making use of some analogies between acoustical and electromagnetic waves. The same results were obtained later by Snakowska and Wyrzykowski [4] who consequently applied to the problem the theory of acoustical field.

In this paper we calculate the impedance of the outlet of the semi-infinite cylindrical wave-guide for any z-axis symmetric Bessel mode propagating towards the end. For this purpose the exact formula for the acoustic field potential [3], [4] has been used.

The obtained numerical results are presented on graphs.

Index of symbols

a radius of wave-guide,
A_l, B_n amplitudes of Bessel modes,
D outlet area,
$H_n^{(1)}(\ )$ n-th order Hankel's function of first kind,
$J_n(\ )$ n-th order Bessel's function,
l, n indices of Bessel modes,
$L_+(\ ), L_-(\ )$ factors of L analytic in upper and lower complex half plane,
N index of the highest Bessel mode allowed in the considered wave-guide,
$N_n(\ )$ n-th order Neuman's function,
p acoustical pressure,
P apparent power,
R_{ln}, R_{ll} reflection and transformation coefficients,
$S(\ )$ function describing transformation coefficient R,
v radial wave number $v = \sqrt{k^2 - w^2}$,
w partial wave number,
γ_n partial wave number of n-th Bessel's wave mode,
ζ_l impedance of the outlet for the l-th mode incident,
$v_{\bar{n}}$ normal velocity of vibration,
θ_{ln} phase of the transformation coefficient R_{ln},
$\varkappa$ diffraction parameter,
μ_n n-th zero of Bessel's function $J_1(\ )$,
ϱ_0 medium density,
Σ wave-guide surface,
$\Phi(\ )$ acoustic potential,
$\Psi(\ )$ acoustical potential discontinuity on Σ,
Ω outlet surface.
Other symbols used in the text are typical and are not listed here.

2. Basic formulae

We will consider the cylindrical wave-guide with an infinitely thin and rigid wall Σ which, in suitable coordinates, can be described as follows:

$$\Sigma = \{(\varrho, z) : \varrho = a,\ z \geqslant 0\}.$$

To simplify the problem, we assume the z-axis symmetry (which means that the acoustic potential $\Phi(\bar{r}, t)$ does not depend on the angle variable φ in cylindrical coordinates) and the dependence on time in the form $\exp(-i\omega t)$.

The time dependent wave equation

$$\left(\Delta - \frac{1}{c^2}\partial_{tt}\right)\Phi(\bar{r}, t) = 0, \tag{1}$$

takes thus the following form:

$$\left(\frac{1}{\varrho}\partial_\varrho(\varrho\partial_\varrho) + \partial_{zz} + k^2\right)\Phi(\varrho, z) = 0. \tag{2}$$

The assumption that the wall Σ is perfectly rigid leads to the following boundary condition:

$$\partial_{\bar{n}}\Phi|_\Sigma = \partial_\varrho \Phi|_\Sigma = 0, \tag{3}$$

which means that the normal component of vibration velocity vanishes at the wave-guide wall.

The solution of the problem consists in finding the function $\Phi(\varrho, z)$ which satisfies Eq. (2) for the boundary condition (3) and, moreover, the Sommerfeld's conditions of radiation [6]. The detailed investigations leading to the solution are enclosed in [3, 5].

The application of the three-dimensional Green free space function and factorization method to the equation of acoustic potential leads to the expression [3, 5]:

$$\Phi(\varrho, z) = \frac{ai}{4}\int_0^\infty \psi(z')\,dz' \int_{-\infty+i\eta}^{\infty+i\eta} v \begin{Bmatrix} H_0^{(1)}(v\varrho)\,J_1(va) \\ H_1^{(1)}(va)\,J_0(v\varrho) \end{Bmatrix} e^{iw(z-z')}\,dw, \qquad \begin{matrix} \varrho > a, \\ \varrho < a, \end{matrix} \tag{4}$$

with the boundary condition taking form of the integral equation [3, 5]:

$$\int_0^\infty \psi(z')\,dz' \int_{-\infty+i\eta}^{\infty+i\eta} v^2 H_1^{(1)}(va)\,J_1(va)\,e^{iw(z-z')}\,dw = 0, \tag{5}$$

where $\psi(z)$ defines the potential discontinuity on the Σ surface

$$\psi(z) = \Phi(\varrho, z)|_{\varrho\to a_+} - \Phi(\varrho, z)|_{\varrho\to a_-}, \tag{6}$$

v being the radial wave number $v = \sqrt{k^2 - w^2}$. The potential discontinuity can be interpreted as the density of the surface sources on Σ.

