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QUASISTATIC EFFECTIVE PARAMETERS OF VIBRATING ELEMENTS OF ELECTRO-ACOUSTIC TRANSDUCERS

ANDRZEJ B. DOBRUCKI, CECYLIA SZMAL

Institute of Telecommunication and Acoustics Wrocław Technical University
50–370 Wrocław, Wybrzeże Wyspiańskiego 27

This paper presents an approximate theory of determining effective parameters determining of surface elements of electro-acoustic transducers. It considers the distribution of displacements of these elements due to a steady load. Furthermore, a critical analysis of existing definitions of active surface was carried out and a new definition was proposed. It is a generalization of hitherto applied definitions. Equivalence of these definitions was proved. Examples of effective parameters calculations for systems with known accurate solutions are given. High consistency of proposed approximate methods with these solutions was demonstrated. Moreover examples of calculations of quasistatic effective parameters of systems previously solved with approximate methods are given.

W pracy przedstawiono przybliżoną teorię określania parametrów czynnych powierzchniowych elementów przetworników elektroakustycznych. Opiera się ona na rozpatrywaniu rozkładu przemieszczeń tych elementów pod wpływem obciążenia statycznego. Ponadto poddano krytycznej analizie istniejące definicje powierzchni czynnej i zaproponowano nową definicję, będącą uogólnieniem dotychczas stosowanych. Wykazano ekwiwalentność tych definicji. Podano przykłady obliczania parametrów czynnych dla układów, dla których znane są rozwiązania ścisłe i wykazano dużą zgodność zaproponowanych metod przybliżonych z tymi rozwiązaniami.Podano również przykłady obliczeń quasistatycznych parametrów czynnych układów, dla których rozwiązania były również znalezione metodami przybliżonymi.

1. Introduction

A standard electro-acoustic transducer consists of an electromechanical transducer which transforms an electric signal into a mechanical one or a mechanical signal into an electric one; and a superficial element – generally a membrane which radiates acoustic waves when it is a loudspeaker or receives them when it is a microphone. Generally, the method of equivalent electric circuits [7], based on analogies of mechanic, electric and acoustic systems, is used in the analysis of loudspeaker functioning. These analogies can be applied to elements with geometric dimensions much smaller than wave length. This is not much of a problem in the case of the electric part of the transducer in the range of acoustic frequencies, because the length of an electromagnetic wave in this frequency range is equal to tens of kilometers. The analysis of mechanic and acoustic equivalent circuits can cause problems. Sound wave velocity in air is equal to about 340 m/s. So, the length of an acoustic wave for frequencies of several thousand Hz is comparable with transducer's dimensions. And even smaller velocities of mechanical waves are found in surface vibrating systems, such as plates and shells. Therefore, concentrated values of these elements are significant in a frequency range up to several hundred Hz.

Concentrated parameters equivalent to individual mechanical and acoustic elements are called effective parameters. In a general case they depend on frequency, because of the distribution variability of amplitudes of vibrations on the surface for various activation frequencies. Such a distribution depends not only on the geometry of the surface element, but also on the activation method. This makes the analysis additionally complicated. This paper proposes such a definition of effective parameters that their values are nearly independent from frequency for a relatively large frequency band.

2. Definitions of effective parameters

The following effective parameters characterize a vibrating surface element: mass, stiffness and surface. The notions of mass and stiffness are related with Rayleigh's method [4] consisting in checking a system with distributed parameters in relation to a system with one degree of freedom. Elements of such a system are calculated on the basis of a comparison of potential and kinetic energy of a real vibrating system with these energies of a system with one degree of freedom. The kinetic and potential energy of a real vibrating system depends on the distribution of displacements on the shell's surface which, in turn, depends on frequency and on the method of activation. In a general case the equation of motion of a homogeneous shell with — a model of a surface vibrating element of a transducer — can be expressed with [2]:

$$L(\mathbf{u}) - \varrho h \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{P}[r, \varphi, z(r, \varphi), t], \tag{1}$$

where ϱ — density of material, h — shell thickness, L — differential operator dependent on shell's shape, $\mathbf{u}$ — vector of displacements of the shell's centre surface, $\mathbf{P}$ — vector of activation.

Fig. 1 presents the shell's geometry. Equation

$$z = z(r, \varphi) \tag{2}$$

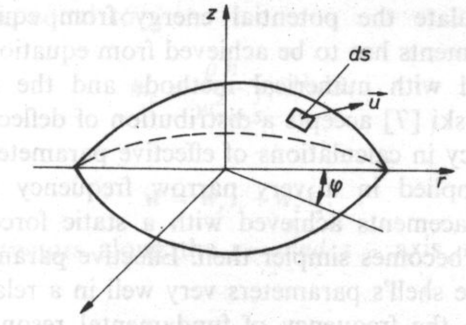

Fig. 1. Geometry of a vibrating shell

describes its shape in the case of axial symmetry. The form of operator L depends on various shapes of the shell.

In case of harmonic excitation

$$\mathbf{P}[r, \varphi, z(r, \varphi), t] = \mathbf{p}[r, \varphi, z(r, \varphi)]e^{j\omega t} \tag{3a}$$

displacement are also sinusoidal

$$\mathbf{u}[r, \varphi, z(r, \varphi), t] = \mathbf{w}[r, \varphi, z(r, \varphi)]e^{j\omega t} \tag{3b}$$

then equation (1) has the following form:

$$L(w) + \omega^2 \varrho h \mathbf{w} = \mathbf{p}[r, \varphi, z(r, \varphi)]. \tag{4}$$

When $\mathbf{p}(r, \varphi, z) = 0$, then the equation of motion is reduced to a eigen problem. Equations (1) and (4) are equations of equilibrium of forces. The first term on the left side describes elastic forces acting on the shell. They are a result of own elasticity. These forces cause a displacement of the shell's elements in relation to the state of equilibrium and impart a certain potential energy to the shell

$$U = \frac{1}{2}\int_s L(\mathbf{w}) \cdot \mathbf{w} dS. \tag{5}$$

Effective stiffness is achieved on the basis of comparison between this energy and the potential energy of a system with one degree of freedom

$$U = \frac{1}{2} k_c w_{\max}^2, \tag{6}$$

where k_c – effective stiffness, $w_{\max}$ – a certain, discriminated displacement of the shell, most frequently equal to the maximum axial deflection, to the displacement of its geometrical centre in the case of activation by a uniformly distributed acoustic pressure, or a displacement at the point of activation by an axial concentrated force.

In order to calculate the potential energy from equation (5), the shell's distribution of displacements has to be achieved from equation (4). This equation is most frequently solved with numerical methods and the solution is frequency dependent. Z. Żyszkowski [7] accepts a distribution of deflections achieved for the first resonance frequency in calculations of effective parameters. Unfortunately this assumption can be applied in a very narrow frequency range. In this paper a distribution of displacements achieved with a static force, i.e. for $\omega = 0$, was accepted. Equation (3) becomes simpler then. Effective parameters calculated from this equation model the shell's parameters very well in a relatively wide frequency range – from zero to the frequency of fundamental resonance.

Neglecting the dynamic term in equation (4) we obtain:

$$L(\mathbf{w}) = \mathbf{p}[r, \varphi, z(r, \varphi)]. \tag{7}$$

Then, equation (5) can be noted as:

$$U = \frac{1}{2}\int_S \mathbf{p}[r, \varphi, z(r, \varphi)]\,\mathbf{w}\,dS. \tag{8}$$

We have to do with an interesting case when the shell is activated by a force concentrated in the medium and acting along the z-axis

$$\mathbf{p}(r, \varphi, z) = P_z \delta[r, \varphi, z(r, \varphi)], \tag{9}$$

where $\delta[r, \varphi, z(r, \varphi)]$ is the δ–Dirac function. Then the potential energy is equal to

$$U = \frac{1}{2} P_z w_{\max}. \tag{10}$$

From comparison of equations (6) and (10) an expression for effective rigidity we see that

$$k_c = \frac{P_z}{w_{\max}}. \tag{11}$$

The second term in equation (4) defines the inertial force related with the kinetic energy of shell's motion.