Further development of the factorization method leads to the following expression for the acoustic potential $\Phi_l(\varrho, z)$ [3, 4]:

$$\Phi_l(\varrho, z) = A_l \frac{J_0\left(\mu_l \frac{\varrho}{a}\right)}{J_0(\mu_l)} e^{-i\gamma_l z} + \sum_{n=0}^{\infty} B_n \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)} e^{i\gamma_n z}. \tag{7}$$

The index l points at the fact that we consider the case of one simple Bessel mode incident. It is well known from the theory of infinite cylindrical wave-guide that the radial wave number of such a mode must be equal to $\gamma_n = \sqrt{k^2 - \left(\frac{\mu_n}{a}\right)^2}$, where μ_n is the n-th zero of the Bessel function $J_1(\,)$ and, moreover, to have γ_n real, the diffraction parameter $\varkappa = ka$ must be not smaller than μ_n. This leads to the conclusion that the index N of the highest mode which can propagate without scattering must fulfill the following condition $\mu_N \leqslant ka < \mu_{N+1}$.

The first component in Eq. (7) represents the l-th Bessel mode which, according to the assumption, propagates towards the wave-guide outlet, where it is partly reflected (a component with $n = l$ index under the sum sign) and, due to diffraction, is transformed into an infinite number of Bessel modes (other components under the sum sign). Analyzing carefully the exponential expression under the sum sign, we can see that for a fixed diffraction parameter ka only a certain number of components will represent the modes which can propagate along the wave-guide because starting from $N+1$, the exponents will became negative real numbers and thus the corresponding components of the sum will represent a disturbance attenuated exponentially with increasing z. Since these disturbances are not the energy carrying waves, they will be neglected in further considerations of impedance.

Reflection and transformation coefficients

According to previous assumptions in further developments, we will take into account the following expression for the acoustic potential inside the wave-guide [3, 4]:

$$\Phi_l(\varrho, z) = A_l \left[\frac{J_0\left(\mu_l \frac{\varrho}{a}\right)}{J_0(\mu_l)} e^{-i\gamma_l z} + \sum_{n=0}^{\infty} R_{ln} \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)} e^{i\gamma_n z} \right], \tag{8}$$

because we usually describe the diffraction phenomena on the outlet introducing the so-called reflection ($R_{ll} = B_l/A_l$) and transformation ($R_{ln} = B_n/A_l, n \neq 1$) coefficients. Detailed calculations [4], [5] lead to the following expression:

$$R_{ln} = \frac{-2\gamma_l}{\gamma_l + \gamma_n} \left(\prod_{\substack{i=0 \\ i \neq l}}^{N} \frac{\gamma_i + \gamma_l}{\gamma_i - \gamma_l} \prod_{\substack{i=0 \\ i \neq n}}^{N} \frac{\gamma_i + \gamma_n}{\gamma_i - \gamma_n} \right)^{1/2} e^{\frac{1}{2}[S(\gamma_l) + S(\gamma_n)]}, \tag{9}$$

$S(w)$ being equal

$$S(w) = \frac{2w}{\pi} \int_{ka}^{i\infty} \frac{1}{w^2 - w'^2} \left(\mathrm{tg}^{-1} \frac{N_1(a\sqrt{k^2 - w'^2})}{J_1(a\sqrt{k^2 - w'^2})} - n\pi \right) dw'. \tag{10}$$

Effective calculation of the values of R_{ln} coefficients as functions of the diffraction parameter ka are only possible by numerical methods because the $S(w)$ function cannot be expressed analytically. In the calculations the generally accepted definition of modules and phase of the wave reflection and transformation coefficient is used [3]:

$$R_{ln} = -|R_{ln}| e^{i\theta_{ln}}. \tag{11}$$

Knowing the explicit expressions for those coefficients, we finally obtained the explicit form of acoustic potential inside the wave-guide, which is necessary to calculate the acoustic impedance of the outlet.