Kinetic energy of the shell is given by expression

$$T = \frac{1}{2}\varrho h \omega^2 \int_S |\mathbf{w}|^2\, dS \tag{12}$$

where

$$\mathbf{w}(r, \varphi)|_\omega \approx \mathbf{w}(r, \varphi)|_{\omega=0}.$$

The effective mass is achieved from a comparison of the kinetic energy determined from expression (12) and the kinetic energy of a system with one degree of freedom

$$T = \frac{1}{2} m_c \omega^2 w_{\max}^2. \tag{13}$$

Thus, effective mass is equal to

$$m_c = \frac{\varrho h}{w_{\max}^2} \int_S |\bar{w}|^2 \, dS. \tag{14}$$

Because:

$$\mathbf{w} = w_r \mathbf{i}_r + w_z \mathbf{i}_z, \tag{15}$$

where $\mathbf{i}_r$, $\mathbf{i}_z$ are unit vectors along the $r-$ *and* $z-$ axis, respectively. Then, we have

$$m_c = \frac{\varrho h}{w_{\max}^2} \int_S w_r^2 \, dS + \frac{\varrho h}{w_{\max}^2} \int_S w_z^2 \, dS. \tag{16}$$

The first term in equation (16) determines the effective mass related to the displacements of the shell in the radial direction, and the second with displacements in the axial direction.

The effective surface is the last among effective parameters which characterize surface systems. Definitions of this quantity given in [5, 7] are not general definitions. According to Żyszkowski the definition refers to plane systems only, while the definition of Makarewicz and Konieczny concerns a specific method of activating the shell, namely – with a plane wave. Therefore, it can not be used for a different activation method i.e. with a spherical wave or by force concentrated in a point. A definition without these shortcomings is presented below.

Effective surface is the surface of a flat piston shifted perpendicularily to its plane by a force of the value equal to the value of a characteristic virtual displacement of the shell due to a uniformly distributed pressure of such a value that performed by it work is equal to the work of forces acting on the shell.

This definition is based on the principle of virtual work [4] and introduces the notion of pressure equivalent to forces really acting on the shell. This pressure does not have to be acoustic pressure; it can also be static pressure.

When the distribution of static displacements achieved from equation (7) is accepted, then a quasistatic effective surface is obtained. Virtual work performed in order to virtually displace the shell by $\delta \mathbf{w}$ is equal to

$$\delta W = \int_S \mathbf{p}[r, \varphi, z(r, \varphi)] \, \delta \mathbf{w} dS. \tag{17}$$

Substituting

$$\delta \mathbf{w} = \delta w_{\max} \frac{\mathbf{w}}{w_{\max}}, \tag{18}$$

we have

$$\delta w = \frac{\delta w_{\max}}{w_{\max}} \int_S \mathbf{p}[r, \varphi, z(r, \varphi)] \, \mathbf{w} dS. \tag{19}$$

The work performed by an uniformly distributed pressure resulting in the same displacement of the shell is equal to

$$\delta W = p\int_S \delta w_n dS = p\frac{\delta w_{max}}{w_{max}}\int_S w_n dS \tag{20}$$

where w_n is the displacement normal to the surface S. We have normal displacement in formula (20), because the force acting on the shell due to an uniformly distributed pressure is perpendicular to the shell's surface. On the basis of a comparison between relationships (19) and (20) we reach an expression for pressure corresponding to force actually acting on the shelll:

$$p = \frac{\int_S \mathbf{p}[r, \varphi, z(r, \varphi)]\, w dS}{\int_S w_n dS}. \tag{21}$$

If such a pressure is exerted on a flat piston with surface equal to the effective surface S_c and shifts it by δw_{max} then work equal to

$$\delta w = pS_c \delta w_{max} \tag{22}$$

is performed.

On the basis of relationships (20), (21) and (22) the effective surface is equal to

$$S_c = \frac{1}{w_{max}}\int_S w_n dS. \tag{23}$$

The following new definition of the effective surface is thus equivalent to the definition given above:

The effective surface is such a surface of a flat piston which ensures a volumetric displacement of the piston equal to the volumetric displacement of the shell.

This definition does not include a direct dependence on the shell activation method. Therefore, it is of more general character than the definition based on the principle of virtual work. It is a well-known fact [6] that the power radiated by the source in the range of low frequencies depends on the volumetric velocity of the source and, on the other hand, it does not depend on its shape. The definition of the effective surface based on the equivalence of volumetric deflections leads to the substitution of a real source by a flat piston which radiated the same power as the source under consideration in an infinitely great baffle. The normal displacement in the equation for effective surface (23) can be expressed with two components – axial and radial:

$$S_c = \frac{1}{w_{max}}\int_S w_n dS = \frac{1}{w_{max}}\left[\int_S w_z \cos(n, z) dS + \int_S w_r \cos(n, r) dS\right] \tag{24}$$

$$= \frac{1}{w_{max}}\left[\int_0^{r(\varphi)}\int_0^{2\pi} w_z r dr d\varphi + \int_0^{r(\varphi)}\int_0^{2\pi} w_r \frac{dz}{dr} r dr d\varphi\right].$$

In a case of an axial activation of the shell the second component of the sum in expression (24) is very small and can be neglected. If the shell's motion is purely axial, then it is equal to zero exactly. If additionally $w_z(r) = \text{const}$, i.e. the shell vibrates like a rigid piston, then we reach a well known result – the effective surface is equal to the surface of the shell's orthogonal projection onto the plane perpendicular to the z – axis. The second term of the sum appearing in Eq. (24) is decisive in the case of radial vibrations of a cylindrical shell. The Żyszkowski and Makarewicz definitions did not take into account the possibility of evaluation of the active surface of such a system.

3. Effective parameters of circular plates

The equation of the circular plate has been solved analytically. Therefore, strictly theoretical results can be compared with results of the theory based on quasistatic displacements of the shell.

Frequently the circular plate is the vibrating element in electroacoustic transducers. It is a planar system.

The equation for vibrations of a plate has the following form

$$\Delta\Delta u + \frac{\varrho h}{B}\frac{\partial^2 u}{\partial t^2} = \frac{P(r, \varphi, t)}{B}, \tag{25}$$

where $u = we^{j\omega t}$ – displacement of the plate, h – plate's thickness, $B = \dfrac{Eh^3}{12(1-\nu^2)}$ – flexural rigidity, ϱ – density of material, E – Young's modulus, ν – Poisson ratio, $P(r, \varphi, t) = p(r, \varphi)e^{j\omega t}$ – force acting on an unit surface, Δ – Laplace operator.

We will consider a case of $p(r, \varphi) = p = \text{const}$. The system are axi-symmetrical. Operator L from equation (1) has the following form

$$L(u) = -\frac{B}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)\right]\right\}. \tag{26}$$

The equation of quasistatic displacements of the plate – equivalent to equation (7) has the following form:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{P}{B}. \tag{27}$$

The solution of this equation depends on boundary conditions. Two typical boundary conditions will be considered:

1. Plate's edge simple-supported

$$w(a) = 0,$$
$$\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0 \quad \text{for } r = a \tag{28}$$

where a — plate's radius.

This equation has the following solution

$$w = w_{\max}\left[1 - \left(\frac{r}{a}\right)^2\right]\left[1 - \frac{1+\nu}{5+\nu}\left(\frac{r}{a}\right)^2\right], \tag{29}$$

where

$$w_{\max} = \frac{pa^4}{64B}\frac{5+\nu}{1+\nu}. \tag{30}$$

The potential energy equivalent to this displacement is equal to:

$$U = \frac{1}{2}\int_S wp\,dS = \frac{1}{2}w_{\max}^2 \frac{64\pi B}{a^2}\frac{(1+\nu)(7+\nu)}{3(5+\nu)^2}. \tag{31}$$

Thus, the effective stiffness corresponding to this potential energy equals:

$$k_c = \frac{64\pi B}{3a^2}\frac{(1+\nu)(7+\nu)}{(5+\nu)^2}. \tag{32}$$

The following equation expresses the kinetic energy corresponding to the quasistatic distribution of velocity on the plate's surface

$$T = \frac{1}{2}\varrho h\omega^2 w_{\max}^2 \int_S \left[1 - \left(\frac{r}{a}\right)^2\right]^2 \left[1 - \frac{1+\nu}{5+\nu}\left(\frac{r}{a}\right)^2\right]^2 dS = \frac{1}{2}M_g v_{\max}^2 \frac{(113+36\nu+3\nu^2)}{15(5+\nu)^2}, \tag{33}$$

where $v_{\max} = \omega w_{\max}$ — plate's velocity in the geometrical centre, $M_g = \varrho h\pi a^2$ — geometrical mass of the plate. The effective mass equivalent to this energy is equal to:

$$m_c = \frac{3\nu^2 + 36\nu + 113}{15(5+\nu)^2} M_g. \tag{34}$$

The resonance frequency calculated on the basis of effective parameters is equal to

$$f_{\text{rez}} = \frac{1}{2\pi}\sqrt{\frac{k_c}{m_c}} = \frac{1}{2\pi a^2}\sqrt{\frac{B}{\varrho h}\frac{320(1+\nu)(7+\nu)}{3\nu^2+36\nu+113}}. \tag{35}$$

The accurate solution, corresponding to the dynamic equation of plate's vibrations, contains ordinary and modified Bessel functions [1]. For $\nu = 0.3$ the fundamental frequency of plate's vibrations, calculated from accurate considerations, is expressed by:

$$f_{01} = \frac{1}{2\pi} \frac{(k_{01}\, a)^2}{a^2} \sqrt{\frac{B}{\varrho h}}, \tag{36}$$

where $k_{01}\ a = 2.221$ is the first eigenvalue.