4. Outlet impedance

To calculate the outlet impedance of the wave-guide, we shall use the formula of apparent acoustical power [7]

$$P = \int_{\Omega} \vartheta_{\bar{n}}(\varrho, 0)\, p(\varrho, 0)\, \mathrm{d}\sigma, \tag{12}$$

which is the surface integral over the outlet from the normal component of the velocity of vibration $\vartheta_{\bar{n}}$ and acoustic pressure p.
The required impedance is related to the apparent power P by the formula [7]:

$$\zeta = \frac{P}{D\langle \vartheta_{\bar{n}}^2 \rangle}, \tag{13}$$

where $\langle \vartheta_n^2 \rangle$ is the quadratic mean of the velocity at the outlet. The two quantities under the integral (12) are connected with the acoustic potential as follows:

$$p = -i\omega\varrho_0 \Phi, \tag{14}$$

$$\vartheta_{\bar{n}} = \partial_{\bar{n}} \Phi. \tag{15}$$

From simple calculation we get

$$p(\varrho, 0) = -i\omega\varrho_0 \sum_{n=0}^{N} (\delta_{ln} + R_{ln}) \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)}, \tag{16}$$

$$\vartheta_{\bar{n}}(\varrho, 0) = iA_l \sum_{n=0}^{N} (-\delta_{ln} + \mathrm{R}_{ln}) \gamma_n \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)}, \tag{17}$$

which leads to the following form of the impedance:

$$\zeta_l = -\omega\varrho_0 \frac{\sum_{n=0}^{N} |R_{ln}|^2 \gamma_n - (1 + 2i\, Im R_{ll}) \gamma_l}{\sum_{n=0}^{N} |R_{ln}|^2 \gamma_n^2 + (1 - 2\, Re\, R_{ll}) \gamma_l^2}. \tag{18}$$

The above result was obtained on the basis of the orthogonality property of the weighted Bessel functions [8]:

$$\int_0^a J_0\left(\mu_n \frac{\varrho}{a}\right) J_0\left(\mu_l \frac{\varrho}{a}\right) \varrho\, d\varrho = \frac{1}{2} \delta_{ln} a^2 J_0^2(\mu_l). \tag{19}$$

The real and imaginary part of the impedance are referred to the specific impedance of environment $\varrho_0 c$:

$$\operatorname{Re} \zeta_l^w = k \frac{\gamma_l - \sum_{n=0}^{N} |R_{ln}|^2 \gamma_n}{\sum_{n=0}^{N} |R_{ln}|^2 \gamma_n^2 + (1 - 2 \operatorname{Re} Rll) \gamma_l^2}, \tag{20}$$

$$\operatorname{Im} \zeta_l^w = k \frac{2 \gamma_l \operatorname{Im} R_l}{\sum_{n=0}^{N} |R_{ln}|^2 \gamma_n^2 + (1 - 2 \operatorname{Re} R_{ll}) \gamma_l^2}. \tag{21}$$

For $N = 0$ we get the case considered by Levine and Schwinger [2]. Expression (18) takes then the well-known form

$$\zeta_0^w = \frac{1 + R_{00}}{1 - R_{00}}, \tag{22}$$

R_{00} being the reflection coefficient of the plane wave.

5. Conclusions

The computer calculations of the real and imaginary part as well as moduli of the outlet impedance have been performed for the diffraction parameter *ka* varying within the range [0,20]. In Figs. 1 and 2 we compare the values of the acoustic impedance (real and imaginary part) of the unbaffled wave-guide for the plane wave outlet incident, respectively, with the results obtained by Rayleigh (Fig. 1) and by LEVINE and SCHWINGER [2].

As it can be seen from Fig. 1, the values of acoustic resistance computed by using the exact formulae are 1.5–2.0 times smaller than those obtained by Lord Rayleigh, although that difference decreases for the value of the diffraction parameter $ka \sim 3.0$. From the physical point of view it is obvious, because the baffle provides better radiation conditions. It is interesting to note that the acoustic reactance proceeds similarly only for $ka < 1.3$, for which value the two curves intersect and afterwards the values computed along the exact formula are about twice as large as Rayleigh's ones.

In Fig. 3 presenting the acoustic reactance for the plane wave and succeeding five Bessel modes incident, the following regularities can be seen:

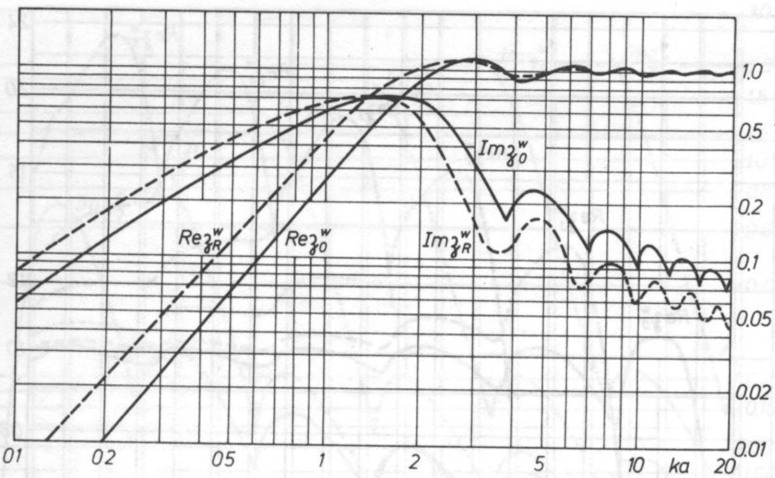

FIG. 1. Acoustic resistance and reactance of the unbaffled wave–guide outlet for the plane wave incident calculated after taking into account the higher Bessel modes which appear, due to diffraction phenomena (continuous line), compared with Rayleigh's resistance (dashed line)

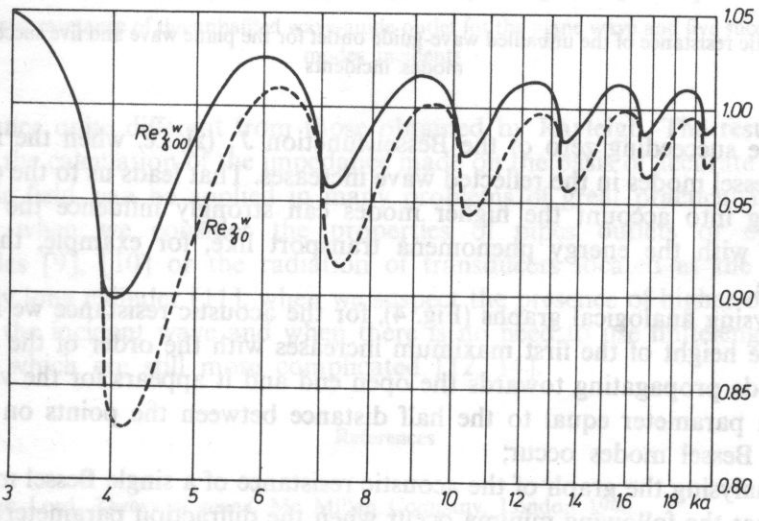

FIG. 2. Acoustic resistance of the unbaffled wave-guide outlet for the plane wave incident calculated after taking into account (dashed line) and neglecting (continuous line) the higher Bessel modes which appear on the open end, due to diffraction phenomena

– the acoustic reactance of the succeeding Bessel modes increases and the maxima appears for such values of diffraction parameter ka for which the adequate mode appears;

– analysing the diagram of the acoustic reactance of a single Bessel mode, it can be seen that the following maxima occur when the diffraction parameter ka