The value corresponding to it in formula (35) is equal to 2.224. Hence, the relative error is equal to 0.15%. It is worth mentioning that active parameters depend on the Poisson ratio which occurs explicitly in boundary conditions. Therefore, the Poisson ratio can be measured indirectly on this basis.

And lastly the effective surface of a supported plate was calculated

$$S_c = \int_S w dS = \pi a^2 \frac{7+\nu}{3(5+\nu)}. \tag{37}$$

For $\nu = 0.3$ its value is equal to 0.459 of the geometrical surface.

2. Plate's edge clamped

In this case boundary conditions have the following form:

$$w(a) = 0, \quad \left.\frac{dw}{dr}\right|_{r=a} = 0. \tag{38}$$

The solution of equation (27) is as follows

$$w(r) = w_{\max}\left[1 - \left(\frac{r}{a}\right)^2\right]^2, \tag{39}$$

where

$$w_{\max} = \frac{pa^4}{64B}. \tag{40}$$

The effective stiffness of this plate equals

$$k_c = \frac{64}{3}\frac{\pi B}{a^2}, \tag{41}$$

whereas, effective mass:

$$m_c = \frac{1}{5} M_g. \tag{42}$$

The resonance frequency, calculated on the basis of these parameters, is equal to

$$f_r = \frac{3.21^2}{2\pi a^2}\sqrt{\frac{B}{\varrho h}}. \tag{43}$$

In this case, the coefficient corresponding to the first eigen value is equal to 3.21, while its value determined from the accurate solution is equal to 3.19. This coefficient does not depend on the Poisson ratio. The Poisson ratio is not present in the boundary conditions.
Therefore, the effective surface:

$$S_c = \frac{1}{3}\pi a^2 \tag{44}$$

is smaller than the analogic surface of a simple-supported plate.

4. Effective parameters of loudspeaker suspension of the voice-coil

Calculation results of effective parameters of non-planar systems will be presented in this paragraph. Apart from simple cases, the form of the Loperator for such systems remains unknown. The distribution of displacements for a quasistatic activation was determined with an approximate method of finite elements [3]. In order to study the influence of chosen geometrical features on active parameters the shape of typical loudspeaker suspension of the voice-coil and of certain model shells was considered in calculations. An unit-value concentrated force, acting on the internal rim of the shell under consideration, was accepted as the activating factor. Fig. 2 presents the shape of the model.

The following standard data were included in calculations:

– Young's modulus $E = 10^9$ N/m^2
– Poisson ratio $\nu = 0.3$
– density of material $\varrho = 1000$ kg/m^3
– the external rigidity of the suspension of the voice-coil is rigidly clamped while the internal rim can move freely along the z–axis only.

The following quantities varied:

– thickness of the suspension of the voice-coil h
– wave length DF
– number of waves n
– wave height A
– internal – R_w and external – R_z radii, at fixed cross-section shape.