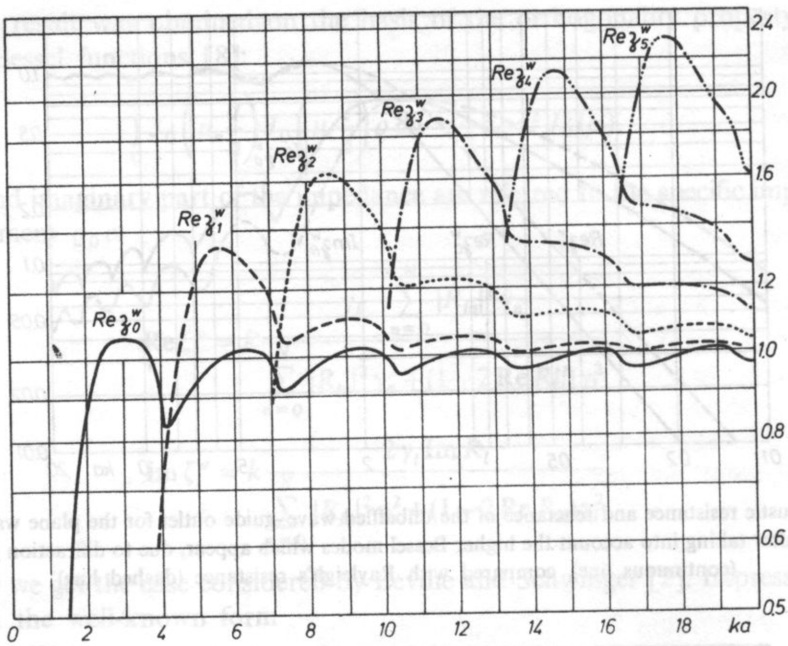

FIG. 3. Acoustic resistance of the unbaffled wave-guide outlet for the plane wave and five succeeding Bessel modes incidents

exceeds the succeeding zero of the Bessel function J_1 (z), i.e. when the number of allowed Bessel modes in the reflected wave increases. That leads us to the conclusion that taking into account the higher modes can strongly influence the quantities connected with the energy phenomena transport like, for example, the acoustic impedance.

Analysing analogical graphs (Fig. 4), for the acoustic resistance we notice that

– the height of the first maximum increases with the order of the considered Bessel mode propagating towards the open end and it appears for the value of the diffraction parameter equal to the half distance between the points on which the following Bessel modes occur;

– analysing the graph of the acoustic resistance of a single Bessel mode, it can be seen that the following minima occur when the diffraction parameter *ka* exceeds the value for which the next Bessel mode appears.

The obtained diagrams show that for wave-length shorter than the diameter of the wave-guide, in the presence of higher order Bessel modes the values of impedance differ considerably from those obtained for a plane wave by Rayleigh. It is obvious that for such a case the plane wave approximation can lead to important errors. It is well known that practically the generation of an ideal plane wave is very difficult, especially for the wave-guide with a large diameter in comparison with the wave length. In such a case we must consider the incident wave as a superposition of all allowed Bessel modes and it is possible that their contribution would lead to a value

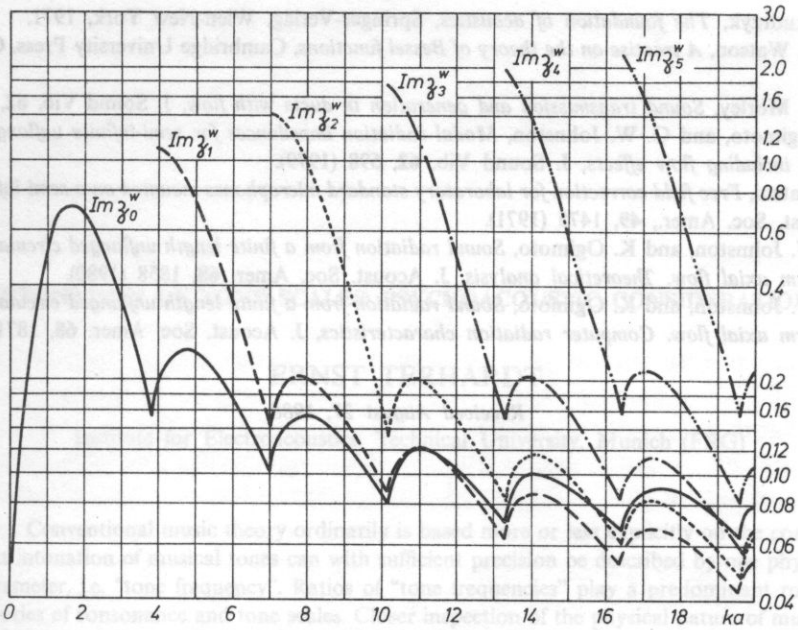

FIG. 4. Acoustic reactance of the unbaffled wave-guide outlet fot the plane wave and five succeeding Bessel modes incidents

of impedance quite different from those obtained by Rayleigh. The results of that paper, i.e. the calculation of the impedance made on the basis of accurate knowledge of acoustic field, can be applied in many problems of great practical importance, especially when we consider the properties of pipes outlets or cylinder-like wave-guides [9], [10] or the radiation of transducers located at the bottom of a relatively long cylinder [11], when we suspect the presence of higher-order Bessel modes in the incident wave and when there is no need to use finitelength cylinder formulae, which are still more complicated [12, 14].

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