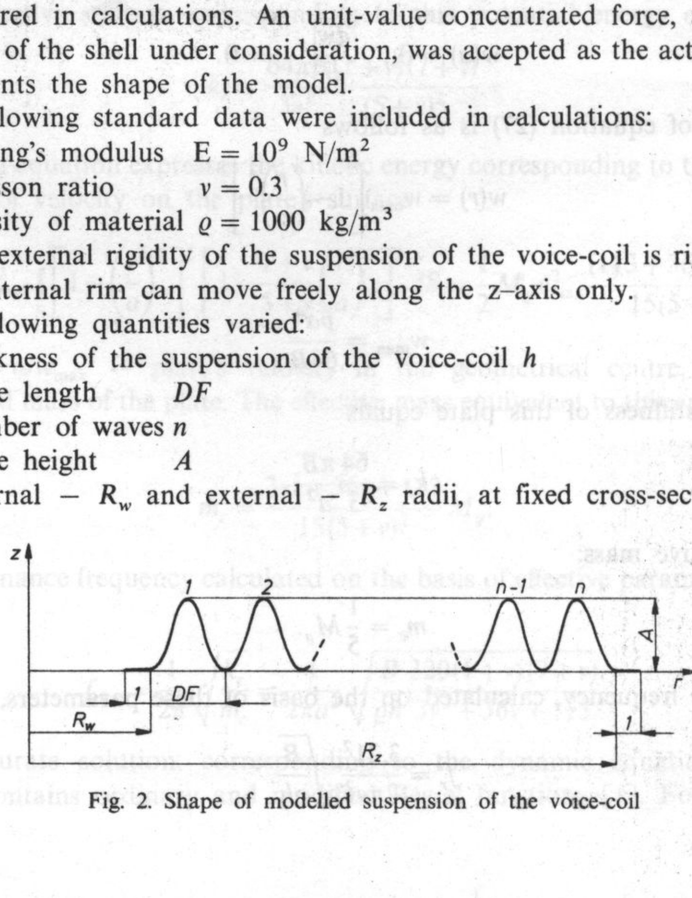

Fig. 2. Shape of modelled suspension of the voice-coil

Table 1. Effective parameters of suspension of the voice-coil

Geometric parameters of the suspension of the voice-coil						Effective parameters of the suspension of the voice-coil				
Lp.	h [mm]	A [mm]	n	DF [mm]	R_w [mm]	R_z [mm]	k_c [N/m]	m_c [g]	m_c/M_g	S_{c_2} [cm^2]	$S_c/\pi(R_z^2-R_w^2)$
1	0.35	2	7	5	24	59	1289	1.262	0.2893	41.61	0.4235
2	0.25	”	”	”	”	”	832	0.929	0.2981	42.77	0.4353
3	0.45	”	”	”	”	”	1848	1.584	0.2823	40.72	0.4144
4	0.35	1	”	”	”	”	481	0.978	0.2680	38.19	0.3969
5	”	3	”	”	”	”	2951	1.621	0.3032	45.64	0.4423
6	”	2	5	”	”	49	2219	0.835	0.2996	27.13	0.4304
7	”	”	9	”	”	69	862	1.762	0.2830	58.67	0.4194
8	”	”	7	4	”	52	1923	1.060	0.2968	32.05	0.4300
9	”	”	”	6	”	66	936	1.496	0.2834	52.46	0.4185
10	”	”	”	5	12	47	1669	0.853	0.2750	28.75	0.4116

Calculation results are presented in Table 1. Effective parameters related to shell's displacements along the radius are not given. They are very small in comparison to adequate effective parameters related with shell's displacements along the axis. For example, in the case of suspension of the voice-coil no 1 the value of the ratio of effective mass related to radial displacement of the suspension of the voice-coil and the geometrical mass is equal to $4.258\cdot10^{-4}$. Whereas, the adequate surface ratio is equal to $1.093\cdot10^{-3}$. A comparison between these data and results presented in Table 1 proves the error due to the neglection of these parameters to be insignificant.

5. Conclusions

A comparison between results of accurate and quasistatic solution proves that the difference between them does not exceed 1%.

Characteristic features of the accurate solution, such as Poisson ratio-dependence or independence of parameters for example, can also be found in the approximate solution.

Values of effective parameters calculated for a non-planar system, such as a loudspeaker suspension of the voice-coil indicate a strong influence of geometrical features on the value of parameters.

To the suspension of the voice-coil designer the suspension voice-coil stiffness is the most interesting feature. It depends on thickness and increases with an increase of thickness — more than h^1 (pure tension) but less than h^3 (pure bending). The index is not constant and depends on thickness. The average value for considered suspension of the voice-coil was equal to 1.35. Hence, material work conditions are closer to tension.

Stiffness increases with the height of the suspension of the voice-coil wave. A plate has the lowest stiffness. Yet it is not applied in loudspeaker suspension of the voice-coil, because of high nonlinearity. Stiffness decreases with an increase of the suspension of the voice-coil wave length and number of waves *n*.

Also other effective parameters, such as mass and surface, depend on geometrical parameters, but the ratio between them and geometrical mass and surface are nearly independent from shape. Changes of absolute values are related with variable amounts of material used to produce the suspension of the voice-coil and with variable geometrical surface of various suspension of the voice-coil structures.

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